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STRONG CONVERGENCE THEOREMS BY HYBRID METHODS FOR SPLIT FEASIBILITY PROBLEMS IN HILBERT SPACES

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Abstract. In this paper, motivated by the idea of the split feasibility problem and results for solving the problem, we consider generalized split feasibility problems and then establish strong convergence theorems by two hybrid methods for the problems. As applications, we get new strong convergence theorems which are connected with fixed point problem and equilibrium problem.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let T be a mapping of C into H. We denote by $F(T)$ the set of fixed points of T. For a constant $\alpha > 0$, the mapping $U : C \to H$ is said to be α-inverse strongly monotone if

$$\langle x-y, Ux-Uy\rangle \geq \alpha \|Ux-Uy\|^2$$

for all $x, y \in C$. An α-inverse strongly monotone mapping is also Lipschitz continuous with a Lipschitz constant $\frac{1}{\alpha}$. A mapping T of C into H is nonexpansive if $\|Tu-Tv\| \leq \|u-v\|$ for all $u, v \in C$. If $T : C \to H$ is a nonexpansive mapping, then $I-T$ is $\frac{1}{2}$-inverse strongly monotone, where I is the identity mapping on H. A nonexpansive mapping $T : C \to H$ with $F(T) \neq \emptyset$ is quasi-nonexpansive; see, for instance, [31]. A mapping S of C into H is nonspreading if

$$2\|Su-Sv\|^2 \leq \|Su-v\|^2 + \|Sv-u\|^2$$

for all $u, v \in C$; see [19, 20]. A mapping S of C into H is hybrid if

$$3\|Su-Sv\|^2 \leq \|Su-v\|^2 + \|Sv-u\|^2 + \|u-v\|^2$$

for all $u, v \in C$; see [32]. Recently, Kocourek, Takahashi and Yao [18] introduced a broad class of nonlinear mappings which contains nonexpansive mappings, nonspreading mappings and hybrid mappings in a Hilbert space. They called such a mapping generalized hybrid; see Section 2. Moreover, Kawasaki and Takahashi [17] defined a more wide class of nonlinear mappings than the class of generalized hybrid mappings. A multi-valued operator $B \subset H \times H$ is said to be monotone if $\langle x-y, u-v\rangle \geq 0$ for all $x, y \in H$, $u \in Bx$ and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. Given set-valued operators $B_i : H_1 \to 2^{H_1}$, $1 \leq i \leq m$, and $G_j : H_2 \to 2^{H_2}$, $1 \leq j \leq n$, respectively, and bounded linear operators

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$A_j : H_1 \to H_2,\ 1 \le j \le n$, the *split common null point problem* [7] is to find a point $z \in H_1$ such that

$$z \in \big(\cap_{i=1}^{m} B_i^{-1}0 \big) \cap \big(\cap_{j=1}^{n} A_j^{-1}(G_j^{-1}0) \big),$$

where $B_i^{-1}0$ and $G_j^{-1}0$ are null point sets of B_i and G_j, respectively. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Then the *split feasibility peoblem* [8] is to find $z \in H_1$ such that $z \in C \cap A^{-1}Q$. Putting $B_i = \partial i_C$ for all i, $G_j = \partial i_Q$ for all j and $A_j = A$ for all j in the split common null point problem, we see that the split feasibility peoblem is a special case of the split common null point problem, where ∂i_C and ∂i_Q are the subdifferentials of the indicator functions i_C of C and i_Q of Q, respectively. Defining $U = A^*(I - P_Q)A$ in the split feasibility peoblem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator, where A^* is the adjoint operator of A and P_C and P_Q are the metric projections of H_1 onto C and H_2 onto Q, respectively. Furthermore, if $C \cap A^{-1}Q$ is non-empty, then $z \in C \cap A^{-1}Q$ is equivalent to $z = P_C(I - \lambda U)z$, where $\lambda > 0$.

In this paper, motivated by the idea of the split feasibility problem and results for solving the problem, we consider generalized split feasibility problems and then establish strong convergence theorems by two hybrid methods for the problems. As applications, we get new strong convergence theorems which are connected with fixed point problem and equilibrium problem.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [31], we know the following basic equality. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.1}$$

We also know that for $x, y, u, v \in H$,

$$2\langle x - y, u - v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.2}$$

Let C be a nonempty, closed and convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. The mapping P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - u \rangle \ge 0$$

for all $x \in H$ and $u \in C$; see [31] for more details.

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a Hilbert space H, define s-Li$_n C_n$ and w-Ls$_n C_n$ as follows: $x \in$s-Li$_n C_n$ if and only if there exists $\{x_n\} \subset H$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in$w-Ls$_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset H$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies

$$C_0 = \text{s-Li}_n C_n = \text{w-Ls}_n C_n, \tag{2.3}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [23] and we write $C_0 =$M-$\lim_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\cap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [23]. We know the following theorem by Tsukada [40].

Theorem 2.1 ([40])**.** *Let H be a Hilbert space. Let $\{C_n\}$ be a sequence of nonempty, closed and convex subsets of H. If $C_0 =M$-$\lim_{n\to\infty} C_n$ exists and nonempty, then for each $x \in H$, $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$, where P_{C_n} and P_{C_0} are the mertic projections of H onto C_n and C_0, respectively.*

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Then, a mapping $T : C \to H$ is called generalized hybrid [18] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{2.4}$$

for all $x, y \in C$. We call such a mapping an (α, β)-generalized hybrid mapping. Notice that the mapping above covers several well-known mappings. For example, an (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. Kawasaki and Takahashi [17] defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping S from C into H is said to be widely more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\|Sx - Sy\|^2 + \beta\|x - Sy\|^2 + \gamma\|Sx - y\|^2 + \delta\|x - y\|^2 \\ &\quad + \varepsilon\|x - Sx\|^2 + \zeta\|y - Sy\|^2 + \eta\|(x - Sx) - (y - Sy)\|^2 \le 0 \end{aligned} \tag{2.5}$$

for all $x, y \in C$. Such a mapping S is called $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid. An $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [18] if $\alpha+\beta = -\gamma-\delta = 1$ and $\varepsilon = \zeta = \eta = 0$. A generalized hybrid mapping with a fixed point is quasi-nonexpansive. However, a widely more generalized hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. We know the following theorem from Kawasaki and Takahashi [17].

Theorem 2.2 ([17])**.** *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let S be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following conditions* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ *and* $\zeta + \eta \ge 0$*;*
(2) $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \beta + \zeta + \eta > 0$ *and* $\varepsilon + \eta \ge 0$.

Then S has a fixed point if and only if there exists $z \in C$ such that $\{S^n z : n = 0, 1, \dots\}$ is bounded. In particular, a fixed point of S is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) *and* (2)*.*

The following lemmas for widely more generalized hybrid mappings are essencial for proving our main theorems.

Lemma 2.3 ([17]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let S be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself such that $F(S) \neq \emptyset$ and it satisfies the conditions* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\zeta + \eta \geq 0$ *and* $\alpha + \beta > 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\varepsilon + \eta \geq 0$ *and* $\alpha + \gamma > 0$.

Then S is quasi-nonexpansive.

Lemma 2.4 ([11]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $S : C \to H$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping. Suppose that it satisfies the following conditions* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$ *and* $\alpha + \gamma + \varepsilon + \eta > 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$ *and* $\alpha + \beta + \zeta + \eta > 0$.

If $x_n \rightharpoonup z$ and $x_n - Sx_n \to 0$, then $z \in F(S)$.

From [37], we also have the following lemmas.

Lemma 2.5. *Let H_1 and H_2 be Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Then a mapping $A^*(I - T)A : H_1 \to H_1$ is $\frac{1}{2\|AA^*\|}$-inverse strongly monotone.*

Lemma 2.6. *Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. Let $\lambda, r > 0$ and $z \in H$. Then the following are equivalent:*

(i) $z = J_\lambda(I - rA^*(I - T)A)z$;
(ii) $0 \in A^*(I - T)Az + Bz$;
(iii) $z \in B^{-1}0 \cap A^{-1}F(T)$.

3. Strong convergence theorems

In this section, using the hybrid method by Nakajo and Takahashi [24], we first prove the following strong convergence theorem in Hilbert spaces.

Theorem 3.1. *Let H_1 and H_2 be Hilbert spaces and let C be a nonempty, closed and convex subset of H_1. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping such that the domain of B is included in C and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let S be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into C which satisfies the conditions* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ *and* $\zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ *and* $\varepsilon + \eta \geq 0$.

Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Suppose that $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. Let

$\{x_n\} \subset H_1$ be a sequence generated by $x \in H_1, x_1 = P_C x$ and

$$\begin{cases} z_n = J_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)Sz_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H_1 onto $C_n \cap Q_n$, and $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy

$$\limsup_{n\to\infty} \alpha_n < 1 \quad \textit{and} \quad 0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{1}{\|AA^*\|}.$$

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap B^{-1}0\cap A^{-1}F(T)}x$, where $P_{F(S)\cap B^{-1}0\cap A^{-1}F(T)}$ is the metric projection of H onto $F(S) \cap B^{-1}0 \cap A^{-1}F(T)$.

Proof. We have from Lemma 2.3 that S is quasi-nonexpansive. Then $F(S)$ is closed and convex. We also know that $B^{-1}0 \cap A^{-1}F(T)$ is closed and convex [28]. Then $F(S)\cap B^{-1}0\cap A^{-1}F(T)$ is closed and convex. Thus there exists the mertic projection $P_{F(S)\cap B^{-1}0\cap A^{-1}F(T)}$ of H onto $F(S) \cap B^{-1}0 \cap A^{-1}F(T)$. Since

$$\begin{aligned} &\|y_n - z\|^2 \leq \|x_n - z\|^2 \\ \Longleftrightarrow &\|y_n\|^2 - \|x_n\|^2 - 2\langle y_n - x_n, z\rangle \leq 0, \end{aligned}$$

we have that C_n, Q_n and $C_n \cap Q_n$ are closed and convex for all $n \in \mathbb{N}$. We next show that $C_n \cap Q_n$ is nonempty. Let $z \in F(S) \cap B^{-1}0 \cap A^{-1}F(T)$. Since $I - T$ is $\frac{1}{2}$-inverse strongly monotone and $z = J_{\lambda_n}(I - \lambda_n A^*(I-T)A)z$, we have from $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{1}{\|AA^*\|}$ and Lemma 2.6 that for any $n \in \mathbb{N}$,

$$\begin{aligned} \|z_n - z\|^2 &= \|J_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n - J_{\lambda_n}(I - \lambda_n A^*(I-T)A)z\|^2 \\ &\leq \|x_n - \lambda_n A^*(I-T)Ax_n - z + \lambda_n A^*(I-T)Az\|^2 \\ &= \|x_n - \lambda_n A^*(I-T)Ax_n - z\|^2 \\ &= \|x_n - z\|^2 - 2\lambda_n\langle x_n - z, A^*(I-T)Ax_n\rangle + (\lambda_n)^2 \|A^*(I-T)Ax_n\|^2 \\ &= \|x_n - z\|^2 - 2\lambda_n\langle Ax_n - Az, (I-T)Ax_n\rangle + (\lambda_n)^2 \|A^*(I-T)Ax_n\|^2 \\ &\leq \|x_n - z\|^2 - \lambda_n \|(I-T)Ax_n\|^2 + (\lambda_n)^2\langle A^*(I-T)Ax_n, A^*(I-T)Ax_n\rangle \\ &= \|x_n - z\|^2 - \lambda_n \|(I-T)Ax_n\|^2 + (\lambda_n)^2\langle AA^*(I-T)Ax_n, (I-T)Ax_n\rangle \\ &\leq \|x_n - z\|^2 - \lambda_n \|(I-T)Ax_n\|^2 + (\lambda_n)^2 \|AA^*\| \|(I-T)Ax_n\|^2 \\ &= \|x_n - z\|^2 + \lambda_n(\lambda_n \|AA^*\| - 1) \|(I-T)Ax_n\|^2 \\ &\leq \|x_n - z\|^2. \end{aligned} \tag{3.1}$$

Since S is quasi-nonexpansive, we have from (3.1) that

$$\begin{aligned} \|y_n - z\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Sz_n - z\|^2 \\ &\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|Sz_n - z\|^2 \\ &\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\ &\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 \\ &= \|x_n - z\|^2. \end{aligned}$$

Thus we have $z \in C_n$ and hence $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \subset C_n$ for all $n \in \mathbb{N}$. Next, we show by induction that $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. From $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \subset Q_1$, it follows that $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \subset C_1 \cap Q_1$. Suppose that $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$. We have from $x_{k+1} = P_{C_k \cap Q_k} x$ that

$$\langle x_{k+1} - z, x - x_{k+1} \rangle \ge 0, \quad \forall z \in C_k \cap Q_k.$$

Since $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \subset C_k \cap Q_k$, we also have

$$\langle x_{k+1} - z, x - x_{k+1} \rangle \ge 0, \quad \forall z \in F(S) \cap B^{-1}0 \cap A^{-1}F(T).$$

This implies $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \subset Q_{k+1}$. Thus we have $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \subset C_{k+1} \cap Q_{k+1}$. By induction, we have $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. This means that $\{x_n\}$ and $\{z_n\}$ are well-defined.

Since $x_n = P_{Q_n} x$ and $x_{n+1} = P_{C_n \cap Q_n} x \subset Q_n$, we have from (2.2) that

$$\begin{aligned} 0 &\le 2\langle x - x_n, x_n - x_{n+1} \rangle \\ &= \|x - x_{n+1}\|^2 - \|x - x_n\|^2 - \|x_n - x_{n+1}\|^2 \\ &\le \|x - x_{n+1}\|^2 - \|x - x_n\|^2. \end{aligned} \tag{3.2}$$

Then we get that

$$\|x - x_n\|^2 \le \|x - x_{n+1}\|^2. \tag{3.3}$$

Furthermore, since $x_n = P_{Q_n} x$ and $z \in F(S) \cap B^{-1}0 \cap A^{-1}F(T) \subset Q_n$, we have

$$\|x - x_n\|^2 \le \|x - z\|^2. \tag{3.4}$$

We have from (3.3) and (3.4) that $\lim_{n\to\infty} \|x - x_n\|^2$ exists. This implies that $\{x_n\}$ is bounded. Hence, $\{y_n\}$, $\{z_n\}$ and $\{Sz_n\}$ are also bounded. From (3.2), we have that

$$\|x_n - x_{n+1}\|^2 \le \|x - x_{n+1}\|^2 - \|x - x_n\|^2$$

and hence

$$\|x_n - x_{n+1}\| \to 0. \tag{3.5}$$

From $x_{n+1} \in C_n$, we have that $\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|$. From (3.5), we have that $\|y_n - x_{n+1}\| \to 0$. Then we have that

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{3.6}$$

From $0 \le \limsup_{n\to\infty} \alpha_n < 1$ and

$$\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1-\alpha_n)Sz_n\| = (1-\alpha_n)\|x_n - Sz_n\|,$$

we have that

$$\|Sz_n - x_n\| \to 0. \tag{3.7}$$

Let us show that $\|Sz_n - z_n\| \to 0$. It follows from (3.1) that

$$\begin{aligned}
\|y_n - z\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Sz_n - z\|^2 \\
&\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\
&\le \alpha_n \|x_n - z\|^2 \\
&\quad + (1-\alpha_n)\{\|x_n - z\|^2 + \lambda_n(\lambda_n \|AA^*\| - 1)\|(I-T)Ax_n\|^2\} \\
&= \|x_n - z\|^2 + (1-\alpha_n)\lambda_n(\lambda_n \|AA^*\| - 1)\|(I-T)Ax_n\|^2
\end{aligned}$$

for all $z \in F(S) \cap B^{-1}0 \cap A^{-1}F(T)$. Thus we have that

$$\begin{aligned}
(1-\alpha_n)\lambda_n(1 - \lambda_n\|AA^*\|)\|(I-T)Ax_n\|^2 &\le \|x_n - z\|^2 - \|y_n - z\|^2 \\
&= (\|x_n - z\| + \|y_n - z\|)(\|x_n - z\| - \|y_n - z\|) \\
&\le (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\|.
\end{aligned}$$

From $\|y_n - x_n\| \to 0$ and $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \frac{1}{\|AA^*\|}$, we have that

$$\lim_{n\to\infty} \|(I-T)Ax_n\| = 0. \tag{3.8}$$

Since J_{λ_n} is firmly nonexpansive, we have that

$$\begin{aligned}
2\|z_n - z\|^2 &= 2\|J_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n - J_{\lambda_n}(I - \lambda_n A^*(I-T)A)z\|^2 \\
&\le 2\langle z_n - z, (I - \lambda_n A^*(I-T)A)x_n - z\rangle \\
&= \|z_n - z\|^2 + \|(I - \lambda_n A^*(I-T)A)x_n - z\|^2 \\
&\quad - \|z_n - (I - \lambda_n A^*(I-T)A)x_n\|^2 \\
&\le \|z_n - z\|^2 + \|x_n - z\|^2 \\
&\quad - \|z_n - (I - \lambda_n A^*(I-T)A)x_n\|^2 \\
&= \|z_n - z\|^2 + \|x_n - z\|^2 - \|z_n - x_n + \lambda_n(A^*(I-T)Ax_n)\|^2 \\
&= \|z_n - z\|^2 + \|x_n - z\|^2 - \|z_n - x_n\|^2 \\
&\quad - 2\lambda_n\langle z_n - x_n, A^*(I-T)Ax_n\rangle - \lambda_n^2\|A^*(I-T)Ax_n\|^2.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
\|z_n - z\|^2 &\le \|x_n - z\|^2 - \|z_n - x_n\|^2 \\
&\quad - 2\lambda_n\langle z_n - x_n, A^*(I-T)Ax_n\rangle - \lambda_n^2\|A^*(I-T)Ax_n\|^2.
\end{aligned}$$

Then we have that

$$\begin{aligned}
\|y_n - z\|^2 &\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|Sz_n - z\|^2 \\
&\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\
&\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\{\|x_n - z\|^2 - \|z_n - x_n\|^2 \\
&\quad - 2\lambda_n\langle z_n - x_n, A^*(I-T)Ax_n\rangle - {\lambda_n}^2\|A^*(I-T)Ax_n\|^2\}.
\end{aligned}$$

This means that

$$\begin{aligned}(1-\alpha_n)\|z_n-x_n\|^2 \le{}& \|x_n-z\|^2-\|y_n-z\|^2\\ &+\|A^*(I-T)Ax_n\|\{2\lambda_n\|z_n-x_n\|+\lambda_n^2\|A^*(I-T)Ax_n\|\}\\ \le{}& (\|x_n-z\|+\|y_n-z\|)\|x_n-y_n\|\\ &+\|A^*(I-T)Ax_n\|\{2\lambda_n\|z_n-x_n\|+\lambda_n^2\|A^*(I-T)Ax_n\|\}.\end{aligned}$$

Since $\lim_{n\to\infty}\|A^*(I-T)Ax_n\|=0$, $\lim_{n\to\infty}\|x_n-y_n\|=0$, and $\{y_n\}$, $\{z_n\}$ and $\{x_n\}$ are bounded, we have

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{3.9}$$

Since $y_n=\alpha_nx_n+(1-\alpha_n)Sz_n$, we have $y_n-Sz_n=\alpha_n(x_n-Sz_n)$. Then, from (3.7) we have

$$\|y_n-Sz_n\|=\alpha_n\|x_n-Sz_n\|\to 0. \tag{3.10}$$

Since

$$\|z_n-Sz_n\|\le\|z_n-x_n\|+\|x_n-y_n\|+\|y_n-Sz_n\|,$$

from (3.6), (3.9) and (3.10) we have

$$\|z_n-Sz_n\|\to 0. \tag{3.11}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}\subset\{x_n\}$ such that $x_{n_i}\rightharpoonup z^*$. We have from (3.9) and $x_{n_i}\rightharpoonup z^*$ that $z_{n_i}\rightharpoonup z^*$. From (3.11) and Lemma 2.4, we have $z^*\in F(S)$. Next, let us show $z^*\in B^{-1}0\cap A^{-1}F(T)$. From the definition of J_{λ_n}, we have that

$$\begin{aligned} z_n &= J_{\lambda_n}(I-\lambda_nA^*(I-T)A)x_n\\ &\Leftrightarrow (I-\lambda_nA^*(I-T)A)x_n\in(I+\lambda_nB)z_n=z_n+\lambda_nBz_n\\ &\Leftrightarrow x_n-z_n-\lambda_nA^*(I-T)Ax_n\in\lambda_nBz_n\\ &\Leftrightarrow \frac{1}{\lambda_n}(x_n-z_n-\lambda_nA^*(I-T)Ax_n)\in Bz_n.\end{aligned}$$

Since B is monotone, we have that for $(u,v)\in B$,

$$\langle z_n-u,\frac{1}{\lambda_n}(x_n-z_n-\lambda_nA^*(I-T)Ax_n)-v\rangle\ge 0$$

and hence

$$\langle z_n-u,\frac{x_n-z_n}{\lambda_n}-(A^*(I-T)Ax_n+v)\rangle\ge 0. \tag{3.12}$$

From $z_{n_i}\rightharpoonup z^*$ and $A^*(I-T)Ax_{n_i}\to 0$, we have that

$$\langle z^*-u,-v\rangle\ge 0. \tag{3.13}$$

Since B is maximal monotone, we have that $0\in Bz^*$. Furthermore, since $I-T$ is $\frac{1}{2}$-inverse strongly monotone, we have that

$$\langle Ax_{n_i}-Az^*,(I-T)Ax_{n_i}-(I-T)Az^*\rangle\quad\ge\frac{1}{2}\|(I-T)Ax_{n_i}-(I-T)Az^*\|^2.$$

From $x_{n_i}\rightharpoonup z^*$ and $(I-T)Ax_{n_i}\to 0$, we have that $\|(I-T)Az^*\|^2\le 0$ and hence $Az^*\in F(T)$. Therefore, $z^*\in B^{-1}0\cap A^{-1}F(T)$.

Put $z_0 = P_{F(S)\cap B^{-1}0\cap A^{-1}F(T)}x$. Since $z_0 = P_{F(S)\cap B^{-1}0\cap A^{-1}F(T)}x \in C_n \cap Q_n$ and $x_{n+1} = P_{C_n\cap Q_n}x$, we have that

$$\|x - x_{n+1}\|^2 \le \|x - z_0\|^2. \tag{3.14}$$

Since $\|\cdot\|^2$ is weakly lower semicontinuous, from $x_{n_i} \rightharpoonup z^*$ we have that

$$\begin{aligned}\|x - z^*\|^2 &= \|x\|^2 - 2\langle x, z^*\rangle + \|z^*\|^2 \\ &\le \liminf_{i\to\infty}(\|x\|^2 - 2\langle x, x_{n_i}\rangle + \|x_{n_i}\|^2) \\ &= \liminf_{i\to\infty}\|x - x_{n_i}\|^2 \\ &\le \|x - z_0\|^2.\end{aligned}$$

From the definition of z_0, we have $z^* = z_0$. Then we obtain $x_n \rightharpoonup z_0$. We finally show that $x_n \to z_0$. We have

$$\|z_0 - x_n\|^2 = \|z_0 - x\|^2 + \|x - x_n\|^2 + 2\langle z_0 - x, x - x_n\rangle, \quad \forall n \in \mathbb{N}.$$

Using (3.14), we have that

$$\begin{aligned}\limsup_{n\to\infty}\|z_0 - x_n\|^2 &= \limsup_{n\to\infty}(\|z_0 - x\|^2 + \|x - x_n\|^2 + 2\langle z_0 - x, x - x_n\rangle) \\ &\le \limsup_{n\to\infty}(\|z_0 - x\|^2 + \|x - z_0\|^2 + 2\langle z_0 - x, x - x_n\rangle) \\ &= \|z_0 - x\|^2 + \|x - z_0\|^2 + 2\langle z_0 - x, x - z_0\rangle \\ &= 0.\end{aligned}$$

Thus we obtain $\lim_{n\to\infty}\|z_0 - x_n\| = 0$. Therefore $\{x_n\}$ converges strongly to z_0. This completes the proof. $\square$

Next, we prove a strong convergence theorem by the shrinking projection method [34] for generalized split feasibility problems in Hilbert spaces.

Theorem 3.2. *Let H_1 and H_2 be Hilbert spaces and let C be a nonempty, closed and convex subset of H_1. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping such that the domain of B is included in C and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let S be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into C which satisfies the conditions* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \beta > 0$ *and* $\zeta + \eta \ge 0$*;*
(2) $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \gamma > 0$ *and* $\varepsilon + \eta \ge 0$.

Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \ne 0$. Suppose that $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \ne \emptyset$. Let $C_1 = C$ and let $\{x_n\}$ be a sequence in H_1 generated by $x \in H_1, x_1 = P_C x$ and

$$\begin{cases} z_n = J_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Sz_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H_1 onto C_{n+1}, and $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ are sequences such that

$$\liminf_{n\to\infty} \alpha_n < 1 \quad \text{and} \quad 0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \frac{1}{\|AA^*\|}.$$

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap B^{-1}0\cap A^{-1}F(T)}x$, where $P_{F(S)\cap B^{-1}0\cap A^{-1}F(T)}$ is the metric projection of H onto $F(S) \cap B^{-1}0 \cap A^{-1}F(T)$.

Proof. As in the proof of Theorem 3.1, $F(S)\cap B^{-1}0\cap A^{-1}F(T)$ is closed and convex. Thus there exists the mertic projection of H onto $F(S) \cap B^{-1}0 \cap A^{-1}F(T)$. We show that C_n are closed and convex for all $n \in \mathbb{N}$. It is obvious from assumption that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex. We know that for $z \in C_k$,

$$\begin{aligned} &\|y_k - z\|^2 \le \|x_k - z\|^2 \\ \Longleftrightarrow &\|y_k\|^2 - \|x_k\|^2 - 2\langle y_k - x_k, z\rangle \le 0. \end{aligned}$$

Then C_{k+1} is closed and convex. By induction, C_n are closed and convex for all $n \in \mathbb{N}$. Next, we show that $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \subset C_n$ for all $n \in \mathbb{N}$. It is obvious from assumption that $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \subset C_1$. Suppose that $F(S)\cap B^{-1}0\cap A^{-1}F(T) \subset C_k$ for some $k \in \mathbb{N}$. Put $z_k = J_{\lambda_k}(I - \lambda_k A^*(I-T)A)x_k$ and take $z \in F(S) \cap B^{-1}0 \cap A^{-1}F(T) \subset C_k$. As in the proof of Theorem 3.1, we have that

$$\begin{aligned} \|z_k - z\|^2 &= \|J_{\lambda_k}(I - \lambda_k A^*(I-T)A)x_k - J_{\lambda_k}(I - \lambda_k A^*(I-T)A)z\|^2 \\ &\le \|x_k - z\|^2 + \lambda_k(\lambda_k \|AA^*\| - 1)\,\|(I-T)Ax_k\|^2 \\ &\le \|x_k - z\|^2 \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \|y_k - z\|^2 &= \|\alpha_k x_k + (1-\alpha_k)Sz_k - z\|^2 \\ &\le \|x_k - z\|^2. \end{aligned}$$

Hence we have $z \in C_{k+1}$. By induction, we have that $F(S)\cap B^{-1}0\cap A^{-1}F(T) \subset C_n$ for all $n \in \mathbb{N}$. Since C_n is nonempty, closed and convex, there exists the metric projection P_{C_n} of H onto C_n. Thus $\{x_n\}$ is well-defined.

Since $\{C_n\}$ is a nonincreasing sequence of nonempty, closed and convex subsets of H with respect to inclusion, it follows that

$$\emptyset \ne F(S) \cap B^{-1}0 \cap A^{-1}F(T) \subset \text{M-}\lim_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} C_n. \tag{3.16}$$

Put $C_0 = \bigcap_{n=1}^{\infty} C_n$. Then, by Theorem 2.1, we have that $\{P_{C_n}x\}$ converges strongly to $x_0 = P_{C_0}x$, i.e.,

$$x_n = P_{C_n}x \to x_0.$$

To complete the proof, it is sufficient to show that $x_0 = P_{F(S)\cap B^{-1}0\cap A^{-1}F(T)}x$.

Since $x_n = P_{C_n}x$ and $x_{n+1} = P_{C_{n+1}}x \in C_{n+1} \subset C_n$, we have from (2.2) that

$$0 \le 2\langle x - x_n, x_n - x_{n+1}\rangle \tag{3.17}$$

$$= \|x - x_{n+1}\|^2 - \|x - x_n\|^2 - \|x_n - x_{n+1}\|^2$$
$$\leq \|x - x_{n+1}\|^2 - \|x - x_n\|^2.$$

Then we get that

$$\|x - x_n\|^2 \leq \|x - x_{n+1}\|^2. \tag{3.18}$$

Furthermore, since $x_n = P_{C_n}x$ and $z \in F(S) \cap B^{-1}0 \cap A^{-1}F(T) \subset C_n$, we have

$$\|x - x_n\|^2 \leq \|x - z\|^2. \tag{3.19}$$

Thus we have that $\lim_{n\to\infty} \|x - x_n\|^2$ exists. This implies that $\{x_n\}$ is bounded. Hence, $\{y_n\}$, $\{z_n\}$ and $\{Sz_n\}$ are also bounded. From (3.17), we have

$$\|x_n - x_{n+1}\|^2 \leq \|x - x_{n+1}\|^2 - \|x - x_n\|^2.$$

Thus we have that

$$\|x_n - x_{n+1}\|^2 \to 0. \tag{3.20}$$

From $x_{n+1} \in C_{n+1}$, we also have that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. Then we get that $\|y_n - x_{n+1}\| \to 0$. Using this, we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{3.21}$$

From $0 \leq \liminf_{n\to\infty} \alpha_n < 1$, we have a subsequence $\{\alpha_{n_i}\}$ of $\{\alpha_n\}$ such that $\alpha_{n_i} \to \gamma$ and $0 \leq \gamma < 1$. From

$$\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1-\alpha_n)Sz_n\| = (1-\alpha_n)\|x_n - Sz_n\|,$$

we have that

$$\|Sz_{n_i} - x_{n_i}\| \to 0. \tag{3.22}$$

Using (3.22), let us show $\|Sz_{n_i} - z_{n_i}\| \to 0$. As in the proof of Theorem 3.1, we have that for any $z \in F(S) \cap B^{-1}0 \cap A^{-1}F(T)$,

$$\begin{aligned} \|y_n - z\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Sz_n - z\|^2 \\ &\leq \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\ &\leq \|x_n - z\|^2 + (1-\alpha_n)\lambda_n(\lambda_n \|AA^*\| - 1)\|(I-T)Ax_n\|^2. \end{aligned}$$

Thus we have

$$\begin{aligned} (1-\alpha_n)\lambda_n(1-\lambda_n \|AA^*\|)\|(I-T)Ax_n\|^2 &\leq \|x_n - z\|^2 - \|y_n - z\|^2 \\ &= (\|x_n - z\| + \|y_n - z\|)(\|x_n - z\| - \|y_n - z\|) \\ &\leq (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\|. \end{aligned}$$

From $\|y_n - x_n\| \to 0$ and $\alpha_{n_i} \to \gamma$, we have that

$$\lim_{i\to\infty} \|(I-T)Ax_{n_i}\| = 0. \tag{3.23}$$

Since J_{λ_n} is firmly nonexpansive, as in the proof of Theorem 3.1, we have that

$$\begin{aligned} 2\|z_n - z\|^2 &= 2\|J_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n - J_{\lambda_n}(I - \lambda_n A^*(I-T)A)z\|^2 \\ &\leq \|z_n - z\|^2 + \|x_n - z\|^2 - \|z_n - x_n\|^2 \\ &\quad - 2\lambda_n\langle z_n - x_n, A^*(I-T)Ax_n\rangle - \lambda_n^2\|A^*(I-T)Ax_n\|^2 \end{aligned}$$

and hence

$$\|z_n - z\|^2 \le \|x_n - z\|^2 - \|z_n - x_n\|^2 - 2\lambda_n \langle z_n - x_n, A^*(I-T)Ax_n\rangle - \lambda_n^2 \|A^*(I-T)Ax_n\|^2.$$

Furthermore, as in the proof of Theorem 3.1, we have

$$\begin{aligned}\|y_n - z\|^2 &\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|Sz_n - z\|^2 \\ &\le \|x_n - z\|^2 - (1-\alpha_n)\|z_n - x_n\|^2 - {\lambda_n}^2(1-\alpha_n)\|A^*(I-T)Ax_n\|^2 \\ &\quad - 2\lambda_n(1-\alpha_n)\langle z_n - x_n, A^*(I-T)Ax_n\rangle.\end{aligned}$$

This means that

$$\begin{aligned}(1-\alpha_n)\|z_n - x_n\|^2 &\le \|x_n - z\|^2 - \|y_n - z\|^2 \\ &\quad + \|A^*(I-T)Ax_n\|\{2\lambda_n\|z_n - x_n\| + \lambda_n^2\|A^*(I-T)Ax_n\|\} \\ &\le (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\| \\ &\quad + \|A^*(I-T)Ax_n\|\{2\lambda_n\|z_n - x_n\| + \lambda_n^2\|A^*(I-T)Ax_n\|\}.\end{aligned}$$

Since $\lim_{i\to\infty}\|(I-T)Ax_{n_i}\| = 0$, $\lim_{n\to\infty}\|x_n - y_n\| = 0$, $\alpha_{n_i} \to \gamma < 1$ and $\{y_n\}$, $\{z_n\}$ and $\{x_n\}$ are bounded, we have

$$\lim_{n\to\infty}\|z_{n_i} - x_{n_i}\| = 0. \tag{3.24}$$

Since $y_n = \alpha_n x_n + (1-\alpha_n)Sz_n$, we have $y_n - Sz_n = \alpha_n(x_n - Sz_n)$. From (3.22) we have

$$\|y_{n_i} - Sz_{n_i}\| = \alpha_{n_i}\|x_{n_i} - Sz_{n_i}\| \to 0. \tag{3.25}$$

Since $\|z_{n_i} - Sz_{n_i}\| \le \|z_{n_i} - x_{n_i}\| + \|x_{n_i} - y_{n_i}\| + \|y_{n_i} - Sz_{n_i}\|$, from (3.21), (3.24) and (3.25) we have

$$\|z_{n_i} - Sz_{n_i}\| \to 0. \tag{3.26}$$

Since $x_{n_i} = P_{C_{n_i}}x \to x_0$, we have from (3.24) that $z_{n_i} \to x_0$. Then $z_{n_i} \rightharpoonup x_0$. From (3.26) and Lemma 2.4 we have $x_0 \in F(S)$. Let us show $x_0 \in B^{-1}0 \cap A^{-1}F(T)$. As in the proof of Theorem 3.1, we have for $(u,v) \in B$,

$$\langle z_n - u, \frac{x_n - z_n}{\lambda_n} - (A^*(I-T)Ax_n + v)\rangle \ge 0. \tag{3.27}$$

from $z_{n_i} \rightharpoonup x_0$, $\|x_{n_i} - z_{n_i}\| \to 0$ and $A^*(I-T)Ax_{n_i} \to 0$, we have $\langle x_0 - u, -v\rangle \ge 0$ and hence $0 \in Bx_0$. Furthermore, since $I - T$ is $\frac{1}{2}$-inverse strongly monotone,

$$\langle Ax_{n_i} - Ax_0, (I-T)Ax_{n_i} - (I-T)Ax_0\rangle \ge \frac{1}{2}\|(I-T)Ax_{n_i} - (I-T)Ax_0\|^2.$$

From $x_{n_i} = P_{C_{n_i}}x \to x_0$ and $(I-T)Ax_{n_i} \to 0$, we have that $(I-T)Ax_0 = 0$. This implies that $Ax_0 \in F(T)$. Therefore, $x_0 \in B^{-1}0 \cap A^{-1}F(T)$. Thus we have $x_0 \in F(S) \cap B^{-1}0 \cap A^{-1}F(T)$. Put $z_0 = P_{F(S)\cap B^{-1}0\cap A^{-1}F(T)}x$. Since $z_0 = P_{F(S)\cap B^{-1}0\cap A^{-1}F(T)}x \in C_{n+1}$ and $x_{n+1} = P_{C_{n+1}}x$, we have that

$$\|x - x_{n+1}\|^2 \le \|x - z_0\|^2. \tag{3.28}$$

Thus we have that

$$\|x - x_0\|^2 = \lim_{n\to\infty} \|x - x_n\|^2 \le \|x - z_0\|^2.$$

Then we get $z_0 = x_0$. Hence $\{x_n\}$ converges strongly to z_0. This completes the proof. $\square$

4. Applications

In this section, we give some applications. Let H be a Hilbert space and let f be a proper, lower semicontinuous and convex function of H into $(-\infty, \infty]$. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y - x\rangle \le f(y),\ \forall y \in H\}$$

for all $x \in H$. From Rockafellar [26], we know that ∂f is maximal monotone. Let C be a nonempty, closed and convex subset of H and let i_C be the indicator function of C, i.e.,

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Since i_C is a proper, lower semicontinuous and convex function on H, the subdifferential ∂i_C of i_C is a maximal monotone operator. Thus we can define the resolvent J_λ of ∂i_C for $\lambda > 0$, i.e.,

$$J_\lambda x = (I + \lambda \partial i_C)^{-1} x$$

for all $x \in H$. We know from [31] that, for any $x \in H$ and $u \in C$,

$$\partial i_C u = N_C u \quad \text{and} \quad J_\lambda x = P_C x,$$

where $N_C u$ is the normal cone to C at u, i.e.,

$$N_C u = \{z \in H : \langle z, v - u\rangle \le 0,\ \forall v \in C\}.$$

Now, using Theorem 3.1, we can obtain the following strong convergence theorem in Hilbert spaces.

Theorem 4.1. *Let H_1 and H_2 be Hilbert spaces and let C be a nonempty, closed and convex subset of H_1. Let S be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into C which satisfies the conditions* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \ge 0$, $\alpha+\beta > 0$ *and* $\zeta+\eta \ge 0$*;*
(2) $\alpha+\beta+\gamma+\delta \ge 0$, $\alpha+\gamma > 0$ *and* $\varepsilon+\eta \ge 0$.

Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \ne 0$. Suppose that $F(S) \cap \cap A^{-1}F(T) \ne \emptyset$. Let $\{x_n\} \subset H_1$ be a sequence generated by $x \in H_1, x_1 = P_C x$ and

$$\begin{cases} z_n = P_C(I - \lambda_n A^*(I - T)A)x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n) S z_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$, and $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy

$$\limsup_{n\to\infty} \alpha_n < 1 \quad \textit{and} \quad 0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \frac{1}{\|AA^*\|}.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap A^{-1}F(T)}x$, where $P_{F(S)\cap A^{-1}F(T)}$ is the metric projection of H onto $F(S) \cap A^{-1}F(T)$.

Proof. Setting $B = \partial i_C$ in Theorem 3.1, we know that $J_{\lambda_n} = P_C$ for all $\lambda_n > 0$. Thus we obtain the desired result by Theorem 3.1. □

Similarly, using Theorem 3.2, we get the following theorem.

Theorem 4.2. *Let H_1 and H_2 be Hilbert spaces and let C be a nonempty, closed and convex subset of H_1. Let S be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into C which satisfies the conditions* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \ge 0$, $\alpha+\beta > 0$ *and* $\zeta+\eta \ge 0$;
(2) $\alpha+\beta+\gamma+\delta \ge 0$, $\alpha+\gamma > 0$ *and* $\varepsilon+\eta \ge 0$.

Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \ne 0$. Suppose that $F(S) \cap A^{-1}F(T) \ne \emptyset$. Let $C_1 = C$ and let $\{x_n\}$ be a sequence in H_1 generated by $x \in H_1, x_1 = P_C x$ and

$$\begin{cases} z_n = P_C(I - \lambda_n A^*(I-T)A)x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)Sz_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H_1 onto C_{n+1}, and $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ are sequences such that

$$\liminf_{n\to\infty} \alpha_n < 1 \quad \textit{and} \quad 0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \frac{1}{\|AA^*\|}.$$

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap A^{-1}F(T)}x$, where $P_{F(S)\cap A^{-1}F(T)}$ is the metric projection of H onto $F(S) \cap A^{-1}F(T)$.

Next, using Theorem 3.1, we consider the problem for finding a common solution of an equilibrium problem and the sets of fixed points of two nonlinear mappings in Hilbert spaces. Let C be a nonempty, closed and convex subset of a Hilbert space and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e. $f(x,y) + f(y,x) \le 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$,

$$\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \le f(x,y);$$

(A4) for all $x \in C$, $f(x,\cdot)$ is convex and lower semicontinuous.

Then, the equilibrium problem (with respect to C) is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \ge 0 \tag{4.1}$$

for all $y \in C$. The set of such solutions $\hat{x}$ is denoted by $EP(f)$. The following lemma appears implicitly in Blum and Oettli [4].

Lemma 4.3 (Blum and Oettli)**.** *Let C be a nonempty, closed and convex subset of H and let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C.$$

The following lemma was also given in Combettes and Hirstoaga [9].

Lemma 4.4. *Assume that $f : C \times C \to \mathbb{R}$ satisfies $(A1)-(A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}.$$

Then, the following hold:

(1) *T_r is single-valued;*
(2) *T_r is a firmly nonexpansive mapping, i.e., for all $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) *$F(T_r) = EP(f)$;*
(4) *$EP(f)$ is closed and convex.*

We call such T_r the resolvent of f for $r > 0$. Using Lemmas 4.3 and 4.4, we know the following theorem from Takahashi, Takahashi and Toyoda [28]. See [2] for a more general result.

Theorem 4.5. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $f : C \times C \to \mathbb{R}$ satisfy $(A1)-(A4)$. Let A_f be a multivalued mapping of H into itself defined by*

$$A_f x = \begin{cases} \{z \in H : f(x, y) \geq \langle y - x, z\rangle, \ \forall y \in C\}, \ x \in C, \\ \emptyset, \ x \notin C. \end{cases}$$

Then $EP(f) = A_f^{-1}0$ and A_f is a maximal monotone operator with $D(A_f) \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r of f coincides with the resolvent of A_f; i.e.,

$$T_r x = (I + rA_f)^{-1}x.$$

Using Theorems 3.1 and 4.5, we obtain the following result.

Theorem 4.6. *Let H_1 and H_2 be Hilbert spaces and let C be a nonempty, closed and convex subset of H_1. Let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let S be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into C which satisfies the conditions* (1) *or* (2)*:*

(1) *$\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and $\zeta + \eta \geq 0$;*
(2) *$\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ and $\varepsilon + \eta \geq 0$.*

Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Suppose that $F(S) \cap EP(f) \cap A^{-1}F(T) \neq \emptyset$. Let $\{x_n\} \subset H_1$ be a sequence generated by $x \in H_1, x_1 = P_C x$ and

$$\begin{cases} z_n = T_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)Sz_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$, and $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy

$$\liminf_{n\to\infty} \alpha_n < 1 \quad \text{and} \quad 0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{1}{\|AA^*\|}.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap EP(f)\cap A^{-1}F(T)}x$, where $P_{F(S)\cap EP(f)\cap A^{-1}F(T)}$ is the metric projection of H onto $F(S) \cap EP(f) \cap A^{-1}F(T)$.

Proof. For the bifunction $f : C \times C \to \mathbb{R}$, we can define A_f in Lemma 4.5. From Theorem 4.5 we also know that $J_{\lambda_n} = T_{\lambda_n}$ for all $n \in \mathbb{N}$. Thus we obtain the desired result by Theorem 3.1. □

As in the proof of Theorem 4.6, we also get similar result from Theorems 3.2 and 4.5,

Theorem 4.7. *Let H_1 and H_2 be Hilbert spaces and let C be a nonempty, closed and convex subset of H_1. Let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1) − (A4). Let S be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into C which satisfies the conditions (1) or (2):*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ *and* $\zeta + \eta \geq 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ *and* $\varepsilon + \eta \geq 0$.

Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Suppose that $F(S) \cap EP(f) \cap A^{-1}F(T) \neq \emptyset$. Let $C_1 = C$ and let $\{x_n\}$ be a sequence in H_1 generated by $x \in H_1, x_1 = P_C x$ and

$$\begin{cases} z_n = T_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)Sz_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H_1 onto C_{n+1}, and $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ are sequences such that

$$\liminf_{n\to\infty} \alpha_n < 1 \quad \text{and} \quad 0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{1}{\|AA^*\|}.$$

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap EP(f)\cap A^{-1}F(T)}x$, where $P_{F(S)\cap EP(f)\cap A^{-1}F(T)}$ is the metric projection of H onto $F(S) \cap EP(f) \cap A^{-1}F(T)$.

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