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FIXED POINTS OF PROXIMINAL VALUED β-ψ-CONTRACTIVE MULTIFUNCTIONS

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Abstract. In this paper, we provide some fixed point results for proximinal valued β-ψ-contractive multifunctions.

1. Introduction

During the last 50 years, a lot of fixed point results appeared on metric spaces and ordered metric spaces by using different notions and distinct methods (see for example, [1–21]). In 2012, Samet et al. introduced the notion of α-ψ-contractive mappings and proved some fixed point results for such mappings ([23]). Afterwards, some authors further investigated this notion and obtained several fixed point results (see, e.g., [9, 10] and [22]). This notion was extended to multifunctions in [5, 7] and [11].

Denote by Ψ the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$. It is well known that $\psi(t) < t$ for all $t > 0$. Let (X, d) be a metric space. Let $\beta : 2^X \times 2^X \to [0, \infty)$ be a mapping and $\psi \in \Psi$. A closed valued multifunction $T : X \to 2^X$ is said to be β-ψ-contractive if $\beta(Tx, Ty)H(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$, where H is the Hausdorff distance. Also, we say that T is β-admissible if $\beta(A, B) \geq 1$ implies $\beta(Tx, Ty) \geq 1$ for all $x \in A$ and $y \in B$, where A and B are subsets of X. A subset A of a metric space (X, d) is called proximinal if for each $x \in X$ there exists $a_0 \in A$ such that $d(x, a_0) = \inf_{a \in A} d(a, x)$. We denote the set of all proximinal subsets of X by $P(X)$. Finally, let $(X, \preceq)$ be an ordered set and $A, B \subseteq X$. We say that $A \preceq B$ if for each $a \in A$ there exists $b \in B$ such that $a \preceq b$. In this paper, we provide some fixed point results for proximinal valued β-ψ-contractive multifunctions.

2. Main results

Now, we are ready to state and prove our main results.

Theorem 2.1. *Let (X, d) be a complete metric space, $\beta : 2^X \times 2^X \to [0, +\infty)$ a mapping and $T : X \to P(X)$ a β-admissible and β-ψ-contractive multifunction.*

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Suppose that there exist $A \subset X$ *and* $x_0 \in A$ *such that* $\beta(A, Tx_0) \geq 1$. *If* $x \to d(x, Tx)$ *is lower semi-continuous, then* T *has a fixed point.*

Proof. Take the subset A of X and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Choose $x_1 \in Tx_0$ such that $d(x_0, x_1) = \inf_{a \in Tx_0} d(a, x_0)$. By continuing this process we can find a sequence $\{x_n\}$ in X such that $d(x_n, x_{n+1}) = \inf_{a \in Tx_n} d(a, x_n)$ and $x_{n+1} \in Tx_n$ for all $n \geq 1$. If $x_n = x_{n+1}$ for some n, then x_n is a fixed point of T. Assume that $x_n \neq x_{n-1}$ for all $n \geq 1$. Since $\beta(A, Tx_0)$ and T is β-admissible, $\beta(Tx_0, Tx_1) \geq 1$. It is easy to get that $\beta(Tx_{n-1}, Tx_n) \geq 1$ for all $n \geq 1$. On the other hand, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq H(Tx_{n-1}, Tx_n) \\ &\leq \beta(Tx_{n-1}, Tx_n) H(Tx_{n-1}, Tx_n) \\ &\leq \psi(d(x_{n-1}, x_n)) \end{aligned}$$

and so $d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n))$ for all n. Now by induction, we get $d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1))$ for all n. Let $\varepsilon > 0$ be given. Choose a natural number N_ε such that $\sum_{n \geq N_\varepsilon} \psi^n(t) < \varepsilon$. Let $m > n \geq N_\varepsilon$. Then,

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \psi^k d(x_0, x_1) \\ &\leq \sum_{n \geq N_\varepsilon} \psi^n d(x_0, x_1) \\ &< \varepsilon. \end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence. Choose $x^* \in X$ such that $x_n \to x^*$. Since $d(x_n, Tx_n) \leq \beta(Tx_{n-1}, Tx_n) H(Tx_{n-1}, Tx_n) \leq \psi(d(x_{n-1}, x_n))$ for all n. we have $\lim_{n \to \infty} d(x_n, Tx_n) = 0$. Lower semi-continuity of $x \to d(x, Tx)$ further implies that $d(x^*, Tx^*) = 0$. Since each proximinal set is closed, we get $x^* \in Tx^*$. □

The following example shows that there are multifunctions satisfying the conditions of Theorem 2.1.

Example 2.2. *Let* $X = [0, \infty)$ *and* $d(x, y) = |x - y|$. *Define the multifunction* $T : X \to P(X)$ *by* $Tx = [\frac{x}{2}, 4]$ *for* $x \leq 4$ *and* $Tx = [2, x]$ *for* $4 < x$. *Also, define the mappings* $\beta : 2^X \times 2^X \to [0, +\infty)$ *and* $\psi : [0, +\infty) \to [0, +\infty)$ *by* $\psi(t) = \frac{t}{2}$ *and* $\beta(A, B) = 1$ *if* A *and* B *are subsets of* $[0, 4]$ *and* $\beta(A, B) = 0$ *otherwise. It is easy to see that* T *is* β*-admissible and* $\beta(Tx, Ty) H(Tx, Ty) \leq \psi(d(x, y))$ *for all* $x, y \in X$. *Let* $A = [1, 2]$ *and* $x_0 = 1$. *Then,* $Tx_0 = [\frac{1}{2}, 4]$ *and* $\beta(A, Tx_0) = 1$. *Also* $x \to d(x, Tx)$ *is lower semi-continuous.*

Corollary 2.3. *Let* (X, d) *be a complete metric space,* $x^* \in X$ *a fixed element and* $T : X \to P(X)$ *a multifunction such that* $H(Tx, Ty) \leq \psi(d(x, y))$ *for all* $x, y \in X$ *with* $x^* \in Tx \cap Ty$. *Suppose that there exist* $A \subset X$ *and* $x_1 \in A$ *such that* $x^* \in Tx_1 \cap A$. *Assume that for each subset* $B \subset X$ *with* $x_0 \in A \cap B$, *we have* $x^* \in Tx \cap Ty$ *for all* $x \in A$ *and* $y \in B$. *If* $x \to d(x, Tx)$ *is lower semi-continuous, then* T *has a fixed point.*

Proof. It is sufficient we define $\beta : 2^X \times 2^X \to [0, \infty)$ by $\beta(A, B) = 1$ if $x^* \in A \cap B$ and $\beta(A, B) = 0$ otherwise, and then we use Theorem 2.1. □

Corollary 2.4. *Let $(X, \preceq, d)$ be a complete ordered metric space and $T : X \to P(X)$ a multifunction such that $H(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$ with $Tx \preceq Ty$. Suppose that there exist $A \subset X$ and $x_0 \in A$ such that $Tx_0 \preceq A$. Assume that for each subset $B \subset X$ with $A \preceq B$, we have $Tx \preceq Ty$ for all $x \in A$ and $y \in B$. If $x \to d(x, Tx)$ is lower semi-continuous, then T has a fixed point.*

Proof. It is sufficient we define $\beta(A, B) = 1$ if $A \preceq B$ and $\beta(A, B) = 0$ otherwise, and then we use Theorem 2.1. □

Now, we give our second result by replacing a distinct condition instead of lower semi-continuity. We call it Condition (R).

Theorem 2.5. *Let (X, d) be a complete metric space and $T : X \to P(X)$ a β-admissible and β-ψ-contractive multifunction. Suppose that there exist $A \subset X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Also, suppose that for each convergent sequence $\{x_n\}$ in X with $x_n \to x$ and $\beta(Tx_{n-1}, Tx_n) \geq 1$ for all n, we have $\beta(Tx_n, Tx) \geq 1$ for all n. Then T has a fixed point.*

Proof. By using a similar argument in the proof of Theorem 2.1, we obtain the Cauchy sequence $\{x_n\}$ in X such that $\beta(Tx_{n-1}, Tx_n) \geq 1$ for all n. Choose an element $x^* \in X$ such that $x_n \to x^*$. Thus, $\beta(Tx_{n-1}, Tx^*) \geq 1$ for all n. Hence,

$$d(x^*, Tx^*) \leq d(x^*, y) \leq d(x^*, z) + d(z, y)$$

for all $y \in Tx^*$ and all $z \in X$ and so $d(x^*, Tx^*) \leq d(x^*, Tx_{n-1}) + d(Tx_{n-1}, Tx^*)$. Thus, we get

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, Tx_{n-1}) + d(Tx_{n-1}, Tx^*) \\ &\leq d(x^*, x_n) + \beta(Tx_{n-1}, Tx^*) d(Tx_{n-1}, Tx^*) \\ &\leq d(x^*, x_n) + \psi(d(x_{n-1}, x^*)) \\ &\leq d(x^*, x_n) + d(x_{n-1}, x^*). \end{aligned}$$

Hence, $d(x^*, Tx^*) = 0$ and so $x^* \in Tx^*$. □

The following examples show that there are multifunctions satisfying Theorem 2.5.

Example 2.6. *Let $X = \mathbb{N} \cup \{0\}$ and $d(x, y) = |x - y|$. Define the proximinal valued multifunction T on $\mathbb{R}$ by $Tx = \cup_{n \geq 1} [x + 2n]_{10}$, where*

$$[x + 2n]_{10} = \{m \in X : x + 2n \equiv m \ (mod 10)\}.$$

If we calculate, then we get that $Tx = \{1, 3, 5, 7, 9, 11, \dots\}$ whenever x is odd and $Tx = \{0, 2, 4, 6, 8, 10, \dots\}$ otherwise. Also, suppose that $\psi \in \Psi$ and define the mapping $\beta : 2^X \times 2^X \to [0, +\infty)$ by $\beta(A, B) = 1$ if A and B does not have any even elements and $\beta(A, B) = 0$ otherwise. Let A and B be subsets of X with $\beta(A, B) \geq 1$. Then A and B have only odd elements and so

$$Tx = Ty = \{1, 3, 5, 7, 9, \dots\}.$$

Hence, $\beta(Tx, Ty) = 1$ for all $x \in A$ and $y \in B$. This implies that T is β-admissible. Now, put $A = \{1, 3\}$ and $x_0 = 1$. Then, $Tx_0 = \{1, 3, 5, 7, 9, \dots\}$ and so $\beta(A, Tx_0) = 1$. Suppose that $\{x_n\}$ is a convergent sequence in X with $x_n \to x^*$ and $\beta(Tx_n, Tx_{n-1}) \geq 1$ for all n. Then for each n, Tx_n has no even element and so x_n is odd for all n. Thus, we conclude that x^* is odd and so $Tx_n = Tx^* = \{1, 3, 5, 7, 9, \dots\}$ for all n. Hence, $\beta(Tx_n, Tx^*) = 1$ for all n. If $\beta(Tx, Ty) = 1$, then $H(Tx, Ty) = 0$ and so $\beta(Tx, Ty)H(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$. Therefore, T is a β-ψ-contractive multifunction.

By using the idea of last example, we can provide an easier one.

Example 2.7. *Let $X = \mathbb{N} \cup \{0\}$ and $d(x, y) = |x - y|$. Define the proximinal valued multifunction T on $\mathbb{R}$ by $Tx = \{1, 3, 5, 7, 9\}$ if x is odd and $Tx = \{0, 2, 4, 6, 8\}$ otherwise. Also, suppose that $\psi \in \Psi$ and define the mapping $\beta : 2^X \times 2^X \to [0, +\infty)$ by $\beta(A, B) = 1$ if A and B does not have even element and $\beta(A, B) = 0$ otherwise. Let A and B be subsets of X with $\beta(A, B) \geq 1$. Then A and B have only odd elements and so $Tx = Ty = \{1, 3, 5, 7, 9\}$ for all $x \in A$ and $y \in B$. Hence, $\beta(Tx, Ty) = 1$ and so T is β-admissible. Now, put $A = \{1, 3\}$ and $x_0 = 1$. Then, $Tx_0 = \{1, 3, 5, 7, 9\}$ and $\beta(A, Tx_0) = 1$. Suppose that $\{x_n\}$ is a convergent sequence in X with $x_n \to x^*$ and $\beta(Tx_n, Tx_{n-1}) \geq 1$ for all n. Then for each n, Tx_n has no even element and so x_n is odd for all n. Thus, x^* is also odd and $Tx_n = Tx^* = \{1, 3, 5, 7, 9\}$ for all n. Hence, $\beta(Tx_n, Tx^*) = 1$ for all n. If $\beta(Tx, Ty) = 1$, then $H(Tx, Ty) = 0$ and so $0 = \beta(Tx, Ty)d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$. Therefore, T is a β-ψ-contractive multifunction.*

Corollary 2.8. *Let (X, d) be a complete metric space, $x_0 \in X$ and $T : X \to P(X)$ a multifunction such that $H(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$ with $x_0 \in Tx \cap Ty$. Suppose that there exist $A \subset X$ and $x_1 \in A$ such that $x_0 \in Tx_1 \cap A$. Assume that for each subset $B \subset X$ with $x_0 \in A \cap B$, we have $x_0 \in Tx \cap Ty$ for all $x \in A$ and $y \in B$. Also, suppose that for each convergent sequence $\{x_n\}$ in X with $x_n \to x$ and $x_0 \in Tx_{n-1} \cap Tx_n$ for all n, we have $x_0 \in Tx_n \cap Tx$ for all n. Then T has a fixed point.*

Corollary 2.9. *Let $(X, \preceq, d)$ be a complete ordered metric space and $T : X \to P(X)$ a multifunction such that $H(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$ with $Tx \preceq Ty$. Suppose that there exist $A \subset X$ and $x_0 \in A$ such that $Tx_0 \preceq A$. Assume that for each subset $B \subset X$ with $A \preceq B$, we have $Tx \preceq Ty$ for all $x \in A$ and $y \in B$. Also, suppose that for each convergent sequence $\{x_n\}$ in X with $x_n \to x$ and $Tx_{n-1} \preceq Tx_n$ for all n, we have $Tx_n \preceq Tx$ for all n. Then T has a fixed point.*

Theorem 2.10. *Let (X, d) be a complete metric space and $T : X \to P(X)$ a β-admissible multifunction. Suppose that there exist $A \subset X$ and x_0 in A such that $\beta(A, Tx_0) \geq 1$. Also, Assume that*

$$\beta(Tx, Ty)H(Tx, Ty) \leq \psi(M(x, y))$$

for all $x, y \in X$, where

$$M(x, y) = \max\Big\{d(x, y), d(x, Tx), d(y, Ty), \Big(\frac{d(y, Tx) + d(x, Ty)}{2}\Big)\Big\}.$$

If $x \to d(x, Tx)$ is lower semi-continuous, then T has a fixed point.

Proof. Let $\{x_n\}$ be the sequence obtained as in the proof of Theorem 2.1. Then,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq H(Tx_{n-1}, Tx_n) \\ &\leq \beta(Tx_{n-1}, Tx_n)H(Tx_{n-1}, Tx_n) \\ &\leq \psi(M(x_{n-1}, x_n)) \end{aligned}$$

for all n. If $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$, then $d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n))$. If $M(x_{n-1}, x_n) = d(x_{n-1}, Tx_{n-1})$, then

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, Tx_{n-1})) \leq \psi(d(x_{n-1}, x_n)).$$

If $M(x_{n-1}, x_n) = d(x_n, Tx_n)$, then

$$d(x_n, x_{n+1}) \leq \psi(d(x_n, Tx_n)) \leq \psi(d(x_n, x_{n+1}))$$

and so we get a contradiction. Thus, $d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n))$. Finally, if $M(x_{n-1}, x_n) = \frac{d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)}{2}$, then $M(x_{n-1}, x_n) \leq d(x_{n-1}, x_n)$ or we have $M(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$. Therefore, we get

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n))$$

for all n. Now, by following the proof of Theorem 2.1, one can show that the sequence $\{x_n\}$ is Cauchy and T has a fixed point. □

Corollary 2.11. *Let (X, d) be a complete metric space, $x^* \in X$ a fixed element and $T : X \to P(X)$ a multifunction such that $H(Tx, Ty) \leq \psi(M(x, y))$ for all $x, y \in X$ with $x^* \in Tx \cap Ty$. Suppose that there exist $A \subset X$ and $x_1 \in A$ such that $x^* \in Tx_1 \cap A$. Assume that for each subset $B \subset X$ with $x^* \in A \cap B$, we have $x^* \in Tx \cap Ty$ for all $x \in A$ and $y \in B$. If $x \to d(x, Tx)$ is lower semi-continuous, then T has a fixed point.*

Corollary 2.12. *Let $(X, \preceq, d)$ be a complete ordered metric space and $T : X \to P(X)$ a multifunction such that $H(Tx, Ty) \leq \psi(M(x, y))$ for all $x, y \in X$ with $Tx \preceq Ty$. Suppose that there exist $A \subset X$ and $x_0 \in A$ such that $Tx_0 \preceq A$. Assume that for each subset $B \subset X$ with $A \preceq B$, we have $Tx \preceq Ty$ for all $x \in A$ and $y \in B$. If $x \to d(x, Tx)$ is lower semi-continuous, then T has a fixed point.*

Similar to Theorem 2.5, we can obtain the following result.

Theorem 2.13. *Let (X, d) be a complete metric space and $T : X \to P(X)$ a β-admissible multifunction. Suppose that there exist $A \subset X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Assume that for each convergent sequence $\{x_n\}$ in X with $x_n \to x$ and $\beta(Tx_{n-1}, Tx_n) \geq 1$ for all n, we have $\beta(Tx_n, Tx) \geq 1$ for all n. Suppose that $\beta(Tx, Ty)H(Tx, Ty) \leq \psi(M(x, y))$ for all $x, y \in X$, where*

$$M(x, y) = \max\Big\{d(x, y), d(x, Tx), d(y, Ty), \Big(\frac{d(y, Tx) + d(x, Ty)}{2}\Big)\Big\}.$$

Then T has a fixed point.

Corollary 2.14. *Let (X, d) be a complete metric space, $x^* \in X$ a fixed element and $T : X \to P(X)$ a multifunction such that $H(Tx, Ty) \leq \psi(M(x, y))$ for all $x, y \in X$ with $x^* \in Tx \cap Ty$. Suppose that there exist $A \subset X$ and $x_1 \in A$ such that $x^* \in Tx_1 \cap A$. Assume that for each subset $B \subset X$ with $x^* \in A \cap B$, we have $x^* \in Tx \cap Ty$ for all $x \in A$ and $y \in B$. Also, suppose that for each convergent sequence $\{x_n\}$ in*

X with $x_n \to x$ and $x_0 \in Tx_{n-1} \cap Tx_n$ for all n, we have $x_0 \in Tx_n \cap Tx$ for all n. Then T has a fixed point.

Corollary 2.15. *Let $(X, \preceq, d)$ be a complete ordered metric space and $T : X \to P(X)$ a multifunction such that $H(Tx, Ty) \leq \psi(M(x, y))$ for all $x, y \in X$ with $Tx \preceq Ty$. Suppose that there exist $A \subset X$ and $x_0 \in A$ such that $Tx_0 \preceq A$. Assume that for each subset $B \subset X$ with $A \preceq B$, we have $Tx \preceq Ty$ for all $x \in A$ and $y \in B$. Also, suppose that for each convergent sequence $\{x_n\}$ in X with $x_n \to x$ and $Tx_{n-1} \preceq Tx_n$ for all n, we have $Tx_n \preceq Tx$ for all n. Then T has a fixed point.*

Finally, we emphasize that by using the techniques of [8] one can obtain similar results by substituting equivalent contractions instead of β-ψ-contractivity.

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