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$\mathcal{I}$-BOUNDED HOLOMORPHIC FUNCTIONS

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Abstract. In this paper we undertake a study of the relationship between the behavior of an entire mapping between complex Banach spaces and its Taylor series expansion by means of ideals of operators. To accomplish this task we introduce the concept of radius of $\mathcal{I}-$boundedness. We also provide a general procedure to construct ideals of holomorphic mappings.

1. Introduction and notation

In this paper we consider the general problem of comparing the behavior of holomorphic functions around the origin with their local behavior around other points, whenever this behavior is related to properties given in terms of operator ideals (see Section 2 for details). This is also connected to the problem of transferring such a property from the function to its Taylor polynomials and viceversa. González and Gutiérrez [9] have obtained a positive solution to these problems for any closed surjective ideal of operators. However, their results cannot be applied to non-closed ideals and new techniques are required. In this paper we focus on this problem from a new perspective that allows us to get some partial solutions. In order to clarify our objectives, let us explain what the problems are by means of an example: the ideal of $p-$compact operators.

In [1] the concept of $p-$compact holomorphic function was defined as a generalization of $p-$compact linear operators introduced by Sinha and Karn [16]. There, the relation between p-compact holomorphic mappings and their Taylor series expansions was discussed.

A holomorphic function $f : E \longrightarrow F$ between Banach spaces E and F is said to be *p-compact* at a point $x \in E$ if there is a neighborhood V_x of x in E such that $f(V_x)$ is relatively p-compact in F. We shall say that f is *p-compact* if it is p-compact at any $x \in E$. It is clear that any linear operator which is $p-$compact at the origin is actually $p-$compact at any point. This is also true for homogeneous polynomials. The question, posed in [1], of whether this holds true for holomorphic functions was recently answered in the negative by Lassalle and Turco [11].

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The question in the previous paragraph was motivated by the good behavior of compact holomorphic mappings. A holomorphic function $f : E \to F$ is said to be *compact* if for all $x \in E$, there exists a neighborhood V_x such that $f(V_x)$ is relatively compact in F. In [4], the first author and R. M. Schottenloher proved that a holomorphic mapping is compact if and only if there is a $0-$neighborhood V_0 in E whose image is relatively compact in F. Further, f is compact if and only if each of its Taylor coefficients at some $x \in E$ takes the ball of E to a relatively compact set in F.

As usual, $\mathcal{H}(E;F)$ is the space of entire mappings from a Banach space E to another Banach space F. Given $f \in \mathcal{H}(E;F)$, let $P_nf(x) : E \to F$ denote the $n-$homogeneous polynomial of the Taylor series expansion of f at $x \in E$. For general background regarding holomorphic functions and homogeneous polynomials we refer to [7] and [12].

Let $\mathcal{L}$ denote the class of all continuous linear operators between Banach spaces. For each E and F, $\mathcal{L}(E;F)$ has the usual sup norm $\|.\|$. Recall that an *operator ideal* $\mathcal{I}$ is a subclass of $\mathcal{L}$ such that for arbitrary Banach spaces E and F the component $\mathcal{I}(E;F) := \mathcal{I} \cap \mathcal{L}(E;F)$ is a linear subspace of $\mathcal{L}(E;F)$ which contains the finite rank operators and satisfies the ideal property: If $T \in \mathcal{L}(E_0;E)$, $S \in \mathcal{I}(E;F)$ and $R \in \mathcal{L}(F;F_0)$, then $R \circ S \circ T \in \mathcal{I}(E_0;F_0)$. An operator ideal $\mathcal{I}$ and a function $\iota : \mathcal{I} \to [0,\infty[$ form a Banach operator ideal, denoted $[\mathcal{I},\iota]$, if:

(a) $\iota(id_{\mathbb{C}}) = 1$, where $id_{\mathbb{C}}$ is the identity map on $\mathbb{C}$.
(b) ι restricted to each component $\mathcal{I}(E;F)$ is a norm that makes $\mathcal{I}(E;F)$ a Banach space.
(c) If $T \in \mathcal{L}(E_0;E)$, $S \in \mathcal{I}(E;F)$ and $R \in \mathcal{L}(F;F_0)$, then $\iota(R \circ S \circ T) \leq \|R\|\iota(S)\|T\|$.

An operator ideal $\mathcal{I}$ is said to be *closed* if for all Banach spaces E and F, the component $\mathcal{I}(E;F)$ is a closed subspace of $\mathcal{L}(E;F)$ endowed with the sup norm $\|.\|$, that is, if $[\mathcal{I},\|.\|]$ is a Banach ideal. An operator ideal $\mathcal{I}$ is said to be *surjective* if T belongs to $\mathcal{I}(E;F)$ whenever $T \circ S \in \mathcal{I}(Z;F)$ for any surjection $S \in \mathcal{L}(Z;E)$. Here, E, F and Z are arbitrary Banach spaces. Equivalently, $\mathcal{I}$ is surjective if T belongs to $\mathcal{I}(E;F)$ whenever $T(B_E) \subset U(B_Z)$ for some $U \in \mathcal{I}(Z;F)$, where B_E and B_Z denote the closed unit balls of E and Z respectively. Compact operators form a closed surjective ideal of operators. For the general theory of operator ideals we refer to [5] and [14].

In order to investigate whether the behavior of holomorphic functions around the origin can spread to local behavior at other points of the domain, we move to a more general setting: we consider (not necessarily closed) Banach ideals of operators $\mathcal{I}$. For example, the ideal $\mathcal{K}_p$ of $p-$compact operators is not closed whenever $1 < p < \infty$, as shown in [1, Example 1]. We consider $\mathcal{K}_p(E;F)$ with its natural norm k_p given by $k_p(T) = \inf\{\|(x_n)_n\|_p\}$, where the infimum is taken over all sequences $(x_n)_n \in \ell_p(F)$ such that $T(B_E) \subset \{\sum_n a_n x_n \mid (a_n) \in B_{\ell'_p}\}$. The norm k_p was introduced by Sinha and Karn [16] and characterized by Delgado, Piñeiro and Serrano [6, Proposition 3.15]. The ideal $[\mathcal{K}_p, k_p]$ is a Banach ideal.

Our techniques provide a method of defining ideals of holomorphic mappings from a given operator ideal. This is based on the ideals of polynomials defined in [2] combined with power series spaces.

2. The radius of $\mathcal{I}-$boundedness

To initiate our task, we introduce the concept of $\mathcal{I}-$bounded holomorphic functions and the radius of $\mathcal{I}-$boundedness.

Let $\mathcal{I}$ be an operator ideal, and let F be a Banach space. Let $C_{\mathcal{I}}(F)$ stand for the collection of subsets $A \subset F$ such that $A \subset T(B_Z)$ for some Banach space Z and some $T \in \mathcal{I}(Z; F)$. We will call any $A \in C_{\mathcal{I}}(F)$ an *$\mathcal{I}-$bounded* set. All $\mathcal{I}-$bounded sets are bounded. The family $C_{\mathcal{I}}(F)$ was first considered in [17] (see also [3] and [9]).

Let E and F be Banach spaces and let $x \in E$. An entire mapping $f : E \to F$ is said to be *locally $\mathcal{I}-$bounded* (or just $\mathcal{I}-$bounded) at x if there is a neighborhood V_x of x such that $f(V_x) \in C_{\mathcal{I}}(F)$. In the above definition different $x \in E$ may be associated with different Banach spaces Z. If f is locally $\mathcal{I}-$bounded at every point of E then it is called *locally $\mathcal{I}-$bounded*. In this case, for each $x \in E$ there are a neighborhood V_x of x, a Banach space Z_x and an operator $T_x \in \mathcal{I}(E; F)$ such that $f(V_x) \subset T_x(B_{Z_x})$. If we consider a direct sum Z of all Z_x (for instance the ℓ_1-sum) then Z can be used for all $x \in E$. However, in general the operators T_x must depend on x. Otherwise, if we assume that there exists an operator T such that $f(V_x) \subset T(B_Z)$ for all $x \in E$ then the entire function f would be bounded and so constant as a consequence of Liouville's theorem. Let $\mathcal{H}_{\mathcal{I}}(E; F)$ denote the linear space of all locally $\mathcal{I}-$bounded entire mappings from E to F. The subspace formed by $k-$homogeneous locally $\mathcal{I}-$bounded polynomials is denoted $\mathcal{P}_{\mathcal{I}}(^kE; F)$. It is easy to prove that a k-homogeneous polynomial $P : E \to F$ is locally $\mathcal{I}-$bounded if and only if $P(B_E) \in C_{\mathcal{I}}(F)$.

Given $P \in \mathcal{P}_{\mathcal{I}}(^mE; F)$, we have

$$P(B_E) \subset T(B_Z) \tag{2.1}$$

for some Banach space Z and some $T \in \mathcal{I}(Z; F)$. If $[\mathcal{I}, \iota]$ is a Banach ideal define

$$\|P\|_{\mathcal{I}} := \inf \iota(T)$$

where T varies among those operators in $\mathcal{I}$ for which (2.1) holds. Notice that

$$\mathcal{I}(E; F) \subset \mathcal{P}_{\mathcal{I}}(^1E; F) =: \mathcal{L}_{\mathcal{I}}(E; F) \subset \mathcal{L}(E; F)$$

and

$$\|T\| \leq \|T\|_{\mathcal{I}} \leq \iota(T)$$

for all $T \in \mathcal{I}(E; F)$. With this notation, it is clear that $\mathcal{I}(E; F) = \mathcal{L}_{\mathcal{I}}(E; F)$ if and only if $\mathcal{I}$ is surjective.

It is proved in [3] that $[\mathcal{P}_{\mathcal{I}}, \|\cdot\|_{\mathcal{I}}]$ is a Banach ideal of polynomials.

Given an entire mapping $f \in \mathcal{H}(E; F)$ which is $\mathcal{I}-$bounded at a point $x \in E$, we define the *radius of $\mathcal{I}-$boundedness* of f at x by

$$r_{\mathcal{I}}(f; x) := \sup\{t > 0 : f(x + tB_E) \in C_{\mathcal{I}}(F)\}.$$

This definition is motivated by the radius of boundedness of an entire mapping. Given an arbitrary entire mapping $f : E \to F$ and $A \subset E$, we shall write $\|f\|_A :=$

$\sup_{x\in A}\|f(x)\|$. Recall that if $f : E \to F$ is an entire mapping and $x \in E$, the *radius of boundedness* of f at x is given by

$$r(f;x) := \sup\{t > 0 : \|f\|_{x+tB_E} < \infty\}$$

and coincides with the *radius of uniform convergence* of f at x defined by

$$r(f;x) = \sup\Big\{t > 0 : \lim_n \Big\|f(x+y) - \sum_{k=0}^{n} P_k f(x)(y)\Big\|_{tB_E} = 0\Big\}.$$

Indeed, it is well known that

$$r(f;x) = \frac{1}{\limsup_n \|P_n f(x)\|^{\frac{1}{n}}}, \tag{2.2}$$

where the norm here denotes the usual sup norm.

In order to establish the relation between the radius of $\mathcal{I}-$boundedness of an entire mapping and its radius of boundedness (or uniform convergence) we need the following characterization of surjective closed ideals (compare with [9, Proposition 3]:

Lemma 2.1. *Let $\mathcal{I}$ be an operator ideal. The following are equivalent:*

(1) *$\mathcal{I}$ is surjective and for every Banach space F, if $A \subset F$ is such that for every $\epsilon > 0$ there is an $\mathcal{I}-$bounded set $A_\epsilon \in C_{\mathcal{I}}(F)$ with $A \subset A_\epsilon + \epsilon B_F$, then $A \in C_{\mathcal{I}}(F)$.*
(2) *For all Banach spaces E and F, if $T \in \mathcal{L}(E;F)$ is such that for every $\epsilon > 0$ there is a Banach space Z_ϵ and an operator $T_\epsilon \in \mathcal{I}(Z_\epsilon;F)$ with $T(B_E) \subset T_\epsilon(B_{Z_\epsilon}) + \epsilon B_F$, then $T \in \mathcal{I}(E;F)$.*
(3) *$\mathcal{I}$ is surjective and closed.*

Proof. (1)⇒(2): Let E and F be Banach spaces and let $T \in \mathcal{L}(E;F)$ be such that for every $\epsilon > 0$ there is a Banach space Z_ϵ and an operator $T_\epsilon \in \mathcal{I}(Z_\epsilon;F)$ so that

$$T(B_E) \subset T_\epsilon(B_{Z_\epsilon}) + \epsilon B_F.$$

As $T_\epsilon(B_{Z_\epsilon})$ is an $\mathcal{I}-$bounded set in F, by (1) it follows from the above inclusion that $T(B_E) \in C_{\mathcal{I}}(F)$. Since $\mathcal{I}$ is surjective, we conclude that $T \in \mathcal{I}(E;F)$.

(2)⇒(3): Clearly $\mathcal{I}$ is surjective. Let us see that it is closed. Suppose that $T \in \mathcal{L}(E;F)$ is the uniform limit of a sequence $(T_n)_n$ in $\mathcal{I}(E;F)$. We have to prove that $T \in \mathcal{I}(E;F)$. Given $\epsilon > 0$ there exists a positive integer n_0 such that $\|T - T_n\| < \epsilon$ for all $n \geq n_0$. Then, $\|T(x) - T_{n_0}(x)\| < \epsilon$, for all $x \in B_E$. Hence,

$$T(B_E) \subset T_{n_0}(B_E) + \epsilon B_F.$$

It follows from (2) that $T \in \mathcal{I}(E;F)$.

(3)⇒(1): This is a consequence of [9, Proposition 3].

□

Lemma 2.1 can be thought as a generalization to arbitrary ideals of well-known characterizations of compact sets and compact operators. Indeed, the ideal of compact operators is surjective and closed. In that case, (1) turns out to be the characterization of compact sets in Banach spaces, whereas (2) is the characterization of compact linear operators.

Theorem 2.2. *Let $\mathcal{I}$ be an operator ideal.*

(1) $r_{\mathcal{I}}(f;x) \le r(f;x)$, *for all $f \in \mathcal{H}_{\mathcal{I}}(E;F)$ and all $x \in E$.*
(2) *If $\mathcal{I}$ is closed and surjective then $r_{\mathcal{I}}(f;x) = r(f;x)$, for all $f \in \mathcal{H}_{\mathcal{I}}(E;F)$ and all $x \in E$.*

Proof. (1) If $t < r_{\mathcal{I}}(f;x)$ then $f(x+tB_E) \in C_{\mathcal{I}}(F)$. Hence $f(x+tB_E)$ is bounded. Thus $t \le r(f;x)$. This proves that $r_{\mathcal{I}}(f;x) \le r(f;x)$.

(2) It suffices to prove that $r(f;x) \le r_{\mathcal{I}}(f;x)$. If $t < r(f;x)$ then

$$f(x+y) = \sum_{k=0}^{\infty} P_k f(x)(y)$$

uniformly in $y \in tB_E$. Hence, given $\epsilon > 0$ there exists a positive integer m_0 such that

$$\Big\| f(x+y) - \sum_{k=0}^{m} P_k f(x)(y) \Big\| < \epsilon \tag{2.3}$$

for all $y \in tB_E$ and all $m \ge m_0$.

On the other hand, since $\mathcal{I}$ is closed and surjective, being $f \in \mathcal{H}_{\mathcal{I}}(E;F)$ it follows from [9, Proposition 5] that $P_k f(x) \in \mathcal{P}_{\mathcal{I}}({}^kE;F)$ for all k. Then, for each m

$$A_m := \sum_{k=0}^{m} P_k f(x)(tB_E) = \sum_{k=0}^{m} t^k P_k f(x)(B_E) \in C_{\mathcal{I}}(F). \tag{2.4}$$

From (2.3) and (2.4)

$$f(x+y) = \Big[f(x+y) - \sum_{k=0}^{m_0} P_k f(x)(y) \Big] + \Big[\sum_{k=0}^{m_0} P_k f(x)(y) \Big] \in \epsilon B_F + A_{m_0}$$

for all $y \in tB_E$. Then, by Lemma 2.1

$$f(x+tB_E) \in C_{\mathcal{I}}(F).$$

Hence $t \le r_{\mathcal{I}}(f;x)$. Thus $r(f;x) \le r_{\mathcal{I}}(f;x)$. □

In [11, Example 3.7] an entire function $f \in \mathcal{H}(\ell_1;\ell_p)$, $1 \le p < \infty$, is constructed such that $r(f;0) = \infty$ but $r_{\mathcal{K}_p}(f;0) = 0$. The proof of part (2) cannot be adapted to surjective ideals that are non-closed as a consequence of Lemma 2.1. Despite this, there are some particular situations in which $r(f;x) = r_{\mathcal{I}}(f;x)$ regardless of the ideal $\mathcal{I}$ as is seen in the next proposition. We need a preliminary lemma.

Given a subset $A \subset E$, the closed absolutely convex hull of A is denoted by $\overline{\Gamma}(A)$.

Lemma 2.3. *Let $\mathcal{I}$ be an operator ideal and F be a finite dimensional Banach space. Then $A \in C_{\mathcal{I}}(F)$ if and only if A is bounded.*

Proof. We only prove the non-trivial implication. Assume that $A \subset F$ is bounded. The operator $T : \ell_1(A) \to F$ given by

$$T((t_x)_{x\in A}) := \sum_{x\in A} t_x x$$

is then well-defined. As F is finite dimensional, T has finite rank. Then T belongs to $\mathcal{I}(\ell_1(A);F)$. Since

$$A \subset \overline{\Gamma}(A) = T(B_{\ell_1(A)})$$

we conclude that $A \in C_{\mathcal{I}}(F)$. □

Proposition 2.4. *Let E and F be Banach spaces and let $\mathcal{I}$ be an operator ideal. If F is finite dimensional, then any entire mapping $f : E \to F$ belongs to $\mathcal{H}_{\mathcal{I}}(E;F)$ and $r(f;x) = r_{\mathcal{I}}(f;x)$ for any $x \in E$.*

Proof. The first assertion follows from the fact that any entire mapping is locally bounded, i.e. for any $x \in E$ there exists a neighborhood V_x of x such that $f(V_x)$ is bounded. Indeed, by Lemma 2.3 $f(V_x) \in C_{\mathcal{I}}(F)$ and then $f \in \mathcal{H}_{\mathcal{I}}(E;F)$. On the other hand, using again the Lemma,

$$\begin{aligned} r(f;x) &= \sup\{t > 0 : f(x + tB_E) \text{ is bounded }\} \\ &= \sup\{t > 0 : f(x + tB_E) \in C_{\mathcal{I}}(F)\} \\ &= r_{\mathcal{I}}(f;x) \end{aligned}$$

□

In [3] the authors constructed the ideal $\mathcal{P}_{\mathcal{I}}$ of $\mathcal{I}$-bounded polynomials. Proposition 2.4 provides a general procedure to construct ideals of holomorphic mappings in the following sense. Let $\mathcal{H}_{\mathcal{I}}$ be the class of all $\mathcal{I}$-bounded holomorphic functions. For Banach spaces E and F, the component $\mathcal{H}_{\mathcal{I}}(E;F)$ satisfies the following conditions:

(i) $\mathcal{H}_{\mathcal{I}}(E;F)$ is a linear subspace of $\mathcal{H}(E;F)$ containing the holomorphic mappings whose range is contained in a finite dimensional subspace.
(ii) The ideal property: if $u \in \mathcal{L}(G;E)$, $f \in \mathcal{H}_{\mathcal{I}}(E;F)$ and $t \in \mathcal{L}(F;H)$, then the composition $t \circ f \circ u$ is in $\mathcal{H}_{\mathcal{I}}(G;H)$.

An operator ideal $[\mathcal{I}, \iota]$ is said to satisfy *Condition* Γ if the closed absolutely convex hull of any $\mathcal{I}-$bounded set is $\mathcal{I}-$bounded. Condition Γ was introduced by the authors in [3], where it was proved that if $\mathcal{I}$ is surjective and satisfies Condition Γ then the ideal of polynomials $\mathcal{P}_{\mathcal{I}}$ coincides with the ideal obtained by composition with $\mathcal{I}$. The authors are unaware of any examples of operator ideals that fail to satisfy Condition Γ.

For the sake of completeness we prove the next proposition (see, e.g, [4, Proposition 3.4]).

Proposition 2.5. *Let $\mathcal{I}$ be an operator ideal that satisfies Condition Γ and let E and F be Banach spaces. If $f \in \mathcal{H}(E;F)$ is locally $\mathcal{I}-$bounded at $x \in E$ then $P_k f(x) \in \mathcal{P}_{\mathcal{I}}({}^kE;F)$ for all k.*

Proof. Let $t_x > 0$ be such that $f(x + t_xB_E) \in C_{\mathcal{I}}(F)$. By Condition Γ, $\overline{\Gamma}(f(x + t_xB_E)) \in C_{\mathcal{I}}(F)$. From [15, Lemma 3.1]

$$P_k f(x)(t_xB_E) \subset \overline{\Gamma}(f(x + t_xB_E))$$

for all k. It follows that $P_k f(x) \in \mathcal{P}_{\mathcal{I}}({}^kE;F)$ for all k. □

Corollary 2.6. *Let $\mathcal{I}$ be an operator ideal that satisfies Condition Γ and let E and F be Banach spaces. If $f \in \mathcal{H}_{\mathcal{I}}(E;F)$ then $P_k f(x) \in \mathcal{P}_{\mathcal{I}}({}^kE;F)$ for all k and all $x \in E$.*

The above result generalizes [9, Proposition 5(c)], which is proved for surjective closed ideals. By [9, Proposition 3(d)] any surjective closed ideal fulfills Condition Γ. However, $\mathcal{K}_p$ is an example of a surjective ideal that fulfills Condition Γ and it is not closed.

Let $(Z_n)_n$ be a sequence of Banach spaces and let $R > 0$. We denote by $\Lambda_R((Z_n)_n)$ the space of all sequences $(z_n)_n \in \Pi_n Z_n$ such that

$$p_r((z_n)_n) := \sum_{n=0}^{\infty} r^n \|z_n\| < \infty,$$

for all $0 < r < R$. It is well known that $\Lambda_R((Z_n)_n)$ is a Fréchet space when endowed with the topology generated by the family of seminorms $\{p_r : 0 < r < R\}$.

Let

$$D_R := \{(z_n)_n \in \Lambda_R((Z_n)_n) : z_n \in R^n B_{Z_n} \text{ for all } n\}.$$

We denote by Z the linear span of the absolutely convex set D_R, and we endow Z with its Minkowski functional given by

$$p_{D_R}((z_n)_n) = \inf\{\lambda > 0 : (z_n)_n \in \lambda D_R\}.$$

Then Z is a Banach space and $Z = \cup_{\lambda>0} \lambda D_R$.

Theorem 2.7. *Let $[\mathcal{I}, \iota]$ be a Banach operator ideal that satisfies Condition Γ and let E and F be Banach spaces. If $f \in \mathcal{H}(E; F)$ is locally $\mathcal{I}-$bounded at $x \in E$ then*

$$r_{\mathcal{I}}(f; x) = \frac{1}{\limsup_n \|P_n f(x)\|_{\mathcal{I}}^{\frac{1}{n}}}. \tag{2.5}$$

Proof. By Proposition 2.5, $P_k f(x) \in \mathcal{P}_{\mathcal{I}}(^k E; F)$ for all k. Write

$$L := \limsup_n \|P_n f(x)\|_{\mathcal{I}}^{\frac{1}{n}}.$$

We start by proving $r_{\mathcal{I}}(f; x) \geq \frac{1}{L}$. Since the result is trivial if $L = \infty$, we may assume $L < \infty$ and take $0 < s < \frac{1}{L}$. Fix r so that $s < r < \frac{1}{L}$. Since $L < \frac{1}{r}$ there exists a positive integer m_0 such that for all $m \geq m_0$, $\|P_m f(x)\|_{\mathcal{I}}^{\frac{1}{m}} < \frac{1}{r}$. Consider

$$c := \max\{r\|P_1 f(x)\|_{\mathcal{I}} + 1, r^2\|P_2 f(x)\|_{\mathcal{I}} + 1, \ldots, r^{m_0}\|P_{m_0} f(x)\|_{\mathcal{I}} + 1\}.$$

Then $\|P_m f(x)\|_{\mathcal{I}} < \frac{c}{r^m}$ for all m. So, if we consider the map $M_s : E \to E$, $M_s(y) := sy$, it follows that $\|P_m f(x) \circ M_s\|_{\mathcal{I}} < c(\frac{s}{r})^m$, for all m. From this inequality, we see that for each m there exist a Banach space Z_m and an operator $T_m \in \mathcal{I}(Z_m; F)$ such that

$$P_m f(x)(sB_E) \subset T_m(B_{Z_m}) \tag{2.6}$$

and

$$\iota(T_m) < c(\frac{s}{r})^m. \tag{2.7}$$

Taking $R = 1$, consider $\Lambda_1((Z_n)_n)$, D_1 and Z as above.

For each n define $S_n : Z \to F$ by

$$S_n((z_j)_j) := T_n(z_n) = T_n \circ \pi_n((z_j)_j),$$

where π_n is the projection on the $n^{th}-$coordinate. By the ideal property, $S_n \in \mathcal{I}(Z;F)$.

Consider now the map $T : Z \to F$ given by

$$T(\lambda(z_n)_n) := \lambda \sum_{n=0}^{\infty} T_n(z_n) = \sum_{n=0}^{\infty} S_n(\lambda(z_j)_j,)$$

for any $\lambda > 0$ and any $(z_n)_n \in D_1$. Then by (2.7)

$$\sum_{n=0}^{\infty} \iota(S_n) \leq \sum_{n=0}^{\infty} \iota(T_n) \leq \sum_{n=0}^{\infty} c(\frac{s}{r})^n < \infty.$$

Since $[\mathcal{I}, \iota]$ is a Banach space, we conclude that $T \in \mathcal{I}(Z;F)$.

On the other hand, by (2.6) we have

$$\|P_m f(x) \circ M_s\| \leq \|T_m\| \leq \iota(T_m) < c\Big(\frac{s}{r}\Big)^m.$$

Then,

$$\|P_m f(x)(sy)\| < c\Big(\frac{s}{r}\Big)^m$$

for all $y \in B_E$. Hence the Taylor series $\sum_{m=0}^{\infty} P_m f(x)$ converges uniformly on sB_E and

$$\begin{aligned} f(x + sB_E) &= \sum_{m=0}^{\infty} P_m f(x)(sB_E) \\ &\subset \sum_{m=0}^{\infty} T_m(B_{Z_m}) \\ &\subset \sum_{m=0}^{\infty} T_m \circ \pi_m(B_Z) \\ &= T(B_Z). \end{aligned}$$

Thus, $s \leq r_{\mathcal{I}}(f;x)$. This shows that

$$r_{\mathcal{I}}(f;x) \geq \frac{1}{\limsup_n \|P_n f(x)\|_{\mathcal{I}}^{\frac{1}{n}}}.$$

We now prove the reverse inequality, $r_{\mathcal{I}}(f;x) \leq \frac{1}{L}$. Let $0 < \lambda < r_{\mathcal{I}}(f;x)$. Then $f(x + \lambda B_E) \in C_{\mathcal{I}}(F)$. Condition Γ implies now that

$$\overline{\Gamma}(f(x + \lambda B_E)) \in C_{\mathcal{I}}(F).$$

Therefore, there exists a Banach space Z and an operator $T \in \mathcal{I}(Z;F)$ such that

$$\overline{\Gamma}(f(x + \lambda B_E)) \subset T(B_Z).$$

By [15, Lemma 3.1], $P_k f(x)(\lambda B_E) \subset \overline{\Gamma}(f(x+\lambda B_E))$ for all k. Then, $\lambda^k P_k f(x)(B_E) \subset T(B_Z)$. Hence $\lambda^k \|P_k f(x)\|_{\mathcal{I}} \leq \iota(T)$. Thus, $\|P_k f(x)\|_{\mathcal{I}}^{\frac{1}{k}} \leq \frac{\iota(T)^{\frac{1}{k}}}{\lambda}$. Since this is valid for all k it follows that

$$\limsup_k \|P_k f(x)\|_{\mathcal{I}}^{\frac{1}{k}} \leq \frac{1}{\lambda}.$$

This proves that

$$r_{\mathcal{I}}(f;x) \leq \frac{1}{\limsup_n \|P_n f(x)\|_{\mathcal{I}}^{\frac{1}{n}}}.$$

□

Whenever $\mathcal{I}$ is taken as the class $\mathcal{L}$ of all continuous linear operators, then $C_{\mathcal{L}}(F)$ is the collection of all bounded sets in a Banach space F. In that case the radius of $\mathcal{L}$-boundedness is the radius of boundedness and Theorem 2.7 reduces to Hadamard's formula. So, the radius of $\mathcal{I}$-boundedness is a generalization of the radius of boundedness for arbitrary ideals $\mathcal{I}$. In [11] a particular case of $r_{\mathcal{I}}$ has been independently introduced for the ideal $\mathcal{I} = \mathcal{K}_p$ of $p-$compact operators. The radius of $\mathcal{K}_p$-boundedness is defined as

$$r_{\mathcal{K}_p}(f;x) = \frac{1}{\limsup_n \|P_n f(x)\|_{\mathcal{K}_p}^{\frac{1}{n}}}$$

and called the $p-$compact radius of convergence. Since $\mathcal{K}_p$ satisfies Condition Γ, Theorem 2.7 shows that $r_{\mathcal{K}_p}$ as defined is a particular case of our general concept.

Theorem 2.7 allows us to consider the radius of $\mathcal{I}$-boundedness as a radius of convergence in the following sense. Given a Banach space E, define the map $M_s : E \to E,\ M_s(y) := sy$.

Corollary 2.8. *Let $[\mathcal{I}, \iota]$ be a Banach operator ideal that satisfies Condition Γ and let E and F be Banach spaces. If $f \in \mathcal{H}(E;F)$ is locally $\mathcal{I}-$bounded at $x \in E$ then*

$$\begin{aligned} r_{\mathcal{I}}(f;x) &= \sup\Big\{r > 0 : \sum_{n=0}^{\infty} \|P_n f(x)\|_{\mathcal{I}} r^n < \infty\Big\} \\ &= \sup\Big\{r > 0 : \sum_{n=0}^{\infty} \|P_n f(x) \circ M_r\|_{\mathcal{I}} < \infty\Big\} \end{aligned}$$

Proof. The first equality follows from Theorem 2.7 and the classical Hadamard's formula. The second one follows from the straightforward fact that $\|P_n f(x) \circ M_r\|_{\mathcal{I}} = r^n \|P_n f(x)\|_{\mathcal{I}}$. □

The ideas of the proof of Theorem 2.7 permit us to prove the following converse to Proposition 2.5.

Theorem 2.9. *Let $[\mathcal{I}, \iota]$ be a Banach ideal of operators. Let E and F be Banach spaces, $f \in \mathcal{H}(E;F)$ and $x \in E$. If $P_m f(x) \in \mathcal{P}_{\mathcal{I}}(^m E;F)$ for all m and there exists $R > 0$ such that $\sum_{n=0}^{\infty} \|P_n f(x)\|_{\mathcal{I}} r^n < \infty$ for all $0 < r < R$ then f is locally $\mathcal{I}-$bounded at x.*

Proof. We will proceed as in the proof of Theorem 2.7. Consider $0 < r < R$ such that $f(x+y) = \sum_{m=0}^{\infty} P_m f(x)(y)$ uniformly for $y \in rB_E$. We may assume without loss of generality that $0 < r < 1$. Let $\epsilon > 0$. Since $P_m f(x) \in \mathcal{P}_{\mathcal{I}}(^m E;F)$, there exist a Banach space Z_m and an operator $T_m \in \mathcal{I}(Z_m;F)$ such that

$$P_m f(x)(B_E) \subset T_m(B_{Z_m}) \tag{2.8}$$

and

$$\iota(T_m) < \|P_m f(x)\|_{\mathcal{I}} + \epsilon. \tag{2.9}$$

As above, consider $\Lambda_1((Z_n)_n)$, D_1 and Z, and for each n define $S_n : Z \to F$ by

$$S_n((z_j)_j) := r^n T_n(z_n) = r^n T_n \circ \pi_n((z_j)_j),$$

(which differs slightly from the analogous operator in the proof of Theorem 2.7). This enables us to define $T : Z \to F$ as

$$T(\lambda(z_j)_j) := \sum_{n=0}^{\infty} S_n(\lambda(z_j)_j) = \lambda \sum_{n=0}^{\infty} r^n T_n(z_n)$$

and to conclude that $T \in \mathcal{I}(Z;F)$. Indeed,

$$\sum_{n=0}^{\infty} \iota(S_n) \leq \sum_{n=0}^{\infty} r^n \iota(T_n) \leq \sum_{n=0}^{\infty} (r^n \|P_n f(x)\|_{\mathcal{I}} + \epsilon r^n) < \infty,$$

which proves that $T := \sum_{n=0}^{\infty} S_n \in \mathcal{I}(Z;F)$.

To conclude the argument, let us prove that $f(x + rB_E) \subset T(B_Z)$. By (2.8) we have

$$f(x+ry) = \sum_{m=0}^{\infty} P_m f(x)(ry) \in \sum_{m=0}^{\infty} r^m T_m(B_{Z_m}) \subset T(D_1) \subset T(B_Z),$$

for all $y \in B_E$. We have proved that $f(x + rB_E) \in C_{\mathcal{I}}(F)$ and therefore f is locally $\mathcal{I}-$bounded at x. □

Corollary 2.10. *Let $[\mathcal{I}, \iota]$ be a Banach ideal of operators. Let E and F be Banach spaces and $f \in \mathcal{H}(E;F)$. If for every $x \in E$, $P_m f(x) \in \mathcal{P}_{\mathcal{I}}(^m E;F)$ for all m and there exists $R > 0$ such that $\sum_{n=0}^{\infty} \|P_n f(x)\|_{\mathcal{I}} r^n < \infty$ for all $0 < r < R$, then $f \in \mathcal{H}_{\mathcal{I}}(E;F)$.*

Let us apply Theorem 2.9 to prove that, given a Banach ideal of operators $[\mathcal{I}, \iota]$, any holomorphic mapping defined on a finite dimensional domain is $\mathcal{I}$-bounded.

Example 2.11. Let $[\mathcal{I}, \iota]$ be a Banach ideal of operators that satisfies Condition Γ. Let E and F be Banach spaces. If E is finite dimensional then $\mathcal{H}(E;F) = \mathcal{H}_{\mathcal{I}}(E;F)$. Moreover, $r_{\mathcal{I}}(f;0) = \infty$ for all $f \in \mathcal{H}(E;F)$.

Proof. For simplicity let us prove the claim for $E = \mathbb{C}$. Let $f : \mathbb{C} \to F$ be an entire mapping. Then f can be written as $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in \mathbb{C}$, where $a_n \in F$ for all n. The Taylor polynomials at 0 are $P_n f(0)(z) = a_n z^n$ and so they have finite rank. Then $P_n f(0) \in \mathcal{P}_{\mathcal{I}}(^n\mathbb{C};F)$. Let us prove that $\|P_n f(0)\|_{\mathcal{I}} \leq |a_n|$ for all n. Consider the linear operator $T_n : \mathbb{C} \to F$ given by $T_n(z) = a_n z$. As T_n has finite rank, $T_n \in \mathcal{I}(\mathbb{C};F)$ and

$$\iota(T_n) \leq |a_n|.$$

Since $P_n f(0)(B_{\mathbb{C}}) \subset T_n(B_{\mathbb{C}})$ we have that

$$\|P_n f(0)\|_{\mathcal{I}} \leq \iota(T_n) \leq |a_n|.$$

Then

$$\sum_{n=0}^{\infty} \|P_n f(0)\|_{\mathcal{I}} \leq \sum_{n=0}^{\infty} |a_n| < \infty.$$

Applying Theorem 2.9 we get that f is locally $\mathcal{I}-$bounded at 0 and by Theorem 2.7

$$r_{\mathcal{I}}(f;0) \geq \frac{1}{\limsup_n \|P_n f(0)\|_{\mathcal{I}}^{\frac{1}{n}}} \geq \frac{1}{\limsup_n |a_n|^{\frac{1}{n}}} = \infty.$$

We conclude that $r_{\mathcal{I}}(f;0) = \infty$ and so $f \in \mathcal{H}_{\mathcal{I}}(\mathbb{C};F)$. □

As a particular case of Example 2.11 we get [1, Proposition 2], where it is shown with different techniques that every entire mapping $f : E \longrightarrow F$ is p-compact for each $1 \leq p \leq \infty$ whenever E is finite dimensional.

For $[\mathcal{I}, \iota]$ satisfying Condition Γ, the description of the radius of $\mathcal{I}$-boundedness given in Corollary 2.8 can be used to endow the space $\mathcal{H}_{\mathcal{I}}(E;F)$ with a Hausdorff locally convex topology with "very good" properties. Take a strictly decreasing null sequence $(R_n)_n$ and consider the Fréchet space $\Lambda_{R_n}((\mathcal{P}_{\mathcal{I}}(^mE;F))_m)$ of all sequences $(P_m)_m \in \Pi_{m=0}^{\infty}\mathcal{P}_{\mathcal{I}}(^mE;F)$ such that $p_r((P_m)_m) := \sum_{m=0}^{\infty} \|P_m\|_{\mathcal{I}} r^m < \infty$ for all $0 \leq r < R_n$. The topology on $\Lambda_{R_n}((\mathcal{P}_{\mathcal{I}}(^mE;F))_m)$ is generated by the family of seminorms $\{p_r : 0 < r < R_n\}$.

To simplify notation, we will write $\Lambda_{R_n}(E;F) := \Lambda_{R_n}((\mathcal{P}_{\mathcal{I}}(^mE;F))_m)$. The topology of $\Lambda_{R_n}(E;F)$ restricted to $\mathcal{P}_{\mathcal{I}}(^mE;F)$ coincides with the topology induced by the norm $\|\cdot\|_{\mathcal{I}}$, and the sequence of Fréchet spaces $(\Lambda_{R_n}(E;F))_n$ is increasing. Then the map that takes each $f \in \mathcal{H}_{\mathcal{I}}(E;F)$ to the sequence $(P_m f(0))_m$ is an injection of $\mathcal{H}_{\mathcal{I}}(E;F)$ into $\cup_n \Lambda_{R_n}(E;F)$. Moreover, for R_1 small enough we can consider that $\mathcal{H}_{\mathcal{I}}(E;F) \cap \Lambda_{R_n}(E;F) \neq \emptyset$, for all $n \geq 1$. Therefore, we can consider the strict inductive limit of the sequence of Fréchet spaces $(\Lambda_{R_n}(E;F))_n$, which we denote $\Lambda_{\mathcal{I}}(E;F)$. That is, $\Lambda_{\mathcal{I}}(E;F) := \cup_n \Lambda_{R_n}(E;F)$ and is endowed with the finest locally convex topology with respect to which each canonical injection $\Lambda_{R_n} \hookrightarrow \Lambda_{\mathcal{I}}$ is continuous. It is easy to check that the space $\Lambda_{\mathcal{I}}$ does not depend on the sequence $(R_n)_n$. We denote this strict inductive limit topology on $\Lambda_{\mathcal{I}}(E;F)$ by $\tau_{\mathcal{I}}$. In other words, $(\Lambda_{\mathcal{I}}(E;F), \tau_{\mathcal{I}})$ is an LF-space. We recall that the strict inductive limit topology induces the original topology on each of the component spaces $\Lambda_{R_n}(E;F)$. In particular $\tau_{\mathcal{I}}$ induces the norm $\|.\|_{\mathcal{I}}$ on each $\mathcal{P}_{\mathcal{I}}(^mE;F)$. We refer to [13] for the basic properties of strict inductive limits. Thus, we can now endow $\mathcal{H}_{\mathcal{I}}(E;F)$ with the restriction of $\tau_{\mathcal{I}}$.

Example 2.12. (i) Let $[\mathcal{I}, \iota]$ be an arbitrary Banach ideal. By Proposition 2.4, $\mathcal{P}_{\mathcal{I}}(^mE) = \mathcal{P}(^mE)$ for all m. In [8] it is proved that $\Lambda_R((\mathcal{P}(^mE))_m)$ is topologically isomorphic to the space $\mathcal{H}_b(R \overset{\circ}{B}_E)$ of all holomorphic functions of bounded type, that is, the space of all holomorphic functions on $R \overset{\circ}{B}_E$ that are bounded on $r \overset{\circ}{B}_E$ for every $0 < r < R$. (Here $\overset{\circ}{B}_E$ denotes the open unit ball of E.)

(ii) The vector case follows when $\mathcal{I}$ is the class of all bounded linear operators. In this case, $\Lambda_R((\mathcal{P}(^mE;F))_m)$ is topologically isomorphic to the space of all holomorphic mappings of bounded type, $\mathcal{H}_b(R \overset{\circ}{B}_E;F)$. We also have $\mathcal{H}_{\mathcal{I}}(E;F) = \mathcal{H}(E;F)$ whereas $\Lambda_{\mathcal{I}}(E;F) = \cup_n \mathcal{H}_b(R_n \overset{\circ}{B}_E;F)$. A base of neighborhoods of 0 for $\tau_{\mathcal{I}}$ in $\mathcal{H}(E;F)$ is given by all absolutely convex sets

U in $\mathcal{H}(E;F)$ for which there exist sequences $(r_n)_n$ and $(\epsilon_n)_n$ of positive numbers decreasing to 0 such that $\cup_n\{f \in \mathcal{H}(E;F) : \sup_{x\in r_n B_E} \|f(x)\| < \epsilon_n\} \subset U$.

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