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THE SP-ITERATION PROCESS FOR NONEXPANSIVE MAPPINGS IN CAT(κ) SPACES

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Abstract. We establish Δ-convergence results of a sequence generated by the SP-iteration process for nonexpansive mappings in complete CAT(κ) spaces. The main results improve and extend some others appeared in the literature.

1. Introduction

Let K be a nonempty subset of a metric space (X, d) and $T : K \to K$ be a mapping. Then T is nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$. The fixed points set of T, denoted by $F(T)$, is the set $\{x \in K : x = Tx\}$. The fixed point theory in CAT(0) spaces for nonexpansive mappings was firstly studied by Kirk [10]. In [10], Kirk also proved the existence of fixed points for nonexpansive mappings in a geodesic space of bounded curvature called a CAT(κ) space (see Section 2 for a definition). Since then there have been many researches concerning the existence and the convergence of fixed points for nonlinear mappings in such spaces, see for examples, [1, 3, 4, 7, 8, 9, 12, 15, 16, 17, 19, 20, 21].

Let us recall some effective iteration processes for solving a fixed point problem in geodesic metric spaces. Mann iteration process was defined by $x_0 \in K$ and

$$x_{n+1} = \alpha_n T x_n \oplus (1 - \alpha_n) x_n, \quad n \geq 0, \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. The Δ-convergence in the sense of Lim [13] for (1.1) was investigated by Dhompongsa and Panyanak [5] (see also [6, 11]) in CAT(0) spaces and subsequently by He et al. [8] in CAT(κ) spaces.

Ishikawa iteration process was defined by $x_0 \in K$ and

$$\begin{aligned} y_n &= \beta_n T x_n \oplus (1 - \beta_n) x_n, \\ x_{n+1} &= \alpha_n T y_n \oplus (1 - \alpha_n) x_n, \quad n \geq 0, \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. Its convergence was discussed in [5, 16] for CAT(0) spaces and in [9] for CAT(κ) spaces.

Recently, Phuengrattana and Suantai [18] defined the SP-iteration as follows: $x_0 \in K$ and

$$\begin{aligned} z_n &= \gamma_n T x_n \oplus (1 - \gamma_n) x_n, \\ y_n &= \beta_n T z_n \oplus (1 - \beta_n) z_n, \\ x_{n+1} &= \alpha_n T y_n \oplus (1 - \alpha_n) y_n, \quad n \geq 0, \end{aligned} \tag{1.3}$$

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where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$. It was significantly shown in [18] that the convergence rate of (1.3) is better than those of Mann and Ishikawa for continuous functions. Very recently, the convergence theorem of (1.3) was subsequently established by Şahin and Başarır [21] in CAT(0) spaces.

In this paper, motivated by Dhompongsa and Panyanak [5], He et al. [8], Phuengrattana and Suantai [18] and Jun [9], we focus in establishing Δ-convergence theorem for the SP-iteration in complete CAT(κ) spaces with $\kappa \geq 0$.

2. Preliminaries and lemmas

In this section, we provide some basic concepts, definitions and lemmas which will be used in the sequel and can be found in [2].

Let (X,d) be a metric space and $x,y \in X$ with $d(x,y) = l$. A geodesic path from x to y is an isometry $c : [0,l] \to X$ such that $c(0) = x$, $c(l) = y$. The image of a geodesic path is called geodesic segment. The space (X,d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is a uniquely geodesic space if every two points of X are jointed by only one geodesic segment. We write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that $d(x,z) = td(x,y)$ and $d(y,z) = (1-t)d(x,y)$ for $t \in [0,1]$. A subset E of X is said to be convex if E includes every geodesic segment joining any two of its points.

Let C be a positive number. A metric space (X,d) is called a C-geodesic space if any two points of X with the distance less than C are joined by a geodesic. If this holds in a convex set E, then E is said to be C-convex. For a constant κ, we denote M_κ by the 2-dimensional, complete, simply connected spaces of curvature κ.

In what follows, we assume that $\kappa \geq 0$ and define the diameter D_κ of M_κ by $D_\kappa = \frac{\pi}{\sqrt{\kappa}}$ for $\kappa > 0$ and $D_\kappa = \infty$ for $\kappa = 0$. It is known that any ball in X with radius less than $D_\kappa/2$ is convex [2]. A geodesic triangle $\Delta(x,y,z)$ in the metric space (X,d) consists of three points x,y,z in X (the vertices of Δ) and three geodesic segments between each pair of vertices. For $\Delta(x,y,z)$ in a geodesic space X satisfying

$$d(x,y) + d(y,z) + d(z,x) < 2D_\kappa,$$

there exist points $\bar{x},\bar{y},\bar{z} \in M_\kappa$ such that $d(x,y) = d_\kappa(\bar{x},\bar{y})$, $d(y,z) = d_\kappa(\bar{y},\bar{z})$ and $d(z,x) = d_\kappa(\bar{z},\bar{x})$ where d_κ is the metric of M_κ. We call the triangle having vertices $\bar{x},\bar{y},\bar{z} \in M_\kappa$ a comparison triangle of $\Delta(x,y,z)$. A geodesic triangle $\Delta(x,y,z)$ in X with $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$ is said to satisfy the CAT(κ) inequality if for any $p,q \in \Delta(x,y,z)$ and for their comparison points $\bar{p},\bar{q} \in \bar{\Delta}(\bar{x},\bar{y},\bar{z})$, then $d(p,q) \leq d_\kappa(\bar{p},\bar{q})$.

Definition 2.1. A metric space (X,d) is called a CAT(κ) space if it is D_κ-geodesic and any geodesic triangle $\Delta(x,y,z)$ in X with $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$ satisfies the CAT(κ) inequality.

Since the results in CAT(κ) spaces can be deduced from those in CAT(1) spaces, we now sufficiently state lemmas on CAT(1) spaces.

Lemma 2.2 ([2])**.** *Let (X,d) be a CAT(1) space and let F be a closed and π-convex subset of X. Then for each point $x \in X$ such that $d(x,F) < \pi/2$, there exists a unique point $y \in F$ such that $d(x,y) = d(x,F)$.*

Lemma 2.3 ([17]). *For a positive number C with $C \leq \pi/2$, let (X, d) be a CAT(1) space and let $p, x, y \in X$ such that $d(p, x) \leq C$, $d(p, y) \leq C$ and $d(x, y) \leq C$. Then for any $t \in [0, 1]$,*

$$d\big((1-t)p \oplus tx, (1-t)p \oplus ty\big) \leq \frac{\sin tC}{\sin C} d(x, y).$$

Lemma 2.4 ([14]). *Let (X, d) be a CAT(1) space. Then there is a constant $M > 0$ such that*

$$d^2\big(x, ty \oplus (1-t)z\big) \leq td^2(x, y) + (1-t)d^2(x, z) - \frac{M}{2}t(1-t)d^2(y, z)$$

for any $t \in [0, 1]$ and any point $x, y, z \in X$ such that $d(x, y) \leq \pi/4$, $d(x, z) \leq \pi/4$ and $d(y, z) \leq \pi/2$.

Let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf \big\{r(x, \{x_n\}) : x \in X\big\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \big\{x \in X : r(\{x_n\}) = r(x, \{x_n\})\big\}.$$

Definition 2.5. A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$.

In this case we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Definition 2.6. For a sequence $\{x_n\}$ in X, a point $x \in X$ is a Δ-cluster point of $\{x_n\}$ if there exists a subsequence of $\{x_n\}$ that Δ-converges to x.

Lemma 2.7 ([8]). *Let (X, d) be a complete CAT(κ) space and let $p \in X$. Suppose that a sequence $\{x_n\}$ in X Δ-converges to x such that $r(p, \{x_n\}) < D_\kappa/2$. Then*

$$d(x, p) \leq \liminf_{n\to\infty} d(x_n, p).$$

Definition 2.8. Let (X, d) be a complete metric space and let F be a nonempty subset of X. Then a sequence $\{x_n\}$ in X is Fejér monotone with respect to F if

$$d(x_{n+1}, q) \leq d(x_n, q)$$

for all $n \geq 0$ and all $q \in F$.

Lemma 2.9 ([8]). *Let (X, d) be a complete CAT(1) space and let F be a nonempty subset of X. Suppose that the sequence $\{x_n\}$ in X is Fejér monotone with respect to F and the asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is less than $\pi/2$. If any Δ-cluster point x of $\{x_n\}$ belongs to F, then $\{x_n\}$ Δ-converges to a point in F.*

Lemma 2.10 ([22]). *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + b_n)a_n.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. Additionally, if there is a subsequence of $\{a_n\}$ which converges to 0, then $\lim_{n\to\infty} a_n = 0$.

3. Main results

To complete our proof, we need the following crucial lemmas.

Lemma 3.1. *Let (X,d) be a complete CAT(1) space and let $T : X \to X$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be generated by (1.3) for $x_0 \in X$ such that $d(x_0, F(T)) \leq \pi/4$. Then there exists a unique point p in $F(T)$ such that $d(y_n,p) \leq d(z_n,p) \leq d(x_n,p) \leq \pi/4$ for all $n \geq 0$.*

Proof. From Theorem 3.4 in [17], we know that $F(T)$ is closed and π-convex. So, by Lemma 2.2, there exists a unique point p in $F(T)$ such that $d(x_0, F(T)) = d(x_0, p)$. Since $d(Tx_0, p) \leq d(x_0, p) \leq \pi/4$ and $B_{\pi/4}[p]$ is convex, we obtain

$$d(z_0, p) \leq \gamma_0 d(Tx_0, p) + (1-\gamma_0) d(x_0, p) \leq d(x_0, p) \leq \pi/4$$

and since $d(Tz_0, p) \leq d(z_0, p) \leq \pi/4$ and $B_{\pi/4}[p]$ is convex, we also obtain

$$d(y_0, p) \leq \beta_0 d(Tz_0, p) + (1-\beta_0) d(z_0, p) \leq d(z_0, p) \leq \pi/4.$$

Suppose that $d(z_k, p) \leq d(y_k, p) \leq d(x_k, p) \leq \pi/4$ for $k \geq 1$. Since $d(Ty_k, p) \leq d(y_k, p) \leq \pi/4$ and $B_{\pi/4}[p]$ is convex, we obtain

$$d(x_{k+1}, p) \leq \alpha_k d(Ty_k, p) + (1-\alpha_k) d(y_k, p) \leq d(y_k, p) \leq \pi/4.$$

Since $d(Tx_{k+1}, p) \leq d(x_{k+1}, p) \leq \pi/4$ and $B_{\pi/4}[p]$ is convex, we obtain

$$d(z_{k+1}, p) \leq \gamma_{k+1} d(Tx_{k+1}, p) + (1-\gamma_{k+1}) d(x_{k+1}, p) \leq d(x_{k+1}, p) \leq \pi/4$$

and also

$$d(y_{k+1}, p) \leq \beta_{k+1} d(Tz_{k+1}, p) + (1-\beta_{k+1}) d(z_{k+1}, p) \leq d(z_{k+1}, p) \leq \pi/4.$$

It follows that $d(y_{k+1}, p) \leq d(z_{k+1}, p) \leq d(x_{k+1}, p) \leq \pi/4$. By mathematical induction, we conclude that $d(y_n, p) \leq d(z_n, p) \leq d(x_n, p) \leq \pi/4$ for all $n \geq 0$. □

Lemma 3.2. *Let (X,d) be a complete CAT(1) space and let $T : X \to X$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be generated by (1.3) for $x_0 \in X$ such that $d(x_0, F(T)) \leq \pi/4$. Then for all $n \geq 0$,*

$$d(x_{n+1}, Tx_{n+1}) \leq (1+\delta_n) d(x_n, Tx_n)$$

where $\delta_n = \alpha_n \gamma_n \Big(2 + \frac{4(1-\alpha_n)C}{\sin C}\Big) + \frac{2\beta_n C}{\sin C}$ *and* $C = 2d(x_0, F(T))$.

Proof. Firstly, it is observed that

$$d(z_n, x_n) = d(\gamma_n Tx_n \oplus (1-\gamma_n) x_n, x_n) = \gamma_n d(x_n, Tx_n)$$

and

$$d(z_n, Tx_n) = d(\gamma_n Tx_n \oplus (1-\gamma_n) x_n, Tx_n) = (1-\gamma_n) d(x_n, Tx_n).$$

Hence we have

$$\begin{aligned} d(y_n, z_n) &= d(\beta_n Tz_n \oplus (1-\beta_n) z_n, z_n) \\ &= \beta_n d(Tz_n, z_n) \\ &\leq \beta_n \big(d(Tz_n, Tx_n) + d(Tx_n, z_n)\big) \\ &\leq \beta_n \big(d(z_n, x_n) + d(Tx_n, z_n)\big) \\ &= \beta_n \big(\gamma_n d(x_n, Tx_n) + (1-\gamma_n) d(x_n, Tx_n)\big) \end{aligned}$$

$$= \beta_n d(x_n, Tx_n).$$

We next compute the following estimation:

$$\begin{aligned} d(Tx_{n+1}, x_{n+1}) &\leq d\big(Tx_{n+1}, T(\alpha_n Tz_n \oplus (1-\alpha_n)x_n)\big) \\ &\quad + d\big(T(\alpha_n Tz_n \oplus (1-\alpha_n)x_n), Tx_n\big) \\ &\quad + d(Tx_n, Tz_n) + d\big(Tz_n, \alpha_n Tz_n \oplus (1-\alpha_n)x_n\big) \\ &\quad + d\big(\alpha_n Tz_n \oplus (1-\alpha_n)x_n, x_{n+1}\big) \\ &\leq 2d\big(\alpha_n Tz_n \oplus (1-\alpha_n)x_n, x_{n+1}\big) + \alpha_n d(Tz_n, x_n) \\ &\quad + d(x_n, z_n) + (1-\alpha_n)d(Tz_n, x_n) \\ &= 2d\big(\alpha_n Tz_n \oplus (1-\alpha_n)x_n, x_{n+1}\big) + d(Tz_n, x_n) + d(x_n, z_n) \\ &\leq 2d\big(\alpha_n Tz_n \oplus (1-\alpha_n)x_n, x_{n+1}\big) + d(x_n, Tx_n) + 2d(x_n, z_n) \\ &= 2d\big(\alpha_n Tz_n \oplus (1-\alpha_n)x_n, x_{n+1}\big) + (1+2\gamma_n)d(x_n, Tx_n). \qquad (3.1) \end{aligned}$$

From Lemma 3.1 we observe that $d(y_n, Ty_n)$, $d(x_n, Ty_n)$, $d(x_n, y_n)$, $d(x_n, Tz_n)$ and $d(Ty_n, Tz_n)$ are all smaller than C. Since $C \leq \pi/2$, by Lemma 2.3, we obtain

$$\begin{aligned} d\big(x_{n+1}, \alpha_n Tz_n \oplus (1-\alpha_n)x_n\big) &\leq d\big(x_{n+1}, \alpha_n Ty_n \oplus (1-\alpha_n)x_n\big) \\ &\quad + d\big(\alpha_n Ty_n \oplus (1-\alpha_n)x_n, \alpha_n Tz_n \oplus (1-\alpha_n)x_n\big) \\ &\leq \frac{\sin(1-\alpha_n)C}{\sin C} d(x_n, y_n) + \frac{\sin \alpha_n C}{\sin C} d(Ty_n, Tz_n) \\ &\leq \frac{(1-\alpha_n)C}{\sin C} d(x_n, y_n) + \frac{\alpha_n C}{\sin C} d(y_n, z_n) \\ &\leq \frac{(1-\alpha_n)C}{\sin C} d(x_n, y_n) + \frac{\alpha_n \beta_n C}{\sin C} d(x_n, Tx_n). \qquad (3.2) \end{aligned}$$

From (3.1) and (3.2), we have

$$d(Tx_{n+1}, x_{n+1}) \leq \frac{2(1-\alpha_n)C}{\sin C} d(x_n, y_n) + \Big(1 + 2\gamma_n + \frac{2\alpha_n\beta_n C}{\sin C}\Big) d(x_n, Tx_n).$$

Multiplying by α_n, we then obtain

(3.3)

$$\alpha_n d(Tx_{n+1}, x_{n+1}) \leq \frac{2\alpha_n(1-\alpha_n)C}{\sin C} d(x_n, y_n) + \Big(\alpha_n + 2\alpha_n\gamma_n + \frac{2\alpha_n^2\beta_n C}{\sin C}\Big) d(x_n, Tx_n).$$

On the other hand, we compute the following estimation:

$$\begin{aligned} d(Tx_{n+1}, x_{n+1}) &\leq d\big(Tx_{n+1}, T(\alpha_n Tx_n \oplus (1-\alpha_n)z_n)\big) \\ &\quad + d\big(T(\alpha_n Tx_n \oplus (1-\alpha_n)z_n), Tz_n\big) \\ &\quad + d\big(Tz_n, Tx_n\big) + d\big(Tx_n, \alpha_n Tx_n \oplus (1-\alpha_n)z_n\big) \\ &\quad + d\big(\alpha_n Tx_n \oplus (1-\alpha_n)z_n, x_{n+1}\big) \\ &\leq 2d\big(\alpha_n Tx_n \oplus (1-\alpha_n)z_n, x_{n+1}\big) + \alpha_n d(Tx_n, z_n) \\ &\quad + d(x_n, z_n) + (1-\alpha_n)d(Tx_n, z_n) \\ &= 2d\big(\alpha_n Tx_n \oplus (1-\alpha_n)z_n, x_{n+1}\big) + d(Tx_n, z_n) + d(x_n, z_n) \\ &= 2d\big(\alpha_n Tx_n \oplus (1-\alpha_n)z_n, x_{n+1}\big) + (1-\gamma_n)d(x_n, Tx_n) \\ &\quad + \gamma_n d(x_n, Tx_n) \end{aligned}$$

$$= 2d\big(\alpha_n Tx_n \oplus (1-\alpha_n)z_n, x_{n+1}\big) + d(x_n, Tx_n). \tag{3.4}$$

Also, by Lemma 3.1, we see that $d(y_n, Ty_n)$, $d(z_n, Ty_n)$, $d(z_n, y_n)$, $d(z_n, Tx_n)$ and $d(Tx_n, Ty_n)$ are all smaller than C. Since $C \leq \pi/2$, by Lemma 2.3, we have

$$\begin{aligned}
d\big(x_{n+1}, \alpha_n Tx_n \oplus (1-\alpha_n)z_n\big) &\leq d\big(x_{n+1}, \alpha_n Ty_n \oplus (1-\alpha_n)z_n\big) \\
&\quad + d\big(\alpha_n Ty_n \oplus (1-\alpha_n)z_n, \alpha_n Tx_n \oplus (1-\alpha_n)z_n\big) \\
&\leq \frac{\sin(1-\alpha_n)C}{\sin C} d(y_n, z_n) + \frac{\sin \alpha_n C}{\sin C} d(Tx_n, Ty_n) \\
&\leq \frac{(1-\alpha_n)C}{\sin C} d(y_n, z_n) + \frac{\alpha_n C}{\sin C} d(x_n, y_n) \\
&\leq \frac{(1-\alpha_n)\beta_n C}{\sin C} d(x_n, Tx_n) + \frac{\alpha_n C}{\sin C} d(x_n, y_n). \qquad (3.5)
\end{aligned}$$

From (3.4) and (3.5), we have

$$d(Tx_{n+1}, x_{n+1}) \leq \frac{2\alpha_n C}{\sin C} d(x_n, y_n) + \Big(1 + \frac{2(1-\alpha_n)\beta_n C}{\sin C}\Big) d(x_n, Tx_n).$$

Multiplying by $(1-\alpha_n)$, we then obtain

$$\begin{aligned}
(1-\alpha_n) d(Tx_{n+1}, x_{n+1}) &\leq \frac{2\alpha_n(1-\alpha_n)C}{\sin C} d(x_n, y_n) \\
&\quad + \Big(1 - \alpha_n + \frac{2(1-\alpha_n)^2 \beta_n C}{\sin C}\Big) d(x_n, Tx_n). \qquad (3.6)
\end{aligned}$$

Adding up (3.3) and (3.6) yields

$$\begin{aligned}
d(Tx_{n+1}, x_{n+1}) &\leq \frac{4\alpha_n(1-\alpha_n)C}{\sin C} d(x_n, y_n) \\
&\quad + \Big(1 + 2\alpha_n\gamma_n + \frac{2\beta_n C}{\sin C}\big(\alpha_n^2 + (1-\alpha_n)^2\big)\Big) d(x_n, Tx_n).
\end{aligned}$$

Noting $d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n) \leq (\beta_n + \gamma_n) d(x_n, Tx_n)$, we thus obtain

$$\begin{aligned}
d(Tx_{n+1}, x_{n+1}) &\leq \frac{4\alpha_n(1-\alpha_n)C}{\sin C} (\beta_n + \gamma_n) d(x_n, Tx_n) \\
&\quad + \Big(1 + 2\alpha_n\gamma_n + \frac{2\beta_n C}{\sin C}\big(\alpha_n^2 + (1-\alpha_n)^2\big)\Big) d(x_n, Tx_n) \\
&= \Big(1 + \alpha_n\gamma_n\big(2 + \frac{4(1-\alpha_n)C}{\sin C}\big) + \frac{2\beta_n C}{\sin C}\Big) d(x_n, Tx_n).
\end{aligned}$$

This completes the proof. □

Lemma 3.3. *Let (X, d) be a complete CAT(1) space and let $T : X \to X$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be generated by (1.3) for $x_0 \in X$ such that $d(x_0, F(T)) \leq \pi/4$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy that (i) $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, (ii) $\sum_{n=0}^{\infty} \alpha_n\gamma_n < \infty$ and (iii) $\sum_{n=0}^{\infty} \beta_n < \infty$. Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

Proof. Using conditions (ii), (iii) and Lemma 3.2, we get that $\lim_{n\to\infty} d(x_n, Tx_n)$ exists by Lemma 2.10. Let p be a unique point in $F(T)$ such that $d(x_0, p) =$

$d(x_0, F(T))$. Note that, by Lemma 3.1, $d(y_n, p) \leq d(z_n, p) \leq d(x_n, p) \leq \pi/4$ for all $n \geq 0$. So, from Lemma 2.4, there exists $M > 0$ such that

$$\begin{aligned} d^2(x_{n+1}, p) &\leq \alpha_n d^2(Ty_n, p) + (1-\alpha_n) d^2(y_n, p) - \frac{M}{2}\alpha_n(1-\alpha_n) d^2(Ty_n, y_n) \\ &\leq d^2(y_n, p) - \frac{M}{2}\alpha_n(1-\alpha_n) d^2(Ty_n, y_n) \\ &\leq d^2(x_n, p) - \frac{M}{2}\alpha_n(1-\alpha_n) d^2(Ty_n, y_n). \end{aligned}$$

This gives

$$\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) d^2(Ty_n, y_n) < \infty.$$

We see that, by condition (ii),

$$\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n)\big(d^2(Ty_n, y_n) + \gamma_n\big) \leq \sum_{n=0}^{\infty} \big(\alpha_n(1-\alpha_n) d^2(Ty_n, y_n) + \alpha_n \gamma_n\big) < \infty.$$

So, by condition (i), we have

$$\liminf_{n\to\infty} \big(d^2(Ty_n, y_n) + \gamma_n\big) = 0.$$

Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\lim_{k\to\infty} d(Ty_{n_k}, y_{n_k}) = 0$$

and $\lim_{k\to\infty} \gamma_{n_k} = 0$. So we obtain

$$\begin{aligned} d(Tx_{n_k}, x_{n_k}) &\leq d(Tx_{n_k}, Ty_{n_k}) + d(Ty_{n_k}, y_{n_k}) + d(y_{n_k}, x_{n_k}) \\ &\leq 2d(y_{n_k}, x_{n_k}) + d(Ty_{n_k}, y_{n_k}) \\ &\leq 2(\beta_{n_k} + \gamma_{n_k}) d(x_{n_k}, Tx_{n_k}) + d(Ty_{n_k}, y_{n_k}), \end{aligned}$$

which implies that $d(x_{n_k}, Tx_{n_k}) \to 0$ as $k \to \infty$. Therefore, by Lemma 2.10, we conclude that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. □

We are now ready to prove our main result.

Theorem 3.4. *Let (X, d) be a complete CAT(κ) space and let $T : X \to X$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be generated by (1.3) for $x_0 \in X$ such that $d(x_0, F(T)) < D_\kappa/4$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy that (i) $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, (ii) $\sum_{n=0}^{\infty} \alpha_n\gamma_n < \infty$ and (iii) $\sum_{n=0}^{\infty} \beta_n < \infty$. Then $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof. Without loss of generality, we assume that $\kappa = 1$. Put $F_0 := F(T) \cap B_{\pi/2}(x_0)$. Let $q \in F_0$. Since $d(Tx_0, q) \leq d(x_0, q)$ and since the open ball in X with center q and radius less than $\pi/2$ is convex, we have

$$d(z_0, q) = d\big(\gamma_0 Tx_0 \oplus (1-\gamma_0)x_0, q\big) \leq d(x_0, q).$$

Since $d(Tz_0, q) \leq d(z_0, q)$, we also have

$$d(y_0, q) = d\big(\beta_0 Tz_0 \oplus (1-\beta_0)z_0, q\big) \leq d(z_0, q) \leq d(x_0, q).$$

So we have

$$d(x_1, q) = d\big(\alpha_0 Ty_0 \oplus (1-\alpha_0)y_0, q\big) \leq d(x_0, q).$$

By mathematical induction, we can show that

$$d(x_{n+1}, q) \leq d(x_n, q) \leq d(x_0, q)$$

for all $n \geq 0$. Hence $\{x_n\}$ is a Fejér monotone sequence with respect to F_0.

In particular, choose $p \in F(T)$ such that $d(x_0, p) < \pi/4$. Then $p \in F_0$ and

$$d(x_{n+1}, p) \leq d(x_n, p) \leq d(x_0, p) < \pi/4. \tag{3.7}$$

This shows that $r(\{x_n\}) < \pi/4$. Thus, by Lemma 2.9, we will show that any Δ-cluster point of $\{x_n\}$ belongs to F_0. Let $x \in X$ be a Δ-cluster point of $\{x_n\}$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which Δ-converges to x. Using (3.7) we have

$$r(p, \{x_{n_k}\}) \leq r(x_0, p) < \pi/4.$$

Using Lemma 2.7, it follows that

$$\begin{aligned} d(x, x_0) &\leq d(x, p) + d(x_0, p) \\ &\leq \liminf_{k\to\infty} d(x_{n_k}, p) + d(x_0, p) \\ &< \pi/2. \end{aligned}$$

This implies that $x \in B_{\pi/2}(x_0)$. Using Lemma 3.3, we get that

$$\begin{aligned} \limsup_{k\to\infty} d(Tx, x_{n_k}) &\leq \limsup_{k\to\infty} d(Tx, Tx_{n_k}) + \limsup_{k\to\infty} d(Tx_{n_k}, x_{n_k}) \\ &\leq \limsup_{k\to\infty} d(x, x_{n_k}), \end{aligned}$$

which yields that $Tx \in A(\{x_{n_k}\})$ and $Tx = x$. Hence $x \in F_0$. We thus complete the proof. □

If $\gamma_n = 0$ for all $n \geq 0$, then we get a convergence result of a new two-step iteration process in CAT(κ) spaces.

Corollary 3.5. *Let (X, d) be a complete CAT(κ) space and let $T : X \to X$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. For $x_0 \in X$ such that $d(x_0, F(T)) < D_K/4$. Let $\{x_n\}$ be generated by*

$$\begin{aligned} y_n &= \beta_n T x_n \oplus (1 - \beta_n) x_n, \\ x_{n+1} &= \alpha_n T y_n \oplus (1 - \alpha_n) y_n, \quad n \geq 0. \end{aligned} \tag{3.8}$$

Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy that (i) $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and (ii) $\sum_{n=0}^{\infty} \beta_n < \infty$. Then $\{x_n\}$ Δ-converges to a fixed point of T.

Remark 3.6. We point out that the iteration process (3.8) is different from Ishikawa iteration process (1.2) studied by Jun [9]. So it is new in CAT(κ) spaces.

If $\beta_n = \gamma_n = 0$ for all $n \geq 0$, then we obtain Theorem 3.1 of He et al. [8].

Corollary 3.7 ([8]). *Let (X, d) be a complete CAT(κ) space and let $T : X \to X$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be generated by (1.3) for $x_0 \in X$ such that $d(x_0, F(T)) < D_K/4$. If $\{\alpha_n\}$ satisfies that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then $\{x_n\}$ Δ-converges to a fixed point of T.*

Remark 3.8. In the case $\kappa = 0$, our results also hold in complete CAT(0) spaces.

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