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FIXED POINT AND ENDPOINTS THEOREMS FOR SET-VALUED CONTRACTION MAPS IN CONE METRIC SPACES

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Abstract. Wardowski [D. Wardowski, End points and fixed points of set-valued contractions in cone metric spaces, J. Nonlinear Analysis, doi:10.1016 j.na.2008. 10.089] introduced the concept of set-valued contractions in cone metric spaces and proved some theorems in normal cone metric spaces. In this work we prove a theorem for nonexpansive set valued maps and by omitting the assumption of normality in some results we generalize some end point and fixed point theorems for set-valued maps.

1. Introduction and preliminary

Long-Guang and Xian [11] defined cone metric spaces by substituting an ordered normed space for the real numbers and proved various fixed point theorems for contractive single valued maps in such spaces. The study of fixed point theorems on cone metric spaces was taken up by other mathematicians, see [1–12]. Wardowski [13] introduced the concept of set-valued contractions in cone metric spaces. In this paper we prove a theorem for nonexpansive set valued maps and by omitting the assumption of normality in some results we generalize several end point and fixed point theorems for set-valued maps.

Let E be a real Banach space, and P a subset of E, P is called a cone in E if it satisfies:

(i) P is closed, nonempty and $P \neq \{0\}$,
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$,
(iii) $x \in P$ and $-x \in P$ imply that $x = 0$.

The space E can be partially ordered by the cone $P \subset E$; that is, $x \leq y$ if and only if $y - x \in P$. Also we write $x \ll y$ if $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

A cone P is called normal if there exists a constant $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$. The least positive such number is called the normal constant of P.

In the following we always suppose that E is a real Banach space, P is a cone in E with $intP \neq \emptyset$ and $\leq$ is the partial ordering induced on E by P.

Definition 1.1 ([11])**.** Let M be a nonempty set. Assume that the mapping $d : M \times M \to E$ satisfies

(i) $0 \leq d(x, y)$ for all $x, y \in M$ and $d(x, y) = 0$ iff $x = y$
(ii) $d(x, y) = d(y, x)$ for all $x, y \in M$

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(iii) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in M$,

then d is called a cone metric on M, and (M, d) is called a cone metric space.

Let (M, d) be a cone metric space. A set $A \subseteq M$ is called *closed* if for any sequence $\{x_n\} \subseteq A$ which is convergent to x, we have $x \in A$, where $\{x_n\}$ converges to x if for every $c \in E$ with $c \gg 0$ there exists N such that for $n \geq N$, we have $d(x_n, x) \ll c$.

A set $A \subseteq M$ is called *sequentially compact* if for any sequence $\{x_n\} \subseteq A$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is convergent to an element of A.

Denote by $N(M)$ the collection of all nonempty subsets of M, $C(M)$ the collection of all nonempty closed subsets of (M) and $K(M)$ the collection of all nonempty sequentially compact subsets of M.

An element $x \in M$ is said to be an *endpoint* of a set-valued map $T : M \to N(M)$, if $Tx = \{x\}$. We denote a set of all endpoints of T by $End(T)$.

An element $x \in M$ is said to be a *fixed point* of a set-valued map $T : M \to N(M)$, if $x \in Tx$. Denote the set of all fixed point of T by $Fix(T) = \{x \in M | x \in Tx\}$.

A map $f : M \to \mathbb{R}$ is called *lower semi continuous,* if for any sequence $\{x_n\}$ in M and $x \in M$, such that $x_n \to x$ as $n \to \infty$, we have $f(x) \leq \liminf_{n\to\infty} f(x_n)$.

A map $f : M \to E$ is said to have the *lower semi continuous condition,* (*l.s.c.c.* for short) if for any sequence $\{x_n\} \subseteq M$ and $x \in M$ such that $x_n \to x$ as $n \to \infty$ and for all $c \in E$ with $c \gg 0$ then there exists $N \in \mathbb{N}$ that $f(x) < f(x_n) + c$ for all $n \geq N$.

P called *minihedral* cone if $\sup\{x, y\}$ exists for all $x, y \in E$, and *strongly minihedral* if every subset of E which is bounded from above has a supremum [8]. Let (M, d) a cone metric space, cone P is strongly minihedral and hence, every subset of P has infimum, so for $A \in C(M)$, we define $d(x, A) = \inf_{y\in A} d(x, y)$.

Definition 1.2. Let (M, d) be a cone metric space. The distance of a point $x \in M$ to a set $A \in C(M)$ is defined to be

$$\rho(x, A) = \inf_{u\in D(x,A)} \|u\|,$$

where $D(x, A) = \{d(x, z) : z \in A\} \subseteq E$.

Distance between sets in $C(M)$, $H : C(M) \times C(M) \to \mathbb{R}$, is defined by,

$$H(A, B) = \max\{\sup_{x\in B} \rho(x, A), \sup_{y\in A} \rho(y, B)\},$$

for every $A, B \in C(M)$.

For a net $\{A_\alpha\} \subseteq C(M)$ and $A_0 \in C(M)$ we say

$$A_\alpha \to A_0 \iff H(A_\alpha, A_0) \to 0.$$

For $A \in C(M)$ let F_A be a non-empty collection of mappings from $[0, 1]$ to $C(M)$ with $f(1) = A$ for all $f \in F_A$. Let $F = \cup_{A\in C(M)} F_A$.

Such a family F is said to be *contractive* if there exists a decreasing map $\varphi : (0, 1) \to (0, 1)$ such that

$$H(f_A(t), f_B(t)) \leq \varphi(t) H(A, B),$$

for all $A, B \in C(M)$ and $t \in (0,1)$.

F said to be *jointly continuous* if $A_\alpha \to A_0$ and $t_\alpha \to t_0$ implies

$$f_{A_\alpha}(t_\alpha) \to f_{A_0}(t_0).$$

2. The results

Lemma 2.1. *If $\rho(x, A) = 0$ then $x \in A$.*

Proof. Suppose $\rho(x, A) = 0$ then, there exists $\{y_n\} \subseteq A$ with $d(x, y_n) \to 0$, so $y_n \to x$ which implies that $x \in A$ because A is closed. □

Theorem 2.2. *Let M be a sequentially compact subset of a complete cone metric space (X, d), with normal cone P of constant one such that every contraction set-valued map $M \to K(M)$ has a fixed point, and there exists a contractive and jointly continuous family F of functions associated to $K(M)$. Then every nonexpansive set-valued map $T : M \to K(M)$ has a fixed point.*

Proof. For all n let $T_n : M \to K(M)$ be defined by $T_n(x) = f_{T(x)}(\frac{n}{n+1})$. So for all $n \in \mathbb{N}$ and $x, y \in M$ we have

$$\begin{aligned} H(T_n x, T_n y) &= H\left(f_{T(x)}\left(\frac{n}{n+1}\right), f_{T(y)}\left(\frac{n}{n+1}\right)\right) \\ &\leq \varphi\left(\frac{n}{n+1}\right) H(Tx, Ty) \\ &\leq \varphi\left(\frac{n}{n+1}\right) \rho(x, y) = \varphi\left(\frac{n}{n+1}\right) \|d(x, y)\|. \end{aligned}$$

Hence every $T_n : M \to K(M)$ is a contraction set-valued map and so has a fixed point. Therefore for all $n \in \mathbb{N}$ there exists $x_n \in M$ such that $x_n \in T_n(x_n)$. By the sequentially compactness of M there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$ for some $x_0 \in M$. Now

$$H(Tx_{n_k}, Tx_0) \leq \rho(x_{n_k}, x_0) = \|d(x_{n_k}, x_0)\| \to 0$$

this implies that $Tx_{n_k} \to Tx_0$ as $k \to \infty$. On the other hand $\lim_{n\to\infty} \frac{n}{n+1} = 1$ and according to the joint continuity of F

$$x_{n_k} \in T_{n_k}(x_{n_k}) = f_{T x_{n_k}}\left(\frac{n}{n+1}\right) \to f_{Tx_0}(1),$$

so $x_0 \in Tx_0$, because for all $z \in Tx_0$ we have

$$d(x_0, z) \leq d(x_0, x_{n_k}) + d(x_{n_k}, z),$$

and hence,

$$\rho(x_0, z) = \|d(x_0, z)\| \leq \rho(x_0, x_{n_k}) + \rho(x_{n_k}, z),$$

because P is a normal cone with constant 1. Now,

$$\rho(x_0, Tx_0) \leq \rho(x_0, z) \leq \rho(x_0, x_{n_k}) + \rho(x_{n_k}, z),$$

for all $z \in Tx_0$, therefore

$$\begin{aligned} \rho(x_0, Tx_0) &\leq \rho(x_0, x_{n_k}) + \inf_{z \in Tx_0} \rho(x_{n_k}, z) \\ &\leq \rho(x_0, x_{n_k}) + \rho(x_{n_k}, Tx_0) \end{aligned}$$

$$\leq \quad \rho(x_0, x_{n_k}) + H(T_{n_k}x_{n_k}, Tx_0),$$

which implies that $x_0 \in Tx_0$. □

Example 2.3. Let $M := [0,1]$ and let $E := \mathbb{R}^2$ with norm $\|(a,b)\| = \max\{|a|,|b|\}$, then $P := \{(x,y) : x, y \geq 0\}$ is normal cone of constant one and (M,d) is a complete cone metric space with $d(x,y) = (|x-y|, \beta|x-y|)$ where $\beta \in (0,1)$.

Define $T : M \to K(M) = C(M)$ with

$$Tx = \begin{cases} \{\frac{1}{2}x\} & x \in [0,1), \\ [0,\frac{1}{2}] & x = 1. \end{cases}$$

T is nonexpansive, since

$$\begin{aligned} H(Tx,Ty) &= H\Big(\frac{1}{2}x, \frac{1}{2}y\Big) = \rho\Big(\frac{1}{2}x, \frac{1}{2}y\Big) \\ &= \Big\|d\Big(\frac{1}{2}x, \frac{1}{2}y\Big)\Big\| = \Big\|\Big(\frac{1}{2}|x-y|, \beta\frac{1}{2}|x-y|\Big)\Big\| = \frac{|x-y|}{2}, \end{aligned}$$

for all $x, y < 1$. On the other hand

$$H(x,y) = \rho(x,y) = \|d(x,y)\| = \|(|x-y|, \beta|x-y|)\| = |x-y|.$$

So for all $x, y < 1$ we have $H(Tx,Ty) \leq H(x,y) = \rho(x,y)$. For $x < 1$ and $y = 1$, we conclude $H(Tx,Ty) = H(\{\frac{x}{2}\}, [0,\frac{1}{2}]) = 0$. And for $x = y = 1$ we have $H(T1,T1) = \rho(1,1) = 0$.

Now we show that every contraction $T : [0,1] \to K([0,1])$ has a fixed point. We note that $H(Tx,Ty) = |Tx - Ty|$ and $\rho(x,y) = |x-y|$ by definition, so every contraction set-valued map $T : [0,1] \to C([0,1])$ has a fixed point, by Nadler's theorem.

Let $f_A : [0,1] \to K([0,1])$ be defined by $f_A(t) = tA = \{tx : x \in A\}$ and $F := \{f_A : A \in K([0,1])\}$. Then F is jointly continuous and we have

$$H(f_A(t), f_B(t)) = H(tA, tB) \leq \varphi(t)H(A,B),$$

for all $A, B \in K([0,1])$ and $\varphi(t) = t$ for all $t \in [0,1]$. Thus, T satisfies the hypothesis of the previous theorem and must have a fixed point. In fact $0 \in T0 = \{0\}$, is a fixed point.

3. Fixed points and endpoints

Let (M,d) be a cone metric space and $T : M \to C(M)$. For $x, y \in M$ we define

$$D(x,Ty) = \{d(x,z) : z \in Ty\},$$

$$S(x,Ty) = \{u \in D(x,Ty) : \|u\| = \inf\{\|v\| : v \in D(x,Ty)\}\}.$$

In the hypothesis of the following lemma and theorems the assumption that the cone is normal is omitted.

Lemma 3.1. *Let (M,d) be a cone metric space and $T : M \to C(M)$. If the function $f(x) = \inf_{y \in Tx} \|d(x,y)\|$, $x \in M$ is lower semi continuous, then $Fix(T)$ is closed.*

Proof. Let $x_n \in Tx$ and $x_n \to x$. We show that $x \in Tx$. We have

$$\begin{aligned} f(x) &\leq \liminf_{n\to\infty} f(x_n) = \liminf_{n\to\infty} \inf_{y\in Tx_n} \|d(x_n, y)\|, \\ &\leq \liminf_{n\to\infty} \|d(x_n, x_n)\| = 0, \end{aligned}$$

so $f(x) = 0$ therefore $\|d(y_n, x)\| \to 0$ for some $y_n \in Tx$. Given $c \in E$ with $c \gg 0$, there exists N such that for $n \geq N$ we have $d(y_n, x) \ll c$. So $y_n \to x$ and since Tx is closed thus $x \in Tx$. □

In the next results we will suppose that P is strongly minihedral cone in E with nonempty interior.

Theorem 3.2. *Let (M, d) be a complete cone metric space, let $T : M \to C(M)$, be a set-valued map and suppose $f(x) = d(x, Tx)$ $x \in M$, satisfies the l.s.c.c. If there exist real numbers $a, b, c, e \geq 0$ and $q > 1$ with $k := aq + b + ceq < 1$ such that*

$$\forall_{x\in M} \exists_{y\in Tx} \exists_{v\in D(y,Ty)} \forall_{u\in D(x,Tx)} \exists_{z\in D(y,Tx)},$$

we have

$$d(x, y) \leq qu,\ z \leq ed(x, y) \text{ and } v \leq ad(x, y) + bu + cz,$$

then $Fix(T) \neq \emptyset$.

Proof. Choose $x_0 \in M$. Take $u_0 \in D(x_0, Tx_0)$. So there exist $x_1 \in Tx_0$ and $u_1 \in D(x_1, Tx_1)$ and $z_0 \in D(x_1, Tx_0)$ such that

$$d(x_0, x_1) \leq qu_0 \text{ and } z_0 \leq ed(x_0, x_1),$$

and

$$u_1 \leq ad(x_0, x_1) + bu_0 + cz_0.$$

Thus $z_0 \leq equ_0$ and $u_1 \leq ku_0$.
Further, for $x_1 \in M$. Take $u_1 \in D(x_1, Tx_1)$. So there exist $x_2 \in Tx_1$ and $u_2 \in D(x_2, Tx_2)$ and $z_1 \in D(x_2, Tx_1)$ such that

$$d(x_1, x_2) \leq qu_1 \text{ and } z_1 \leq ed(x_1, x_2),$$

and

$$u_2 \leq ad(x_1, x_2) + bu_1 + cz_1.$$

Thus $z_1 \leq equ_1$ and $u_2 \leq ku_1 \leq k^2u_0$.
Continuing in this way we obtain sequence $\{u_n\}$ and $\{x_n\}$ with, $u_n \leq k^n u_0$ and $d(x_n, x_{n+1}) \leq qu_n \leq qk^n u_0$.

So, for every $n > m$

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq qk^{n-1}u_0 + qk^{n-2}u_0 + \cdots + qk^m u_0, \\ &\leq qk^m(1 + k + k^2 + \cdots)u_0 \\ &\leq q\frac{k^m}{1-k}u_0. \end{aligned}$$

So, for every $c \gg 0$ and $c \in E$ there exists N such that for $n, m \geq N$ we have $d(x_n, x_m) \leq q\frac{k^m}{1-k}u_0 \ll c$.

Therefore $\{x_n\}$ is Cauchy in a complete cone metric space, so $x_n \to x^*$ for some $x^* \in M$. Now we claim $x^* \in Tx^*$.

Let $u_n \in D(x_n, Tx_n)$ hence there exists $t_n \in Tx_n$ such that $0 \leq u_n = d(x_n, t_n) \leq k^n u_0$ for all $u_0 \in D(x_0, Tx_0)$. Now $k^n u_0 \to 0$ as $n \to \infty$, so again, for all $0 \ll c$ there exists $N \in \mathbb{N}$ such that $u_n \leq k^n u_0 \ll c$ for all $n \geq N$.

Now, from the *l.s.c.c.* of f, for all $c \gg 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$f(x^*) < f(x_n) + c = \inf_{y \in Tx_n} d(x_n, y) + c \leq u_n + c \ll 2c.$$

So $0 \leq f(x^*) \ll 2c$ for all $c \gg 0$. Hence, $f(x^*) = 0$ and so $d(y_n, x^*) \to 0$ for some $y_n \in Tx^*$, and since Tx^* is closed we have $x^* \in Tx^*$. □

Corollary 3.3. *Let (M, d) be a complete cone metric space, let $T : M \to K(M)$ be a set-valued map and suppose the function $f(x) = d(x, Tx)$, for $x \in M$, has the l.s.c.c.*

(i) *If there exist real numbers $a, b, c, e \geq 0$ and $q > 1$ with $k := aq + b + ceq < 1$ such that*

$$\forall_{x \in M} \exists_{y \in Tx} \exists_{v \in S(y,Ty)} \forall_{u \in S(x,Tx)} \exists_{z \in S(y,Tx)},$$

with

$$d(x, y) \leq qu, \;\; z \leq ed(x, y) \;\; \text{and} \;\; v \leq ad(x, y) + bu + cz,$$

then $Fix(T) \neq \emptyset$.

(ii) *If there exist real numbers $a, b, c, e \geq 0$ and $q > 1$ with $k := aq + b + ceq < 1$ such that*

$$\forall_{x \in M} \forall_{y \in Tx} \exists_{v \in S(y,Ty)} \forall_{u \in S(x,Tx)} \exists_{z \in D(y,Tx)},$$

we have

$$d(x, y) \leq qu, \;\; z \leq ed(x, y) \;\; \text{and} \;\; v \leq ad(x, y) + bu + cz,$$

then $Fix(T) = End(T) \neq \emptyset$.

Corollary 3.4. *Let (M, d) be a complete cone metric space, let $T : M \to C(M)$ be a set-valued map and suppose that the function defined by $f(x) = d(x, Tx)$, for $x \in M$, satisfies the l.s.c.c. If there exist real numbers $a, b \geq 0$ and $q > 1$ with $aq + b < 1$ such that*

$$\forall_{x \in M} \exists_{y \in Tx} \exists_{v \in D(y,Ty)} \forall_{u \in D(x,Tx)},$$

we have

$$d(x, y) \leq qu \;\; \text{and} \;\; v \leq ad(x, y) + bu,$$

then $Fix(T) \neq \emptyset$.

Corollary 3.5. *Let (M, d) be a complete cone metric space, let $T : M \to C(M)$ be a set-valued map and suppose that the function defined by $f(x) = d(x, Tx)$, for $x \in M$, satisfies the l.s.c.c. If there exist real numbers $0 \leq \lambda < 1$, $\lambda < b \leq 1$ such that*

$$\forall_{x \in M} \exists_{y \in Tx} \exists_{v \in D(y,Ty)} \forall_{u \in D(x,Tx)},$$

we have

$$bd(x, y) \leq u \;\; \text{and} \;\; v \leq \lambda d(x, y)],$$

then $Fix(T) \neq \emptyset$.

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