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# GENERIC CONTRACTIVITY OF NONEXPANSIVE MAPPINGS WITH UNBOUNDED DOMAINS

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*Dedicated to Professor Sompong Dhompongsa on his 65th birthday*

Abstract. Given a nonempty, bounded, closed and convex subset $K$ of a hyperbolic complete metric space, we studied in previous papers of ours the class of nonexpansive self-mappings of $K$ endowed with a natural metric. Using the Baire category approach and the notion of porosity, we showed that most elements of this class are contractive. In the present paper we prove a variant of this result for unbounded sets. Namely, we show that most nonexpansive mappings are contractive on all bounded subsets.

## 1. Introduction and preliminaries

For more than fifty years now, there has been considerable interest in the fixed point theory of nonexpansive mappings in metric and Banach spaces. See, for example, the papers and books by de Blasi and Myjak [2, 3], Goebel and Kirk [4], Goebel and Reich [5] and Kirk [6], as well as the references mentioned therein. This interest originates in the classical Banach theorem [1] regarding the existence of a unique fixed point for a strict contraction. Since that seminal result, many developments have taken place in this area. We mention, for instance, existence results for fixed points of nonexpansive mappings which are not strictly contractive [4, 5]. Such results were obtained for general nonexpansive mappings in special Banach spaces, while for self-mappings of general complete metric spaces existence results were established for, the so-called, contractive mappings [9]. For general nonexpansive mappings in general Banach spaces the existence of a unique fixed point was established in the generic sense, using the Baire category approach [2, 3, 13]. More precisely, in these papers the space $\mathcal{A}$ of all nonexpansive self-mappings of a closed and convex set $K$ in a Banach space was endowed with the natural metric of uniform convergence on bounded subsets, and it was shown that there exists a subset $\mathcal{A}' \subset \mathcal{A}$, which is a countable intersection of open and everywhere dense subsets of $\mathcal{A}$, such that every mapping in $\mathcal{A}'$ has a unique fixed point. Note that in [2, 3] the set $K$ was assumed to be bounded, while in [13] this assumption was removed.

Now let $(X, \rho)$ be a metric space and let $R^1$ denote the real line. We say that a mapping $c : R^1 \to X$ is a metric embedding of $R^1$ into $X$ if $\rho(c(s), c(t)) = |s - t|$ for all real $s$ and $t$. The image of $R^1$ under a metric embedding will be called a

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metric line. The image of a real interval $[a,b] = \{t \in R^1 : a \le t \le b\}$ under such a mapping will be called a metric segment.

Assume that $(X,\rho)$ contains a family $M$ of metric lines such that for each pair of distinct points $x$ and $y$ in $X$, there is a unique metric line in $M$ which passes through $x$ and $y$. This metric line determines a unique metric segment joining $x$ and $y$. We denote this segment by $[x,y]$. For each $0 \le t \le 1$, there is a unique point $z$ in $[x,y]$ such that

$$\rho(x,z) = t\rho(x,y) \text{ and } \rho(z,y) = (1-t)\rho(x,y).$$

This point will be denoted by $(1-t)x \oplus ty$. We will say that $X$, or more precisely $(X,\rho,M)$, is a *hyperbolic space* if

$$\rho\Big(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z\Big) \le \frac{1}{2}\rho(y,z)$$

for all $x, y$ and $z$ in $X$. An equivalent requirement is that

$$\rho\Big(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}w \oplus \frac{1}{2}z\Big) \le \frac{1}{2}(\rho(x,w) + \rho(y,z))$$

for all $x, y, z$ and $w$ in $X$. A set $K \subset X$ is called $\rho$-convex if $[x,y] \subset K$ for all $x$ and $y$ in $K$.

It is clear that all normed linear spaces are hyperbolic in this sense. A discussion of more examples of hyperbolic spaces and, in particular, of the Hilbert ball can be found, for example, in [5, 11, 12, 17].

Let $(X,\rho,M)$ be a complete hyperbolic space and let $K$ be a nonempty, closed and $\rho$-convex subset of $X$.

For each $x \in K$ and each $r > 0$, set

$$B(x,r) = \{y \in K : \ \rho(x,y) \le r\}.$$

Denote by $\mathcal{A}$ the set of all mappings $A : K \to K$ such that

$$\rho(Ax, Ay) \le \rho(x,y) \text{ for all } x, y \in K. \tag{1.1}$$

Fix some $\theta \in K$.

We equip the set $\mathcal{A}$ with the uniformity determined by the base

$$\mathcal{U}(n) = \{(A,B) \in \mathcal{A} \times \mathcal{A} : \ \rho(Ax, Bx) \le n^{-1} \text{ for all } x \in B(\theta, n)\}, \tag{1.2}$$

where $n$ is a natural number. Clearly, the uniform space $\mathcal{A}$ is metrizable and complete.

Let $A \in \mathcal{A}$. The mapping $A$ is called *contractive* if there exists a decreasing function $\phi : [0,\infty) \to [0,1]$ such that

$$\phi(t) < 1 \text{ for all } t > 0$$

and

$$\rho(Ax, Ay) \le \phi(\rho(x,y))\rho(x,y) \text{ for all } x, y \in K.$$

According to the Rakotch theorem [9], every contractive mapping possesses a unique fixed point.

In previous papers of ours [14, 15] we studied the space $\mathcal{A}$ in the case where the set $K$ is bounded. Using the Baire category approach and the notion of porosity, we showed that most elements of the space $\mathcal{A}$ are contractive. If the set $K$ is unbounded, it is known [18] that our results no longer hold. Nevertheless, in the present paper we prove a variant of our results for unbounded sets by showing that most mappings in this class are contractive on all *bounded* subsets.

Suppose that $A \in \mathcal{A}$ and $\epsilon \geq 0$. A point $x \in K$ is called an $\epsilon$-approximate fixed point of $A$ if $\rho(x, Ax) \leq \epsilon$ [7, 8, 10, 16, 17].

We say that $A$ has the bounded approximate fixed point property (or the BAFP property, for short) if there is a nonempty bounded set $K_0 \subset K$ such that for each $\epsilon > 0$, the mapping $A$ has an $\epsilon$-approximate fixed point in $K_0$.

In [16] we proved the following result.

**Proposition 1.1.** *Assume that $A \in \mathcal{A}$ and that $K_0 \subset K$ is a nonempty, bounded, closed and $\rho$-convex subset of $K$ such that*

$$A(K_0) \subset K_0. \tag{1.3}$$

*Then $A$ has the BAFP property.*

Proposition 1.1 immediately implies the following result.

**Proposition 1.2.** *Assume that $K$ is bounded. Then any $A \in \mathcal{A}$ has the BAFP property.*

Obviously, Proposition 1.2 no longer holds if the set $K$ is unbounded. For example, if $K$ is a Banach space and $A$ is a translation operator, then $A$ does not possess the BAFP propery. Nevertheless, the following theorem is true [16].

**Theorem 1.3.** *There exists an open and everywhere dense set $\mathcal{F} \subset \mathcal{A}$ such that each $A \in \mathcal{F}$ has the BAFP property.*

Theorem 1.3 is the main result of [16], but actually in [16] we prove the following stronger result, which will be used in the sequel.

**Theorem 1.4.** *There exists an open and everywhere dense set $\mathcal{F} \subset \mathcal{A}$ such that for each $A \in \mathcal{F}$, there exists a nonempty, bounded, closed and $\rho$-convex set $K_A \subset K$ such that $A(K_A) \subset K_A$.*

Now we are ready to state our main result. Its proof is given in Section 2.

**Theorem 1.5.** *There exists a set $\mathcal{F}_* \subset \mathcal{A}$ which is a countable intersection of open and everywhere dense sets in $\mathcal{A}$ such that for each $A \in \mathcal{F}_*$, the following two properties hold:*

(i) *there exists a unique point $x_A \in K$ such that $Ax_A = x_A$;*

(ii) *for each $r > 0$, there exists a decreasing function $\phi : [0, \infty) \to [0, 1]$ such that*

$$\phi(t) < 1 \text{ for all } t > 0$$

*and*

$$\rho(Ax, Ay) \leq \phi(\rho(x, y))\rho(x, y) \text{ for all } x, y \in B(x_A, r).$$

## 2. Proof of Theorem 1.5

We may assume without any loss of generality that $K$ is not a singleton.

By Theorem 1.4, there exists an open and everywhere dense set $\mathcal{F}_0 \subset \mathcal{A}$ such that the following property holds:

(P1) for each $A \in \mathcal{F}_0$, there exists a nonempty, bounded, closed and $\rho$-convex set $K_A \subset K$ such that $A(K_A) \subset K_A$.

Choose

$$\kappa \in (0,1)$$

for which there exist two points $u_1, u_2 \in K$ such that $\rho(u_1, u_2) \geq \kappa$.

For each natural number $n$, denote by $\mathcal{F}_n$ the set of all $A \in \mathcal{A}$ such that

$$\sup\{\rho(Ax, Ay)[\rho(x,y)]^{-1} : \ x, y \in B(\theta, n), \ \rho(x,y) \geq n^{-1}\kappa\} < 1. \tag{2.1}$$

**Lemma 2.1.** *For each natural number $n$, the set $\mathcal{F}_n$ contains an open and everywhere dense set.*

*Proof.* Let $n$ be a natural number, $A \in \mathcal{A}$ and let $k$ be a natural number. In order to prove the lemma, it is sufficient to show that there exist $\tilde{A} \in \mathcal{A}$ and a natural number $p$ such that

$$(A, \tilde{A}) \in \mathcal{U}(k) \tag{2.2}$$

and

$$\{B \in \mathcal{A} : \ (B, \tilde{A}) \in \mathcal{U}(p)\} \subset \mathcal{F}_n. \tag{2.3}$$

To this end, choose $\gamma \in (0,1)$ such that

$$\gamma(k + \rho(A\theta, \theta)) < (2k)^{-1} \tag{2.4}$$

and a natural number $p$ such that

$$p > n, \ 4p^{-1}n < \kappa\gamma. \tag{2.5}$$

Define

$$\tilde{A}x = (1-\gamma)Ax \oplus \gamma\theta, \ x \in K. \tag{2.6}$$

By (2.6), we have for all $x, y \in K$,

$$\begin{aligned} \rho(\tilde{A}x, \tilde{A}y) &= \rho((1-\gamma)Ax \oplus \gamma\theta, (1-\gamma)Ay \oplus \gamma\theta) \\ &\leq (1-\gamma)\rho(Ax, Ay) \leq (1-\gamma)\rho(x,y). \end{aligned} \tag{2.7}$$

By (2.4) and (2.6), for all $x \in B(\theta, k)$, we have

$$\begin{aligned} \rho(\tilde{A}x, Ax) &= \rho((1-\gamma)Ax \oplus \gamma\theta, Ax) \leq \gamma\rho(Ax, \theta) \leq \gamma\rho(Ax, A\theta) + \gamma\rho(A\theta, \theta) \\ &\leq \gamma\rho(x, \theta) + \gamma\rho(A\theta, \theta) \leq \gamma k + \gamma\rho(A\theta, \theta) < (2k)^{-1}. \end{aligned}$$

Thus (2.2) holds.

Assume now that $B \in \mathcal{A}$ satisfies

$$(B, \tilde{A}) \in \mathcal{U}(p). \tag{2.8}$$

Let

$$x, y \in B(\theta, n) \text{ with } \rho(x,y) \geq n^{-1}\kappa. \tag{2.9}$$

It follows from (2.8), (2.9), (2.5) and (2.7) that

$$\begin{aligned} \rho(Bx, By) &\leq \rho(Bx, \tilde{A}x) + \rho(\tilde{A}x, \tilde{A}y) + \rho(\tilde{A}y, By) \\ &< p^{-1} + \rho(\tilde{A}x, \tilde{A}y) + p^{-1} \leq 2p^{-1} + (1-\gamma)\rho(x,y). \end{aligned}$$

When combined with (2.9) and (2.5), this implies that

$$\rho(Bx, By)[\rho(x, y)]^{-1} \le 2p^{-1}n\kappa^{-1} + (1 - \gamma) < 1 - \gamma/2$$

and

$$\sup\{\rho(Bx, By)[\rho(x, y)]^{-1} : \ x, y \in B(\theta, n), \ \rho(x, y) \ge n^{-1}\kappa\} < 1 - \gamma/2.$$

Thus $B \in \mathcal{F}_n$ and (2.3) holds. Lemma 2.1 is proved. □

*Completion of the proof of Theorem 1.5*

By Lemma 2.1, for any natural number $n$, there exists an open and everywhere dense set $\mathcal{F}'_n$ in $\mathcal{A}$ such that

$$\mathcal{F}'_n \subset \mathcal{F}_n. \tag{2.10}$$

Set

$$\mathcal{F}_* = \mathcal{F}_0 \cap (\cap_{n=1}^{\infty} \mathcal{F}'_n). \tag{2.11}$$

Let

$$A \in \mathcal{F}_*.$$

In order to complete the proof of Theorem 1.5, it is sufficient to show that properties (i) and (ii) hold.

Since $A \in \mathcal{F}_0$, it follows from (P1) that there exists a nonempty, bounded, closed and $\rho$-convex set $K_A \subset K$ such that

$$A(K_A) \subset K_A. \tag{2.12}$$

Let $n$ be a natural number such that

$$K_A \subset B(\theta, n). \tag{2.13}$$

Since $A \in \mathcal{F}_n$ (see (2.1)), it follows from (2.13) that

$$\sup\{\rho(Ax, Ay)[\rho(x, y)]^{-1} : \ x, y \in K_A, \ \rho(x, y) \ge n^{-1}\kappa\}$$
$$\le \sup\{\rho(Ax, Ay)[\rho(x, y)]^{-1} : \ x, y \in B(\theta, n), \ \rho(x, y) \ge n^{-1}\kappa\} < 1.$$

Since the above relation holds for any natural number $n$ satisfying (2.13), we conclude that there exists a decreasing function $\phi : [0, \infty) \to [0, 1]$ such that

$$\phi(t) < 1 \text{ for all } t > 0,$$
$$\rho(Ax, Ay) \le \rho(x, y)\phi(\rho(x, y)) \text{ for all } x, y \in K_A.$$

By [9], there is $x_A \in K_A$ such that

$$Ax_A = x_A. \tag{2.14}$$

Assume that $z \in K$ satisfies

$$Az = z. \tag{2.15}$$

We claim that $z = x_A$. Assume the contrary. Then there exists a natural number $n$ such that

$$K_A \subset B(\theta, n), \ z \in B(\theta, n), \ n^{-1}\kappa < \rho(x_A, z). \tag{2.16}$$

Since $A \in \mathcal{F}_n$ (see (2.1)), it follows from (2.14)–(2.16) that

$$1 > \sup\{\rho(Ax, Ay)[\rho(x, y)]^{-1} : \ x, y \in B(\theta, n), \ \rho(x, y) \ge n^{-1}\kappa\}$$

$$\geq \rho(Ax_A, Az)[\rho(x_A, z)]^{-1} = 1,$$

a contradiction. The contradiction we have reached proves that $z = x_A$ and property (i) holds.

Now we show that property (ii) holds.

Let $r > 0$. Since $Ax_A = x_A$, we have

$$A(B(x_A, r)) \subset B(x_A, r). \tag{2.17}$$

Let a natural number $n$ satisfy

$$B(x_A, r) \subset B(\theta, n). \tag{2.18}$$

The inclusion $A \in \mathcal{F}_n$ and (2.18) imply that

$$\begin{aligned} 1 &> \sup\{\rho(Ax, Ay)[\rho(x, y)]^{-1} : \ x, y \in B(\theta, n), \ \rho(x, y) \geq n^{-1}\kappa\} \\ &\geq \sup\{\rho(Ax, Ay)[\rho(x, y)]^{-1} : \ x, y \in B(x_A, r), \ \rho(x, y) \geq n^{-1}\kappa\}. \end{aligned}$$

Since the above relations hold for any natural number $n$ satisfying (2.18), we conclude that there exists a decreasing function $\phi : [0, \infty) \to [0, 1]$ as in property (ii). This completes the proof of Theorem 1.5.

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