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COMMON FIXED POINTS ITERATION PROCESSES FOR TWO ASYMPTOTIC POINTWISE NONEXPANSIVE MAPPINGS IN A HADAMARD SPACE

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Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday

Abstract. In this paper, we introduce and study the new type of Ishikawa iteration schemes for approximating common fixed points of two asymptotic pointwise nonexpansive mappings in a Hadamard space. Weak and strong convergence of iterates of two asymptotic pointwise nonexpansive mappings without any condition on the rate of convergence, namely $\Sigma_{n=1}^{\infty}(c_n(x)-1) < \infty$, associated with the two maps in a Hadamard space are established. Some results have been obtained which generalize and unify many important known results in recent literature.

1. Introduction and preliminaries

Iterative construction of fixed point for asymptotic nonexpansive mappings (but not asymptotic pointwise nonexpansive maps) including Mann and Ishikawa iteration processes in Hilbert, Banach and metric spaces have been studied extensively by various authors, see e.g. [1, 4–7, 9–15, 17, 19, 20, 23, 26–29, 31, 32, 34, 36–38, 40] and the references therein.

In 1991, Schu [38] proved the weak convergence of the modified Mann iteration process to a fixed point of asymptotic nonexpansive mappings in uniformly convex Banach spaces with the Opial property [32] and the strong convergence for compact asymptotic nonexpansive mappings in uniformly convex Banach spaces. Since then, Schu's iteration process has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert space or Banach spaces (see [33, 35, 36, 39]).

In 1994, Tan and Xu [40] proved the weak convergence of modified Mann and Ishikawa iteration processes for asymptotic nonexpansive mappings in uniformly convex Banach spaces, which have either a Fréchet differentiable norm or Opial property.

Note that the rate of convergence condition, namely $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, has remained in extensive use to prove both weak and strong convergence theorems to approximate fixed points of asymptotic nonexpansive mappings. The conditions like Opial's condition, Kadec-Klee property or Fréchet differentiable norm have remained key to prove weak convergence theorems. Also Tan and Xu [40] remarked:

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we do not know whether our Theorem 3.1 remains valid if k_n (the sequence associated with the asymptotic nonexpansive mapping T) is allowed to approach 1 slowly enough so that $\sum_{n=1}^{\infty}(k_n - 1)$ diverges.

In 2010, Khan and Fukhar-Ud-Din [21] established weak convergence of Ishikawa iterates of two asymptotic nonexpansive mappings without any condition on the rate of convergence associated with the two mappings. They also got the new weak convergence theorem which does not require any of the Opial condition, Kadec-Klee property or Fréchet differentiable norm.

The class of asymptotic nonexpansive self-mappings is a natural generalization of the important class of nonexpansive mappings. Goebel and Kirk [14] proved that if C is a nonempty closed convex and bounded subset of a real uniformly convex Banach space, then every asymptotic nonexpansive self-mapping has a fixed point.

T is said to be an asymptotic pointwise nonexpansive mapping if there exists a sequence of maps $a_n : C \to [0, \infty)$ such that

$$d(T^n(x), T^n(y)) \leq a_n(x)d(x, y)$$

for all $x, y \in C, n \geq 1$, where $\limsup_{n\to\infty} a_n(x) \leq 1$. Denote $c_n(x) = \max(a_n(x), 1)$. Then note that without any loss of generality, T is an asymptotic pointwise nonexpansive mapping if

$$d(T^n(x), T^n(y)) \leq c_n(x)d(x, y)$$

for all $x, y \in C, n \geq 1$, where $c_n(x) \geq 1$ and $\lim_{n\to\infty} c_n(x) = 1$. Moreover, we recall that $T : C \to C$ is uniformly L-Lipschitzian if for some $L > 0$ we have that

$$d(T^n x, T^n y) \leq Ld(x, y)$$

for $x, y \in K$ and $n \geq 1$. T is an asymptotic nonexpansive mapping if there is a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y)$$

for all $x, y \in C$ and $n \geq 1$. T is said to be semi-compact (completely continuous) if for any bounded sequence $\{x_n\}$ in C with $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x \in C$ as $i \to \infty$.

Let $S, T : C \to C$ be asymptotic pointwise nonexpansive mappings with function sequences $\{a_n(x) \geq 1\}$ and $\{b_n(x) \geq 1\}$ satisfying $\lim_{n\to\infty} a_n(x) = 1$ and $\lim_{n\to\infty} b_n(x) = 1$, respectively. Set $c_n(x) = max[a_n(x), b_n(x)]$. Then

$$\lim_{n\to\infty} c_n(x) = \lim_{n\to\infty} a_n(x) = \lim_{n\to\infty} b_n(x) = 1.$$

Throughout the paper, we shall take $\tau(C)$ as the class of all asymptotic pointwise nonexpansive self-mappings T and C with function sequence $\{c_n(x) \geq 1\}$ with $\lim_{n\to\infty} c_n(x) = 1$ for every $T \in \tau(C)$. Also F will stand for the set of common fixed points of the two maps $S, T : C \to C$. We assume that c_n is a bounded function for every $n \geq 1$ and all the functions c_n are not bounded by a common constant, therefore an asymptotic pointwise nonexpansive mapping is not uniformly Lipschitzian. However, an asymptotic nonexpansive mapping is an asymptotic pointwise nonexpansive mapping as well as uniformly Lipschitzian.

A strictly increasing sequence $\{n_i\}$ of natural numbers is quasi-periodic if the sequence $\{n_{i+1} - n_i\}$ is bounded or equivalently if there exists a natural number q

such that any block of q consecutive natural numbers must contain a term of the sequence $\{n_i\}$ The smallest of such numbers q will be called a quasi-period of $\{n_i\}$.

In 2008, Kirk and Xu [25] studied the existence of fixed points of asymptotic pointwise nonexpansive self-mapping T on C in Banach spaces defined by:

$$d(T^n(x), T^n(y)) \leq c_n(x)d(x, y)$$

for all $x, y \in C$, where $\limsup_{n\to\infty} c_n(x) \leq 1$. Their main result (see [25], Theorem 3.5) states that every asymptotic pointwise nonexpansive self-mapping of a nonempty closed bounded convex subset C of a uniformly convex Banach space has a fixed point. This result of Kirk and Xu is a generalization of Goebel and Kirk fixed point theorem [14] for a narrower class of maps, the class of asymptotic nonexpansive mappings, where (using our notation) every function c_n is a constant function.

In 2009, the results of [25] has been generalized by Hussain and Khamsi [16] to metric spaces. As pointed out by Kirk and Xu in [25], asymptotic pointwise mappings seem to be a natural generalization of nonexpansive maps. The conditions on c_n can be, for instance, expressed in terms of the derivatives of iterations of T for differentiable T. Hussain and Khamsi [16] have shown that if X is a Hadamard space and C is a nonempty bounded closed convex subset of X, then any pointwise asymptotic nonexpansive self-mapping on C has a fixed point. Moreover, this fixed point set is closed and convex. The proof of this important theorem is of the existential nature and does not describe any algorithm for constructing a fixed point of an asymptotic pointwise nonexpansive mapping. It is well known that the iteration processes for generalized nonexpansive mappings have been successfully used to develop efficient and powerful numerical methods for solving various nonlinear equations and variational problems.

Espinola et al. [8] examined the convergence of iterates for asymptotic pointwise contractions in uniformly convex metric spaces. Kozlowski [26] proved convergence to a fixed point of some iterative algorithms applied to asymptotic pointwise mappings in Banach spaces.

For more on metric fixed point theory, the reader may consult the book of Khamsi and Kirk [18].

Recently, Khan and Fukhar-ud-din [22] used the concept of unique geodesic path denoted by $\alpha x \oplus (1-\alpha)y$ of two points x, y in geodesic space and define Ishikawa iterative process $I(S, T, \alpha_k, \beta_k, n_k)$ of two pointwise asymptotic nonexpansive mappings in a geodesic space. It is given as follows:

$$\begin{aligned} x_{k+1} &= (1-\alpha_k)x_k \oplus \alpha_k S^{n_k} y_k, \\ y_k &= (1-\beta_k)x_k \oplus \beta_k T^{n_k} x_k, \ k \geq 1, \end{aligned} \tag{1.1}$$

where $\{n_k\}$ is an increasing sequence of natural numbers, $0 \leq \alpha_k, \beta_k \leq 1$ and $I(S, T, \alpha_k, \beta_k, n_k)$ is well -defined if $\limsup_{k\to\infty} c_{n_k}(x_k) = 1$. They studied the weak and strong convergence of the scheme (1.1) under proper conditions.

Inspired and motivated by the recent works, we introduce and study a new type of Ishikawa iterative schemes in this paper. The scheme is defined as follows:

Let C be a nonempty and convex subset of a geodesic space X. Let $S, T : C \to C$ be asymptotic pointwise nonexpansive mappings and let $\{n_k\}$ be an increasing sequence of natural numbers and $0 \leq \alpha_k, \beta_k \leq 1$. Then the new type of Ishikawa iteration process denoted by $I(S, T, \alpha_k, \beta_k, n_k)$ in a geodesic space X is as under:

$$\begin{aligned} x_{k+1} &= (1-\alpha_k)y_k \oplus \alpha_k S^{n_k} y_k, \\ y_k &= (1-\beta_k)x_k \oplus \beta_k T^{n_k} x_k, \ k \geq 1. \end{aligned} \tag{1.2}$$

The iterative schemes (1.1) and (1.2) are independent: neither reduces to the other.

The purpose of this paper is to construct an iteration scheme for approximating common fixed points of two asymptotic pointwise nonexpansive mappings and to prove some strong and weak convergence theorems for such mappings in a Hadamard space.

Now, we recall some well known concepts and results.

Let (X, d) be a metric space. (X, d) is said to be a length space if any two points of X are joined by a rectifiable path (that is, a path of finite length) and the distance between any two points of X is taken to be the infimum of the lengths of all rectifiable paths joining them. In this case, d is known as *length metric* (otherwise an *inner metric* or *intrinsic metric*). In case, no rectifiable path joins two points of the space, the distance between them is taken to be ∞. A *geodesic space* is a metric space such that every $x, y \in X$ can be joined by a *geodesic* map $c : [0, l] \to X$ were $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. Moreover, c is an isometry and $d(x, y) = l$. X is said to be *uniquely geodesic* if for every $x, y \in X$ there is exactly one geodesic joining them, which will be denoted by $[x, y]$, and called the segment joining x to y.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the *vertices* of Δ) and three geodesic segments between each pair of vertices (the edges of Δ). A *comparison triangle* for $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [2]).

A geodesic metric space is a $CAT(0)$ space if every geodesic triangle satisfies the following $CAT(0)$ inequality:

$$d(x, y) \leq d(\bar{x}, \bar{y})$$

for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$. If x, y, z are points of a $CAT(0)$ space and $y_0 = \frac{x \oplus y}{2}$ is the midpoint of the segment $[x, y]$, then the $CAT(0)$ inequality implies:

$$d^2(z, y_0) \leq \frac{1}{2}d^2(z, x) + \frac{1}{2}d^2(z, y) - \frac{1}{4}d^2(x, y),$$

which is the (CN) inequality of Bruhat and Tits [3]. For any $\alpha \in [0, 1]$ and $x, y, z \in X$, Dhompongsa and Panyanak [7] modified the (CN) inequality of Bruhat and Tits [3] as

$$\begin{aligned} d^2(z, \alpha x \oplus (1-\alpha)y) &\leq \alpha d^2(z, x) + (1-\alpha)d^2(z, y) \\ &\quad -\alpha(1-\alpha)d^2(x, y). \end{aligned} \tag{1.3}$$

If $\alpha = \frac{1}{2}$, then (1.3) reduces to the original (CN) inequality of Bruhat and Tits [4].

Let us recall that a *geodesic metric space* is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see [2], p. 163). Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [24]) and if $x, y, z \in X$ and $\alpha \in [0,1]$, then there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \quad \leq \quad \alpha d(z, x) + (1-\alpha)d(z, y). \tag{1.4}$$

A subset C of a $CAT(0)$ space X is convex if for any $x, y \in C$, we have $[x, y] \subset C$.

Let $\{x_k\}$ be a bounded sequence in a metric space X. For $x \in X$, define $r(x, \{x_k\}) = \limsup_{k\to\infty} d(x_k, x)$. The asymptotic radius $r(\{x_k\})$ of $\{x_k\}$ is given by:

$$r(\{x_k\}) = \inf\{r(x, \{x_k\}) : x \in X\}.$$

The asymptotic center of a bounded sequence $\{x_k\}$ with respect to $C \subseteq X$ is defined by

$$A_c(\{x_k\}) = \{x \in X : r\{(x, \{x_k\}) \leq r(y, \{x_k\})$$

for any $y \in C\}$. If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_k\})$.

A bounded sequence $\{x_k\}$ in X is said to be regular if $r(\{x_k\}) = r(\{u_k\})$ for every subsequence $\{u_k\}$ of $\{x_k\}$. Recall that a sequence $\{x_k\}$ converges weakly to w (written as $x_k \rightharpoonup w$) if and only if $r(w, \{x_k\}) = \inf_{x\in C} r(x, \{x_k\})$, where C is a closed and convex subset containing the bounded sequence $\{x_k\}$. Moreover, a sequence $\{x_k\} \subseteq X$ $\Delta-$converges to $x \in X$ if x is the unique asymptotic center of $\{u_k\}$ for every subsequence $\{u_k\}$ of $\{x_k\}$. In this case, we write $\Delta - \lim_n x_n = x$ and x is called $\Delta -$ limit of $\{x_n\}$.

In a Banach space setting, $\Delta-$convergence coincides with weak convergence. A connection between weak convergence and $\Delta-$convergence in geodesic spaces is characterized in the following lemma due to Nanjaras and Panyanak [30].

Lemma 1.1 (see [30], Proposition 3.12)**.** Let $\{x_k\}$ be a bounded sequence in a $CAT(0)$ space X and let C be a closed and convex subset of X which contains $\{x_k\}$.Then

(1) $\Delta - \lim_k x_k = x$ *implies that* $x_k \rightharpoonup x$,

(2) *the converse of (1) is true if* $\{x_k\}$ *is regular.*

The following demiclosed principle in $CAT(0)$ spaces due to Hussain and Khamsi [16] plays an important role in the study of the weak convergence theorem.

Lemma 1.2 (see [22])**.** Let C be a nonempty bounded closed convex set in a $CAT(0)$ space X and let $T : C \to C$ be an asymptotic pointwise nonexpansive mapping. Let $\{x_k\}$ be a sequence in C such that $\{x_k\} \rightharpoonup \omega$ and $\lim_{k\to\infty} d(x_k, Tx_k) = 0$. Then $T(\omega) = \omega$.

2. Main results

In this section, we prove weak and strong convergence theorems of the iterative scheme given in (1.2) to a common fixed point for two asymptotic pointwise nonexpansive mappings in a Hadamard space. In order to prove our main results, the following lemmas are needed.

Lemma 2.1. *Let X be a geodesic space and let C be a nonempty closed convex bounded subset in X. Let $S, T \in \tau(C)$ and let $\{n_k\}$ be an increasing sequence of natural numbers such that the sequence $\{x_k\}$ in (1.2) is well defined. If the set $\Omega = \{j : n_{j+1} = n_j + 1\}$ is quasi-periodic and*

$$\lim_{k\to\infty} d(x_k, S^{n_k}x_k) = 0 = \lim_{k\to\infty} d(x_k, T^{n_k}x_k), \tag{2.1}$$

then

$$\lim_{k\to\infty} d(x_k, Sx_k) = 0 = \lim_{k\to\infty} d(x_k, Tx_k).$$

Proof. Set $c_k = d(x_k, S^{n_k}x_k)$ and $d_k = d(x_k, T^{n_k}x_k)$. Using (1.2) and (1.4), we have

$$\begin{aligned} d(x_k, y_k) &= d(x_k, (1-\beta_k)x_k \oplus \beta_k T^{n_k}x_k) \\ &\le \beta_k d(x_k, T^{n_k}x_k) + (1-\beta_k)d(x_k, x_k) \\ &= \beta_k d(x_k, T^{n_k}x_k) \\ &\le d(x_k, T^{n_k}x_k). \end{aligned} \tag{2.2}$$

From (2.1) and (2.2), we have

$$\lim_{k\to\infty} d(x_k, y_k) = 0. \tag{2.3}$$

Using (1.2) and (1.4), we have

$$\begin{aligned} d(x_k, x_{k+1}) &= d(x_k, (1-\alpha_k)y_k \oplus \alpha_k S^{n_k}y_k) \\ &\le \alpha_k d(x_k, S^{n_k}y_k) + (1-\alpha_k)d(x_k, y_k) \\ &\le d(x_k, S^{n_k}x_k) + d(S^{n_k}x_k, S^{n_k}y_k) + d(x_k, y_k) \\ &\le d(x_k, S^{n_k}x_k) + c_{n_k}(x_k)d(x_k, y_k) + d(x_k, y_k). \end{aligned} \tag{2.4}$$

It follows from (2.1), (2.3) and (2.4) that

$$\lim_{k\to\infty} d(x_k, x_{k+1}) = 0. \tag{2.5}$$

In addition,

$$\begin{aligned} d(x_k, Sx_k) &\le d(x_k, x_{k+1}) + d(x_{k+1}, Sx_k) \\ &\le d(x_k, x_{k+1}) + d(x_{k+1}, S^{n_{k+1}}x_{k+1}) \\ &\quad + d(S^{n_{k+1}}x_{k+1}, S^{n_{k+1}}x_k) + d(S^{n_k+1}x_k, Sx_k) \\ &\le d(x_k, x_{k+1}) + d(x_{k+1}, S^{n_{k+1}}x_{k+1}) \\ &\quad + c_{n_{k+1}}(x_{k+1})d(x_{k+1}, x_k) + c_1(x_1)d(S^{n_k}x_k, x_k) \\ &= d(x_{k+1}, S^{n_{k+1}}x_{k+1}) + (1 + c_{n_{k+1}}(x_{k+1}))d(x_{k+1}, x_k) \\ &\quad + c_1(x_1)d(S^{n_k}x_k, x_k). \end{aligned} \tag{2.6}$$

By taking lim sup on both sides of inequality (2.6) and using (2.1) and (2.5), we obtain $\limsup_{k\to\infty} d(x_k, Sx_k) \le 0$ and hence

$$\lim_{k\to\infty} d(x_k, Sx_k) = 0.$$

Similarly, we may show that

$$\lim_{k\to\infty} d(x_k, Tx_k) = 0.$$

That is, $\lim_{k\to\infty} d(x_k, Sx_k) = 0 = \lim_{k\to\infty} d(x_k, Tx_k)$. □

Lemma 2.2. *Let X be a Hadamard space and let C be a nonempty closed convex bounded subset in X. Let $S, T \in \tau(C)$, let $\alpha_k, \beta_k \in (\delta, 1-\delta)$ for some $\delta \in (0, \frac{1}{2})$ and let $\{n_k\}$ be an increasing sequence of natural numbers such that the sequence $\{x_k\}$ in (1.2) is well-defined. If the set $\Omega = \{j : n_{j+1} = n_j + 1\}$ is quasi-periodic and $F \neq \emptyset$, then*

$$\lim_{k\to\infty} d(x_k, Sx_k) = 0 = \lim_{k\to\infty} d(x_k, Tx_k).$$

Proof. Let $p \in F$. Using (CN) inequality (1.3) and (1.2), we have

$$\begin{aligned}
d^2(x_{k+1}, p) &= d^2((1-\alpha_k)y_k \oplus \alpha_k S^{n_k} y_k, p) \\
&\leq \alpha_k d^2(S^{n_k} y_k, p) + (1-\alpha_k) d^2(y_k, p) - \alpha_k(1-\alpha_k) d^2(y_k, S^{n_k} y_k) \\
&\leq \alpha_k c_{n_k}^2(p) d^2(y_k, p) + (1-\alpha_k) d^2(y_k, p) \\
&\quad -\alpha_k(1-\alpha_k) d^2(y_k, S^{n_k} y_k) \\
&\leq \alpha_k c_{n_k}^2(p) d^2(y_k, p) + c_{n_k}^2(p)(1-\alpha_k) d^2(y_k, p) \\
&\quad -\alpha_k(1-\alpha_k) d^2(y_k, S^{n_k} y_k) \\
&= c_{n_k}^2(p) d^2(y_k, p) - \alpha_k(1-\alpha_k) d^2(y_k, S^{n_k} y_k) \\
&= c_{n_k}^2(p) d^2((1-\beta_k)x_k \oplus \beta_k T^{n_k} x_k, p) \\
&\quad -\alpha_k(1-\alpha_k) d^2(y_k, S^{n_k} y_k) \\
&\leq c_{n_k}^2(p)(\beta_k c_{n_k}^2(p) d^2(x_k, p) + (1-\beta_k) d^2(x_k, p) \\
&\quad -\beta_k(1-\beta_k) d^2(x_k, T^{n_k} x_k)) - \alpha_k(1-\alpha_k) d^2(y_k, S^{n_k} y_k) \\
&\leq \beta_k c_{n_k}^4(p) d^2(x_k, p) + (1-\beta_k) c_{n_k}^4(p) d^2(x_k, p) \\
&\quad -c_{n_k}^2(p)\beta_k(1-\beta_k) d^2(x_k, T^{n_k} x_k) - \alpha_k(1-\alpha_k) d^2(y_k, S^{n_k} y_k) \\
&= c_{n_k}^4(p) d^2(x_k, p) - c_{n_k}^2(p)\beta_k(1-\beta_k) d^2(x_k, T^{n_k} x_k) \\
&\quad -\alpha_k(1-\alpha_k) d^2(y_k, S^{n_k} y_k) \\
&= d^2(x_k, p) + c_{n_k}^4(p) d^2(x_k, p) - d^2(x_k, p) \\
&\quad -c_{n_k}^2(p)\beta_k(1-\beta_k) d^2(x_k, T^{n_k} x_k) - \alpha_k(1-\alpha_k) d^2(y_k, S^{n_k} y_k). \qquad (2.7)
\end{aligned}$$

Since C is bounded, there exists $B_r[x_0] = \{x \in X : d(x, x_0) \leq r\}$ such that $C \subset B_r[x_0]$ for some $r > 0$. Therefore the inequality (2.7) becomes

$$\begin{aligned}
d^2(x_{k+1}, p) &\leq d^2(x_k, p) + r^2(c_{n_k}^4(p) - 1) \\
&\quad -\delta^2 d^2(x_k, T^{n_k} x_k) - \delta^2 d^2(y_k, S^{n_k} y_k). \qquad (2.8)
\end{aligned}$$

Form (2.8), we obtain the following two important inequalities:

$$d^2(x_{k+1}, p) \leq d^2(x_k, p) + r^2(c_{n_k}^4(p) - 1) - \delta^2 d^2(y_k, S^{n_k} y_k), \qquad (2.9)$$

$$d^2(x_{k+1}, p) \leq d^2(x_k, p) + r^2(c_{n_k}^4(p) - 1) - \delta^2 d^2(x_k, T^{n_k} x_k). \qquad (2.10)$$

Now, we prove that

$$\lim_{k\to\infty} d(y_k, S^{n_k} y_k) = 0 = \lim_{k\to\infty} d(x_k, T^{n_k} x_k).$$

Assume that $\limsup_{k\to\infty} d(y_k, S^{n_k}y_k) > 0$. Then, there exist a subsequence (use the same notation for subsequence as for the sequence) of $\{x_k\}$ and $\mu > 0$ such that $d(y_k, S^{n_k}y_k) \geq \mu > 0$. Form (2.9), we have

$$\begin{aligned} d^2(x_{k+1}, p) &\leq d^2(x_k, p) + r^2(c^4_{n_k}(p) - 1) - (\delta\mu)^2 \\ &= d^2(x_k, p) + r^2((c^4_{n_k}(p) - 1) - \frac{(\delta\mu)^2}{2r^2}) - \frac{(\delta\mu)^2}{2}. \end{aligned} \tag{2.11}$$

In addition, $c^4_{n_k}(p) \to 1$ and $\frac{(\delta\mu)^2}{2r^2} > 0$; there exists $k_0 \geq 1$ such that $(c^4_{n_k}(p) - 1) < \frac{(\delta\mu)^2}{2r^2}$ for all $k \geq k_0$. From (2.11) we obtain

$$\frac{(\delta\mu)^2}{2} \leq d^2(x_k, p) - d^2(x_{k+1}, p) \tag{2.12}$$

for all $k \geq k_0$. Let $l \geq k_0$. It follows from (2.12) that

$$\begin{aligned} \frac{(\delta\mu)^2}{2}(l - k_0) &\leq d^2(x_{k_0}, p) - d^2(x_{l+1}, p) \\ &\leq d^2(x_{k_0}, p). \end{aligned} \tag{2.13}$$

By letting $l \to \infty$ in (2.13), we obtain

$$\infty \leq d^2(x_{k_0}, p) < \infty,$$

which contradicts the reality. This proves that $\mu = 0$. Thus,

$$\limsup_{k\to\infty} d(y_k, S^{n_k}y_k) \leq 0.$$

Consequently, we have

$$\lim_{k\to\infty} d(y_k, S^{n_k}y_k) = 0. \tag{2.14}$$

Similarly, using (2.10), we may show that

$$\lim_{k\to\infty} d(x_k, T^{n_k}x_k) = 0. \tag{2.15}$$

Using (2.3) and (2.14), we have

$$\begin{aligned} d(x_k, S^{n_k}x_k) &\leq d(x_k, y_k) + d(y_k, S^{n_k}x_k) \\ &\leq d(x_k, y_k) + d(y_k, S^{n_k}y_k) + d(S^{n_k}x_k, S^{n_k}y_k) \\ &\leq d(x_k, y_k) + d(y_k, S^{n_k}y_k) + b_{n_k}(x_k)d(x_k, y_k) \\ &\quad \to 0 \ \ (\text{as } k \to \infty). \end{aligned}$$

That is,

$$\lim_{k\to\infty} d(x_k, S^{n_k}x_k) = 0. \tag{2.16}$$

Finally, using (2.15) and (2.16), Lemma 2.1 appeals that

$$\lim_{k\to\infty} d(x_k, Sx_k) = 0 = \lim_{k\to\infty} d(x_k, Tx_k).$$

□

Next, we deal with the weak convergence of the sequence $\{x_k\}$ defined by (1.2) in a Hadamard space.

Theorem 2.3. *Let X be a Hadamard space and let C be a nonempty closed convex bounded subset in X. Let $S, T \in \tau(C)$, let $\alpha_k, \beta_k \in (\delta, 1-\delta)$ for some $\delta \in (0, \frac{1}{2})$ and let $\{n_k\}$ be an increasing sequence of natural numbers such that the sequence $\{x_k\}$ in (1.2) is well defined. If the set $\Omega = \{j : n_{j+1} = n_j + 1\}$ is quasi-periodic and $F \neq \emptyset$, then $\{x_k\}$ converges weakly to a point in F.*

Proof. Let $\omega_w(x_k)$ be the set of all weak subsequential limits of $\{x_k\}$. That is, $\omega_w(x_k) = \{y \in C : x_{k_i} \rightharpoonup y \text{ for } \{x_{k_i}\} \subseteq \{x_k\}\}$. Since C is a nonempty bounded closed convex subset of a Hadamard space X, there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that x_{k_i} converges weakly to $p \in \omega_w(x_k)$. This show that $\omega_w(x_k) \neq \emptyset$ and, using Lemma 2.1,

$$\lim_{i\to\infty} d(x_{k_i}, Sx_{k_i}) = 0 = \lim_{i\to\infty} d(x_{k_i}, Tx_{k_i}).$$

It follows from Lemma 1.2 that $Sp = p = Tp$. Therefore $\omega_w(x_k) \subset F$. Next, we follow the idea of Chang *et al.* [5]. For any $p \in \omega_w(x_k)$, there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that

$$x_{k_i} \rightharpoonup p \text{ (as } i \to \infty). \tag{2.17}$$

It follows from (2.15) and (2.17) that

$$T^{n_{k_i}} x_{k_i} \rightharpoonup p \text{ (as } i \to \infty). \tag{2.18}$$

Now, from (1.2), (2.17) and (2.18), we get that

$$y_{k_i} = (1-\beta_{k_i})x_{k_i} \oplus \beta_{k_i} T^{n_{k_i}} \rightharpoonup p \text{ (as } i \to \infty). \tag{2.19}$$

Also, from (2.14) and (2.19), we have

$$S^{n_{k_i}} y_{k_i} \rightharpoonup p \text{ (as } i \to \infty). \tag{2.20}$$

It follows from (1.2), (2.19) and (2.20) that

$$x_{k_i+1} = (1-\alpha_{k_i})y_{k_i} \oplus \alpha_{k_i} S^{n_{k_i}} y_{k_i} \rightharpoonup p \text{ (as } i \to \infty).$$

Continuing in this way, by induction, we can prove that, for any $l \geq 0$,

$$x_{k_i+l} \rightharpoonup p.$$

By induction, one can prove that $\bigcup_{l=0}^{\infty}\{x_{k_j+l}\}$ converges weakly to p as $j \to \infty$; in fact, $\{x_k\}_{k=k_1}^{\infty} = \bigcup_{l=0}^{\infty}\{x_{k_j+l}\}_{j=1}^{\infty}$ gives that $x_k \rightharpoonup p$ as $k \to \infty$. This completes the proof. □

Remark 2.4. If $\{x_k\}$ is reqular in a geodesic space, then $\{x_k\}$ is $\Delta-$convergent.

Our strong convergence theorem is as follows. We do not use the rate of convergence condition namely $\sum_{k=1}^{\infty}(c_{n_k}(x) - 1) < \infty$ in its proof.

Theorem 2.5. *Let X be a Hadamard space and let C be a nonempty closed convex bounded subset in X. Let $S, T \in \tau(C)$, let $\alpha_k, \beta_k \in (\delta, 1-\delta)$ for some $\delta \in (0, \frac{1}{2})$ and let $\{n_k\}$ be an increasing sequence of natural numbers such that the sequence $\{x_k\}$ in (1.2) is well defind. If the set $\Omega = \{j : n_{j+1} = n_j + 1\}$ is quasi-periodic and $F \neq \emptyset$ and either S or T is semi-compact (completely continuous), then x_k converges strongly to a point in F.*

Proof. Suppose that S is semi-compact. By Lemma 2.2, we have

$$lim_{k\to\infty} d(x_k, Sx_k) = 0.$$

Since S is semi-compact, there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that

$$x_{k_i} \to p \text{ (as } i \to \infty).$$

Now Lemma 2.3 guarantees that $\lim_{i\to\infty} d(x_{k_i}, Sx_{k_i}) = 0$. By the continuity of S and T, we obtain that $p \in F$. The rest of the proof follows by replacing $\rightharpoonup$ with $\to$ in Theorem 2.3 and we, therefore, omit the details. □

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