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CONIC EFFICIENCY AND DUALITY IN NONDIFFERENTIABLE MULTIOBJECTIVE MATHEMATICAL PROGRAMMING

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Abstract. The aim of this paper is to study nondifferentiable constrained multiobjective programs where the partial order in the image space is induced by a closed, convex pointed solid cone C. In particular, a new necessary optimality condition is stated for C-efficient and C-weakly efficient solutions. Assumptions guaranteeing that every weak critical point is a C-efficient or C-weakly efficient solution are investigated, improving in an unifying framework many definitions and results appeared in the recent vector optimization literature. Finally, some weak, strong and converse duality results are stated.

1. Introduction

It is well known that optimality conditions and objective functions properties play a key role in mathematical programming and its applications (see [12, 19, 33]). In these light, many efforts appeared in the literature with the aim to study the optimality of critical points [25]. Generalized convexity has been introduced in order to generalize the well known properties of convex functions related to the optimality of local optima and of critical points. The concept of scalar invex functions has been introduced (see [6, 14, 15, 21]) in order to determine assumptions guaranteeing the equivalence of global minima and stationary points. The use of invexity concepts is quite limited in the case of scalar functions, where generalized invexity has been proved to be almost unuseful. In the case of multiobjective functions generalized invexity is not trivial at all and deserved lots of attention in the recent literature.

Recently, Arana et al. [2, 4, 5] provided some generalizations of the equivalence of stationary points and efficient or weakly efficient solutions by means of the use of new classes of vector valued functions named pseudoinvex-I and pseudoinvex-II functions. Nondifferentiability often occurs in many applicative problems (economics, engineering design...), which can be only described by locally Lipschitz functions (see [13]). In this regard, the previous vector optimization results have been generalized to nondifferentiable multiobjective problems by Arana et al. [1].

On the other hand, the study of efficiency under conic relations requires specific necessary optimality conditions. In this light, sufficient optimality conditions have been studied by Cambini [8, 9], Cambini and Martein [11] and Cambini and Carosi [10], where new cone generalized concavity classes of functions have been introduced.

Furthermore, the recent literature showed an increasing interest in the study of nondifferentiable multiobjective programming problems involving these kinds of functions. Giorgi and Guerraggio [20], Kaul et al. [22], Arana et al. [3], Kim and

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Schaible [23] or Nobakhtian [28, 29] derived several sufficient optimality conditions under various generalized nondifferentiable invexity assumptions. Mishra et al. [26] derived the sufficiency of Kuhn-Tucker necessary optimality conditions for a weakly Pareto efficient solution involving generalized d-invexity functions. Using cones, similar results were obtained by Suneja et al. [31] for weakly efficient points.

The aim of this paper is to state new optimality results which extend to general C-efficiency the results proved by Arana et al. [1] and Suneja et al. [31]. Also the results given by Cambini and Martein [11] are generalized from the vector differentiable case to the vector nondifferentiable one under locally Lipschitz assumptions. New generalized convexity concepts, based on a partial ordering induced by a convex cone, are introduced and studied in order to characterize the optimality of critical points.

This work is organized as follows. Section 2 presents the notations and definitions of generalized derivatives and gradients of a locally Lipschitz functions, as well as other basic results. In Section 3, concepts of efficient solutions under a partial order induced by a generic closed, convex, pointed, solid cone are introduced and new optimality conditions are stated. In Section 4, new classes of generalized convex vector functions are proposed and their use in characterizing the efficiency or weak efficiency of critical point is stated (some examples are also given). In Section 5, various duality results are finally provided.

2. Preliminaries

In this section we recall the generalized gradient of a Lipschitz function and some related concepts, as well as some concepts and results that will be needed in the following sections. For a more extensive treatment of the theory, motivations, applications and extensive references of generalized gradients we let the reader to refer to the book by Clarke [13].

Let $Y = \mathbb{R}^p$ be the p-dimensional Euclidean vectorial space, with the usual norm $\|.\|$ and topology; and Y^* the dual space of Y. We denote by $\langle ., . \rangle$ the usual pairing between Y and Y^*. Let us consider a closed, convex cone C in Y with nonempty interior, and recall the notion of the dual cone to C, denoted as C^*, and given by

$$C^* := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \forall y \in C\}.$$

Since $Y = \mathbb{R}^p$, each element in Y^* can be represented as a p-dimensional vector. So, if w represents an element in Y^*, then we can write $\langle w, y\rangle$ as $w^T y$, for all $y \in Y$. Further, following this notation, if $B \subseteq Y$, we write the subset $\{\langle w, b\rangle : b \in B\}$ as $w^T B$. If $A \in \mathbb{R}^{p\times n}$ is a matrix, $w^T A$ denotes the classical product between a vector w and a matrix A. If $D \subseteq \mathbb{R}^{p\times n}$ then the subset $\{w^T d : d \in D\}$ is written as $w^T D$.

Let from now on we denote with X a nonempty open subset of $\mathbb{R}^n$.

Definition 2.1. A function $\theta : X \to \mathbb{R}$ is said to be Lipschitz near $x \in X$ if for some $k > 0$,

$$|\theta(y) - \theta(z)| \leq k\|y - z\|, \quad \forall y, z \text{ within a neighbourhood of x.}$$

We say that $\theta : X \to \mathbb{R}$ is locally Lipschitz on X (or simply, locally Lipschitz) if it is Lipschitz near any point of X.

Let $f = (f_1, \dots f_p) : \mathbb{R}^n \to \mathbb{R}^p$ be a vector valued function. Then f is said to be locally Lipschitz on $\mathbb{R}^n$ if each f_i is locally Lipschitz on $\mathbb{R}^n$.

Definition 2.2. If $\theta : X \to \mathbb{R}$ is locally Lipschitz at $x \in X$, the generalized derivative (in the sense of Clarke) of θ at $x \in X$ in the direction $v \in \mathbb{R}^n$, denoted by $\theta^0(x; v)$, is given by

$$\theta^0(x; v) = \limsup_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{\theta(y + \lambda v) - \theta(y)}{\lambda}$$

Definition 2.3. The Clarke's generalized gradient of θ at $x \in X$ or the Clarke's generalized subdifferential of θ at x, denoted by $\partial\theta(x)$, is the set:

$$\partial\theta(x) := \{\xi \in \mathbb{R}^n : \theta^0(x; v) \geq \langle \xi, v \rangle, \forall v \in \mathbb{R}^n\}$$

For any $v \in \mathbb{R}^n$, it follows:

$$\theta^0(x; v) = \max\{\langle \xi, v \rangle : \xi \in \partial\theta(x)\}$$

such that ξ is called a subgradient of θ at x.

If θ is locally Lipschitz, then its Clarke's generalized gradient has got some properties: $\partial\theta(x)$ is a nonempty, convex and compact subset, for any $x \in X$.

If we extend this derivative to the vectorial case, we have that the generalized derivative of a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^p$ at $x \in \mathbb{R}^n$ in the direction v is given by

$$f^0(x; v) = \{f_1^0(x; v), f_2^0(x; v), \dots, f_p^0(x; v)\}$$

The generalized gradient of f at x is the set

$$\partial f(x) = \partial f_1(x) \times \dots \times \partial f_p(x) \subset \mathbb{R}^{p \times n}$$

where $\partial f_i(x)$ is the generalized gradient of f_i at x for $i = 1, 2, \dots, p$.

In order to establish optimality conditions let us now consider the closed, convex cone $C \subset \mathbb{R}^p$ with nonempty interior, which induces a partial order on Y. Let us also denote C^0 as $C \setminus \{0\}$. Thus,

$$\begin{array}{lll} x \leqq_C y & \text{iff} & y - x \in C, \\ x \leq_C y & \text{iff} & y - x \in C^0, \\ x <_C y & \text{iff} & y - x \in intC. \end{array}$$

The definitions of $\geqq_C, \geq_C$ and $>_C$ are straightforward from the previous ones. Following Küçük et al. [24], we have that $\leqq_C$ is a partial order induced by the cone C, but the relations $\leq_C$ and $<_C$ are not reflexive, that is, $x \leq_C x$ and $x <_C x$ are not fulfilled for any x. So, and although we can not properly consider these relations ($x \leq_C x$ and $x <_C x$) as partial orders, they provide classical efficiency relations in vector optimization.

A first interesting nondifferentiable vector optimization problem is as follows:

(VP) C-Minimize $f(x)$
subject to
$x \in \mathbb{R}^n$,

where $f = (f_1, \dots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$, is a locally Lipschitz vector valued function.

Although it is usual to request the incorporation of some constraints to our vector optimization problem. In such a way, we are going to focus our optimization study on this one:

$$(VPC) \quad \begin{array}{l} C\text{-Minimize } f(x) \\ \text{subject to} \\ -g(x) \in Q, \end{array}$$

where $g = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$, is a locally Lipschitz vector valued function, and Q is a closed convex cone with nonempty interior in $\mathbb{R}^m$. For the sake of convenience, let us denote with X_0 the feasible solution set of problem (VPC).

The minimal solutions for (VP) will be studied under a relation induced by C, what extends the classical concept of efficiency introduced by Pareto [30]. We will center our attention in C-efficient solutions and C-weakly efficient solutions for (VP):

Definition 2.4. A feasible point, $\bar{x}$, is said to be a C-efficient solution of (VPC) if there does not exist another feasible point, x, such that $f(x) \leq_C f(\bar{x})$, that is, $f(x) - f(\bar{x}) \in -C^0$.

Definition 2.5. A feasible point, $\bar{x}$, is said to be a C-weakly efficient solution of (VPC) if there does not exist another feasible point, x, such that $f(x) <_C f(\bar{x})$, that is, $f(x) - f(\bar{x}) \in -intC$.

Observe that all C-efficient solutions are C-weakly efficient solutions, and the reverse is not true in general. In the particular case $C = \mathbb{R}^p_+ = \{y = (y_1, \ldots, y_p) : y_1 \geq 0, i = 1, \ldots, p\}$, note that the definitions of C-efficient and C-weakly efficient solutions reduce to the classical efficient and weakly efficient solutions. Let us denote X_0 the set of feasible points for our vector optimization problem (VPC).

3. Optimality conditions

In order to locate minimal solutions for (VP) and (VPC), it is useful to use the stationary or vector critical points as optimality conditions. For the first vector optimization problem (VP), we have the following necessary optimality condition (see [13, 17]).

Theorem 3.1. *Let us consider the vector optimization problem (VP). If $\bar{x}$ is a C-weakly efficient solution then $\bar{x}$ is a C-critical point for (VP), that is that there exists $\lambda \in C^*$ such that*

$$0 \in \lambda^T \partial f(\bar{x}) \tag{3.1}$$

Recently, Arana et al. [1] have formulated the minimal property that the objective function has got for a C-critical point to be a C-weakly efficient solution of (VP). For that, they have introduced the class of functions (C_1, C_2)-pseudoinvex, where C_1 and C_2 are two nonempty and convex cones in $\mathbb{R}^p$. In particular, if we denote the interior of C as *int* C, we have:

Definition 3.2 ([1])**.** The vector function f is said to be ($int\ C$, $int\ C$)-pseudoinvex at $\bar{x}$ if there exists a vector function $\eta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{p\times n} \to \mathbb{R}^n$ such that for all $x, \bar{x} \in \mathbb{R}^n$

$$f(x) \in f(\bar{x}) - int\ C \Rightarrow \xi\eta(x, \bar{x}, \xi) \in -int\ C, \quad \forall \xi \in \partial f(\bar{x})$$

Theorem 3.3. *Every C-critical point is a C-efficient solution of (VP) if and only if f is $(int\ C, int\ C)$-pseudoinvex.*

In the case of the vector problem with constraints (VPC), researchers are looking for optimality conditions in terms of critical points, as a generalization of those given when $C = \mathbb{R}^p_+$ and $Q = \mathbb{R}^m_+$. In this last situation, Fritz John type optimality conditions are necessary, and so, a fine start point to obtain candidates for weakly efficient solutions. But, in (VPC), these conditions are not necessary. In this regard, it is necessary to require additional properties to the involved functions to ensure that a Fritz John type optimality condition leads us to a C-weakly efficient solution. Yen and Sach [32] proposed the following class of functions:

Definition 3.4. f is said to be C-generalized invex at $\bar{x} \in \mathbb{R}^n$ if there exists $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for $x \in \mathbb{R}^n$ and $A \in \partial f(\bar{x})$,

$$f(x) - f(\bar{x}) - A\eta(x, \bar{x}) \in C.$$

There exists a first relation between C-generalized invexity and $(int\ C, int\ C)$-pseudoinvex in the case that the parameter functionals η are chosen.

Proposition 3.5. *If f is a C-generalized invex function at $\bar{x} \in \mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, then f is $(int\ C, int\ C)$-pseudoinvex with respect to $\bar{\eta} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{p\times n} \to \mathbb{R}^n$, with $\bar{\eta}(x, \bar{x}, \xi) = \eta(x, \bar{x})$, for all $x \in \mathbb{R}^n$, $\xi \in \partial f(\bar{x})$.*

Proof. Let f be a C-generalized invex function at $\bar{x} \in \mathbb{R}^n$ with respect to η, and be $x \in \mathbb{R}^n$ such that $f(x) \in f(\bar{x}) - int\ C$, that is,

$$-f(x) + f(\bar{x}) \in int\ C. \tag{3.2}$$

On the other hand, f be a C-generalized invex, so we have that for all $\xi \in \partial f(\bar{x})$,

$$f(x) - f(\bar{x}) - \xi\eta(x, \bar{x}) \in C. \tag{3.3}$$

Since C is a cone, we sum (3.2) and (3.3), obtaining that

$$-\xi\eta(x, \bar{x}) \in int\ C. \tag{3.4}$$

We just define $\bar{\eta}(x, \bar{x}, \xi) = \eta(x, \bar{x})$, and it follows from (3.4) that $\xi\bar{\eta}(x, \bar{x}, \xi) = \xi\eta(x, \bar{x}) \in -int\ C$. Therefore, f is $(int\ C, int\ C)$-pseudoinvex with respect to $\bar{\eta}$. □

The reverse of this result is not true in general. For that, let us consider the case in which $C = \mathbb{R}^p_+$ and f is differentiable. In this particular case, the class of $(int\ C, int\ C)$-pseudoinvex functions reduces to the class of pseudoinvex-I introduced by Arana et al. [2], and C-generalized invex reduces to invex. Arana et al. [2] proved that the class of invex functions is contained into the class of pseudoinvex-I functions, and they both are not equivalent.

Under C-generalized invexity, and from Suneja et al. [31], we have the following Fritz John type optimality result:

Theorem 3.6. *Let f be C-generalized invex and g be Q-generalized invex at $\bar{x} \in X_0$ with respect to the same $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. If (VPC) attains a C-weakly efficient solution at $\bar{x}$, then there exist $\lambda \in C^*$, $\mu \in Q^*$ not both zero such that*

$$0 \in \lambda^T \partial f(\bar{x}) + \mu^T \partial g(\bar{x}), \tag{3.5}$$

$$\mu^T g(\bar{x}) = 0. \tag{3.6}$$

Note that C-generalized invexity has been required to ensure that a C-weakly efficient solution fulfills a Fritz John type condition. Following, we will prove that if we remove any kind of generalized convexity requirement, we are able to obtain a new necessary optimality condition.

Theorem 3.7 (Weak necessary optimality condition)**.** *If $\bar{x}$ is a C-weakly efficient solution for (VPC), then $\bar{x}$ is a weak critical point for (VPC), that is, $\bar{x}$ is a feasible point for (VPC) such that there exist $\lambda \in C^*$, $\mu \in Q^*$ not both zero such that*

$$0 \in \lambda^T f\partial(\bar{x}) + \mu^T g\partial(\bar{x}). \tag{3.7}$$

Proof. For that, let $\bar{x}$ be a C-weakly efficient solution for our vector optimization problem (VPC), and let us formulate an auxiliary problem:

(AP) $\quad (C \times Q)$-Minimize $(f(x),\ g(x))$
subject to
$x \in \mathbb{R}^n$.

Firstly, we prove that if $\bar{x}$ is a C-weakly efficient solution for (VPC) then $\bar{x}$ is a $(C \times Q)$-weakly efficient solution for (AP). For this purpose, let us suppose that $\bar{x}$ is not a $(C \times Q)$-weakly efficient solution for (AP), that is, there exists $x \in \mathbb{R}^n$ such that $(f(x),\ g(x)) - (f(\bar{x}),\ g(\bar{x})) \in (-int\ C) \times (-int\ Q)$, i.e.,

$$f(x) - f(\bar{x}) \in -int\ C, \tag{3.8}$$

$$g(x) - g(\bar{x}) \in -int\ Q. \tag{3.9}$$

By assumption, $\bar{x}$ is feasible for (VPC), and so $g(\bar{x}) \in -Q$. Adding $g(\bar{x})$ to (3.9), and from the properties of cones (see [7]), we have that

$$g(x) - g(\bar{x}) + g(\bar{x}) \in -int\ Q,$$

that is,

$$g(x) \in -int\ Q \subseteq -Q. \tag{3.10}$$

Therefore, by (3.10), x is a feasible point for (VPC), and then by (3.8), it follows that $\bar{x}$ is not a C-weakly efficient solution for (VPC), which stands in contradiction with our initial assumption. Consequently, we have proved that if $\bar{x}$ is a C-weakly efficient solution for (VPC), then $\bar{x}$ is a $(C \times Q)$-weakly efficient solution for (AP). Thus, by Theorem 3.1, there exists $\nu \in (C \times Q)^*$ such that

$$0 \in \nu^T \partial(f, g)(\bar{x}). \tag{3.11}$$

Since $(C \times Q)^* = C^* \times Q^*$ (see [7]), we have that there exist $\lambda \in C^*$, $\mu \in Q^*$, such that $\nu = (\lambda, \mu)$. Then, (3.11) is equivalent to

$$0 \in \lambda^T \partial f(\bar{x}) + \mu^T \partial g(\bar{x}). \tag{3.12}$$

Therefore, by (3.12), $\bar{x}$ is a weak critical point for (VPC), and the result is proved. □

Since all efficient solutions are weakly efficient solutions, Theorem 3.7 derives the following result.

Corollary 3.8. *If $\bar{x}$ is a C-efficient solution for (VPC), then $\bar{x}$ is a weak critical point for (VPC), that is, $\bar{x}$ is a feasible point for (VPC) such that there exist $\lambda \in C^*$, $\mu \in Q^*$ not both zero such that equation (3.7) is fulfilled.*

In the particular case that the constraints are removed from (VPC) and $C = \mathbb{R}^p_+$, the previous optimality condition reduces to (3.1) given by Clarke [13].

4. A necessary and sufficient property

Let us recall that a weak critical point provides us a necessary optimality condition to locate C-efficient and C-weakly efficient solutions of our vector problem (VPC), but it is not a sufficient condition. To ensure the reverse, that is, a weak critical point is a C-efficient or C-weakly efficient solution, we need the objective function to satisfy some additional properties. In this regard, and following, we study the sutitable properties of the involved vector functions to get this sufficiency. Furthermore, we will obtain the minimal properties to state this sufficiency.

For this purpose, we introduce the following new class of nondifferentiable vector functions, inspired on definitions given by Arana et al. [4] in the nondifferentiable case, as well as, from classes under relations induced by C, by Cambini [8, 9], Cambini and Martein [11], Cambini and Carosi [10], and Arana et al. [1].

Definition 4.1. Let us consider, again, $C \subseteq \mathbb{R}^p$ and $Q \subseteq \mathbb{R}^m$ two closed and convex cones, with nonempty interiors. Let $\bar{x}$ be $\in \mathbb{R}^n$. The pair of vector functions (f, g) is said to be (C, Q)-W-pseudoinvex-I at $\bar{x}$ if there exists a vector function $\eta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{p\times n} \times \mathbb{R}^{m\times n} \to \mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$

$$f(x) - f(\bar{x}) \in -int\, C \Rightarrow \begin{cases} \xi\eta(x, \bar{x}, \xi, \zeta) \in -int\, C \\ \zeta\eta(x, \bar{x}, \xi, \zeta) \in -int\, Q \end{cases}, \tag{4.1}$$

$\forall \xi \in \partial f(\bar{x}), \forall \zeta \in \partial g(\bar{x})$.

Definition 4.2. Let us consider, again, $C \subseteq \mathbb{R}^p$ and $Q \subseteq \mathbb{R}^m$ two closed and convex cones, with nonempty interiors. Let $\bar{x}$ be $\in \mathbb{R}^n$. The pair of vector functions (f, g) is said to be (C, Q)-W-pseudoinvex-II at $\bar{x}$ if there exists a vector function $\eta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{p\times n} \times \mathbb{R}^{m\times n} \to \mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$

$$f(x) - f(\bar{x}) \in C^0 \Rightarrow \begin{cases} \xi\eta(x, \bar{x}, \xi, \zeta) \in -int\, C \\ \zeta\eta(x, \bar{x}, \xi, \zeta) \in -int\, Q \end{cases}, \tag{4.2}$$

$\forall \xi \in \partial f(\bar{x}), \forall \zeta \in \partial g(\bar{x})$.

In relation to these definitions, we say that (f, g) is (C, Q)-W-pseudoinvex-I(II) at $\bar{x}$ on a subset $D \subseteq \mathbb{R}^n$ if the definition above is fulfilled for $x \in D$ instead of $\mathbb{R}^n$. We say that (f, g) is (C, Q)-W-pseudoinvex-I(II) on a subset $D \subseteq \mathbb{R}^n$ if (f, g) is (C, Q)-W-pseudoinvex-I(II) at $\bar{x}$ on D, for all $\bar{x} \in D$. We say that (f, g) is (C, Q)-W-pseudoinvex-I(II), if it is on $\mathbb{R}^n$.

Remark 4.3. Observe that if the constraints given by g are removed, then (C,Q)-W-pseudoinvexity-I and II coincide with the $(int\ C, int\ C)$-pseudoinvex and $(C^0, int\ C)$-pseudoinvex classes of functions, respectively, introduced in [1].

Our outline is to find and formulate classes of vector functions that make up our nondifferentiable vector problem (VPC), such that any class of functions which is characterized by having every weak critical point as a C-weakly efficient solution of (VPC) must be equivalent to these classes of functions. For that, we will need the following generalized alternative result (see [7, 16]):

Lemma 4.4. *Let A be an element of $\mathbb{R}^{p\times n}$. C^* is the dual cone to C. Then one and only one of the following statements is true:*

(i) *There exists $u \in \mathbb{R}^n$ such that $Au <_C 0$.*
(ii) *There exists $y \in C^*$, $y \neq 0$, such that $y^T A = 0$.*

Theorem 4.5. *Every weak critical point is a C-weakly efficient solution of (VPC) if and only if (f, g) is (C,Q)-W-pseudoinvex-I on X_0.*

Proof. (i) Let us begin with the proof of the sufficiency of the (C,Q)-W-pseudoinvex-Iity on X_0 of (f, g). So, let $\bar{x}$ be a weak critical point and (f, g) be (C,Q)-W-pseudoinvex-I on X_0. We have to prove that $\bar{x}$ is C-weakly efficient solution of (VPC). For that, let us suppose that $\bar{x}$ is not C-weakly efficient solution of (VPC). Then, there exists $x \in X_0$ such that $f(x) - f(\bar{x}) \in -C \setminus \{0\}$. Since (f, g) is (C,Q)-W-pseudoinvex-I on X_0, there exists a vector function η such that

$$\left\{ \begin{array}{l} \xi\eta(x,\bar{x},\xi,\zeta) \in -int\ C \\ \zeta\eta(x,\bar{x},\xi,\zeta) \in -int\ Q \end{array} \right. , \tag{4.3}$$

for all $\xi \in \partial f(\bar{x})$ and for all $\zeta \in \partial g(\bar{x})$. Further, by assumption, we have that $\bar{x}$ is a weak critical point, and then there exist $\lambda \in C^*$, $\mu \in Q^*$ not both zero such that

$$0 \in \lambda^T f\partial(\bar{x}) + \mu^T g\partial(\bar{x}), \tag{4.4}$$

that is, there exist $\xi \in \partial f(\bar{x})$, $\zeta \in \partial g(\bar{x})$, such that

$$\lambda^T\xi + \mu^T\zeta = 0. \tag{4.5}$$

Multiplying (4.5) by $\eta(x,\bar{x},\xi,\zeta)$, we get

$$(\lambda^T\xi + \mu^T\zeta)\eta(x,\bar{x},\xi,\zeta) = 0 \tag{4.6}$$

On the other hand, since $\lambda \in C^*$ and $\xi\eta(x,\bar{x},\xi,\zeta) \in -int\ C$, it is known (see [7]) that

$$\lambda^T\xi\eta(x,\bar{x},\xi,\zeta) < 0, \tag{4.7}$$

and the same for $\mu \in Q^*$ and $\zeta\eta(x,\bar{x},\xi,\zeta) \in -int\ Q$,

$$\mu^T\zeta\eta(x,\bar{x},\xi,\zeta) < 0. \tag{4.8}$$

Now, adding (4.7) and (4.8), it follows

$$\lambda^T\xi\eta(x,\bar{x},\xi,\zeta) + \mu^T\zeta\eta(x,\bar{x},\xi,\zeta) < 0, \tag{4.9}$$

which stands in contradiction with (4.6). Therefore, $\bar{x}$ is a C-weakly efficient solution of (VPC). Thus, we have proved that (C,Q)-W-pseudoinvexity-I on X_0 of (f,g) is a sufficient property for all weak critical point to be a C-weakly efficient solution of (VPC). Now, we prove that, moreover, (C,Q)-W-pseudoinvexity-I is necessary in order that every weak critical point is a C-weakly efficient solution of (VPC).

(ii) For that, let us assume this last, i.e., every weak critical point is a C-weakly efficient solution; and let us suppose that there exist two points x and $\bar{x}$ in X_0, such that $f(x) - f(\bar{x}) \in -C \setminus \{0\}$, since otherwise (f,g) would be (C,Q)-W-pseudoinvexity-I on X_0, and the result would be proved. This means that $\bar{x}$ is not a C-weakly efficient solution, and by the assumptions, $\bar{x}$ is not a weak critical point, that is, given any $\xi \in \partial f(\bar{x})$ and $\zeta \in \partial g(\bar{x})$, the following equation

$$\lambda^T \xi + \mu^T \zeta = 0 \tag{4.10}$$

has no solution $\lambda \in C^*$, $\mu \in Q^*$, not both zero. (4.10) can be written as follows:

$$[\lambda\, \mu] \begin{bmatrix} \xi \\ \zeta \end{bmatrix} = 0.$$

By Lemma 4.4, there exists $u \in \mathbb{R}^n$ such that

$$\begin{bmatrix} \xi \\ \zeta \end{bmatrix} u \in (-int\ C \times -int\ Q),$$

which can be written as follows,

$$\begin{cases} \xi\, u \in -int\ C \\ \zeta\, u \in -int\ Q. \end{cases} \tag{4.11}$$

If we define $\eta(x,\bar{x},\xi,\zeta) = u \in \mathbb{R}^n$, and replace u by η in (4.11), we get

$$\begin{cases} \xi\eta(x,\bar{x},\xi,\zeta) \in -int\ C \\ \zeta\eta(x,\bar{x},\xi,\zeta) \in -int\ Q. \end{cases} \tag{4.12}$$

Therefore, we have proved that (f,g) is (C,Q)-W-pseudoinvex-I on X_0. □

In a similar way, (C,Q)-W-pseudoinvexity-II is the minimal property of the functions involved in our vector problem (VPC) in order that all weak critical points are C-efficient solutions, such as we have as follows. Its proof is analogous to the previous one.

Theorem 4.6. *Every weak critical point is a C-efficient solution of (VPC) if and only if (f,g) is (C,Q)-W-pseudoinvex-II on X_0.*

Let us note again, that no additional requirements have been needed, such as, for instance, C-generalized invexity (see [31, 32]). Further, this characterization result can be considered a generalization of those given by Arana et al. [1,2], Cambini [8,9], Cambini and Martein [11] and Cambini and Carosi [10] to the nondifferenctiable case, and to the conic relations. In fact, (C,Q)-W-pseudoinvexity is the minimal property of the involved functions in (VPC) we need to obtain C-weakly efficient solutions from weak critical points.

From the previous results, we obtain the following consequence.

Corollary 4.7. *If there is no weak critical point for (VPC), then (f, g) is (C, Q)-W-pseudoinvex-I and II on X_0.*

Example 4.8. Let us consider the following constrained vector optimization problem:

(V_{Ex}) C-Minimize $(x + 1, x^3 - 5)$
subject to
$-(-x^3, 8) \in Q$,

where $f = (f_1, f_2) = (x + 1, x^3 - 5) : \mathbb{R} \to \mathbb{R}^2$, $g = (g_1, g_2) = (-x^3, 8) : \mathbb{R} \to \mathbb{R}^2$. These functions are differentiable, so $\partial f(x)$ and $\partial g(x)$ reduce to $\nabla f(x)$ and $\nabla g(x)$, respectively. It follows that $\nabla f(x) = (1, 3x^2)$ and $\nabla g(x) = (-3x^2, 0)$. The cones are defined as follows:

$$C = \{(x, y) \in \mathbb{R}^2 : 4x + y \geq 0 \text{ and } x + 3y \geq 0\}$$

and

$$Q = \{(x, y) \in \mathbb{R}^2 : 5x + y \leq 0 \text{ and } 5x - y \leq 0\}.$$

Then, by easy operations,

$$C^* = \{(x, y) \in \mathbb{R}^2 : 3x - y \geq 0 \text{ and } -x + 4y \geq 0\}$$

and

$$Q^* = \{(x, y) \in \mathbb{R}^2 : x + 5y \leq 0 \text{ and } x - 5y \leq 0\}.$$

Let us show that there exists no weak critical point for (V_{Ex}). For that, let us calculate the values for $\lambda = (\lambda_1, \lambda_2) \in C^*$ and $\mu = (\mu_1, \mu_2) \in Q^*$ not both zero such that the equation (3.7) is fulfilled, that is, such that

$$\lambda_1 + 3\lambda_2 x^2 - 3\mu_1 x^2 + 0\mu_2 = 0. \tag{4.13}$$

Observe that $(1, 3x^2)$ belongs to the first orthant, in whose interior C^* is strictly contained, and $(-3x^2, 0)$ belongs to the interior of Q^*, for all $x \neq 0$. Then, on one hand, $\lambda_1 + 3\lambda_2 x^2 > 0$, and on the other hand, $-3\mu_1 x^2 + 0\mu_2 > 0$, for all $x \neq 0$. Adding these two inequalities, with $x \neq 0$, we have that

$$\lambda_1 + 3\lambda_2 x^2 - 3\mu_1 x^2 + 0\mu_2 > 0,$$

for all $\lambda = (\lambda_1, \lambda_2) \in C^*$, $\mu = (\mu_1, \mu_2) \in Q^*$ not both zero, in contradiction with (4.13). The case $x = 0$ provides $g(0) = (0, 8)$, with $-g(0) = (0, -8) \notin Q$, what means that $x = 0$ is not a feasible solution of (V_{Ex}). Therefore, there is no weak critical point for our vector problem. In consequence, by Corollary 4.7, (f, g) is (C, Q)-W-pseudoinvex-I and II on the feasible set.

5. Duality

Duality is a classical and fine approach to get optimal vector solutions for our (VPC), by solving another vector optimization problem. So, and following, we propose a duality between our multiobjective problem (VCP) and an associated and dual problem of the Mond-Weir [27] type, defined for the multiobjective case by Egudo and Hanson [18], but with the convenient modifications suited to nondifferentiability and the partical orders induced by cones. For this purpose, we have the dual of (VPC), formulated as follows:

$$
\begin{array}{ll}
(DVPC) & C\text{-Maximize } f(u) \\
& \text{subject to} \\
& 0 \in \lambda^T \partial f(\bar{u}) + \mu^T \partial g(\bar{u}), \\
& \lambda \in C^*,\ \mu \in Q^*,\ (\lambda, \mu) \neq 0, \\
& u \in \mathbb{R}^n.
\end{array}
$$

Let us denote U_0 the feasible set of (DVPC). Observe that this dual problem is formulated in terms of maximization instead of minimization. It implies that we have to reconsider the definition of C-efficient and C-weakly efficient solution for this type of problems. But it is natural and easy. It is enough to say that $\bar{u} \in U_0$ is a C-weakly efficient solution of (DVPC) is there exists no $u \in U_0$ such that $f(\bar{u}) - f(u) \in -int\ C$ is not verified. That is, just the reverse definition for a minimization process. And the same for C-efficient solution of (DVPC).

Let us begin with weak duality for the study of C-weakly efficient solutions.

Theorem 5.1 (Weak Duality)**.** *Let x be a feasible point for (VPC), and (u, λ, μ) a feasible point for (DVPC). If (f, g) is (C, Q)-W-pseudoinvex-I, then $f(x) - f(u) \in -int\ C$ is not verified.*

Proof. Let us suppose (f, g) is (C, Q)-W-pseudoinvex-I with respect to a vector function η. Let x be a feasible point for (VPC), and (u, λ, μ) a feasible point for (DVPC), that is, $x \in X_0,\ u \in U_0$, such that $f(x) - f(u) \in -int\ C$, if not, the result would be proved. Since $u \in U_0$, there exist $\lambda \in C^*,\ \mu \in Q^*$, not both zero, such that

$$0 \in \lambda^T f\partial(u) + \mu^T g\partial(u), \tag{5.1}$$

that is, there exist $\xi \in \partial f(u)$, $\zeta \in \partial g(u)$, such that

$$\lambda^T \xi + \mu^T \zeta = 0. \tag{5.2}$$

Multiplying (5.2) by $\eta(x, u, \xi, \zeta)$, we get

$$(\lambda^T \xi + \mu^T \zeta)\eta(x, u, \xi, \zeta) = 0 \tag{5.3}$$

On the other hand, since $\lambda \in C^*$ and $\mu \in Q^*$, and $\xi\eta(x, u, \xi, \zeta) \in -int\ C$ and $\zeta\eta(x, u, \xi, \zeta) \in -int\ Q$, it follows (see [7]) that

$$\lambda^T \xi\eta(x, u, \xi, \zeta) < 0, \quad \mu^T \zeta\eta(x, u, \xi, \zeta) < 0. \tag{5.4}$$

Now, adding these two inequalities in (5.4), it follows

$$\lambda^T \xi\eta(x, u, \xi, \zeta) + \mu^T \zeta\eta(x, u, \xi, \zeta) < 0, \tag{5.5}$$

which stands in contradiction with (5.3), and therefore, $f(x) - f(u) \in -int\ C$ is not verified. □

Thanks to the weak duality result we can now prove the strong duality result as follows.

Theorem 5.2 (Strong Duality)**.** *Let (f, g) be (C, Q)-W-pseudoinvex-I. If $\bar{x}$ is a C-weakly efficient solution for (VPC), then there exist λ, μ, such that $(\bar{x}, \lambda, \mu)$ is a C-weakly efficient solution for (DVPC).*

Proof. Let us suppose that $\bar{x}$ is a C-weakly efficient solution for (VPC). Then, by Theorem 3.7, there exist $\lambda \in C^*$, $\mu \in Q^*$, not both zero, such that

$$0 \in \lambda^T f\partial(u) + \mu^T g\partial(u).$$

Then $\bar{x}$ is a feasible point for (DPVC), and from the weak duality theorem it follows that the vector inequality $f(\bar{x}) - f(u) \in -int\ C$ is not verified, where u is a feasible point for (DPVC). Therefore, $\bar{x}$ is a C- weakly efficient solution of (DPVC). □

The converse result is also verified as we demonstrate below.

Theorem 5.3 (Converse Duality). *Let (f, g) be (C, Q)-W-pseudoinvex-I on X_0, and $\bar{x}$ a feasible point for (VPC). If $(\bar{x}, \lambda, \mu)$ is a feasible point for (DVPC), then $\bar{x}$ is a C-weakly efficient solution of (VPC).*

Proof. Let us suppose that $\bar{x}$ is a feasible point for (VPC). If $(\bar{x}, \lambda, \mu)$ is a feasible point for (DVPC), then $\lambda \in C^*$, $\mu \in Q^*$, not both zero, such that

$$0 \in \lambda^T f\partial(u) + \mu^T g\partial(u),$$

and therefore $\bar{x}$ is a weak critical point. Since (f, g) is (C, Q)-W-pseudoinvex-I on X_0, from Theorem 4.5 it follows that $\bar{x}$ is a C-weakly efficient solution of (VPC). □

We proceed as before for the study of C-efficient solutions. For that, we require (C, Q)-W-pseudoinvexity-II. The proofs of the following duality results are very similar to the previous and respective theorems.

Theorem 5.4 (Weak Duality). *Let x be a feasible point for (VPC), and (u, λ, μ) a feasible point for (DVPC). If (f, g) is (C, Q)-W-pseudoinvex-II, then $f(x) - f(u) \in -C^0$ is not verified.*

Theorem 5.5 (Strong Duality). *Let (f, g) be (C, Q)-W-pseudoinvex-II. If $\bar{x}$ is a C-efficient solution for (VPC), then there exists λ, μ, such that $(\bar{x}, \lambda, \mu)$ is a C-efficient solution for (DVPC).*

Theorem 5.6 (Converse Duality). *Let (f, g) be (C, Q)-W-pseudoinvex-II on X_0, and $\bar{x}$ a feasible point for (VPC). If $(\bar{x}, \lambda, \mu)$ is a feasible point for (DVPC), then $\bar{x}$ is a C-efficient solution of (VPC).*

6. Conclusions

In nondifferentiable multiobjective mathematical programming problems, with an induced partial conic order, new necessary optimality conditions have been obtained. New classes of vector valued locally Lipschitz generalized invex functions have been proposed for a suitable study of minimal solutions. In this regard, the introduced functions allow to characterize that every weak critical point is a minimal solution.

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