Journal of Nonlinear and Convex Analysis
Volume 16, Number 1, 2015, 119–125

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

GENERALIZED CARISTI FIXED POINT RESULTS IN PARTIAL METRIC SPACES

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Abstract. In this paper, we prove Caristi type fixed point results for multivalued maps in the setting of partial metric spaces. Our results improve and generalize a number of known fixed point results.

1. Introduction

A number of extensions of the Banach contraction principle have appeared in the literature. One of its most important extensions is known as Caristi's fixed point theorem. It is well-known that Caristi's fixed point theorem is equivalent to Ekland variational principle [5], which is nowadays an important tool in nonlinear analysis. Many authors have studied and generalized Caristi's fixed point theorem to various directions. For example, see [3, 8, 9, 10, 14, 16] and others. The existence of fixed point for multivalued contractions was first studied by Nadler [13], in which he established a multivalued version of the Banach contraction principle. While, in [6] Jachymski proved that Nadler's fixed point theorem yield from the Caristi's fixed point theorem.

In 1992, Matthews [11] introduced a notion of partial metric space which is a generalization of the usual metric spaces. Among others results, he proved the Banach contraction principle for partial metric spaces. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation. In the setting of partial metric spaces, an existence of fixed points for various single-valued maps have been studied by many authors. Among others, Karapinar [7] generalized Caristi fixed point theorem on partial metric spaces for single-valued maps. While Acar-Altum [1, 2], studied Bae and Suzuki type generalizations of Caristi fixed point theorem for partial metric space. In this paper, we prove some results on the existence of fixed points for multivalued Caristi type maps in the setting of partial metric spaces. Our results improve and extend the corresponding known fixed point results.

2. Preliminaries

Let X be a metric space with metric d. We use 2^X to denote the collection of all nonempty subsets of X. A point $x \in X$ is called a fixed point of a map $f : X \to X$ ($T : X \to 2^X$) if $x = f(x)$ ($x \in T(x)$).

2010 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. Partial metric space, multivalued map, Caristi's fixed point theorem.

In 1976, Caristi [4] obtained the following fixed point theorem on complete metric spaces, known as Caristi's fixed point theorem.

Theorem 2.1. *Let X be a complete metric space with metric d. Let $\mu : X \to \mathbb{R}^+$ be a lower semicontinuous function and let $f : X \to X$ be a single valued map such that for any $x \in X$*

$$d(x, f(x)) \leq \mu(x) - \mu(f(x)). \tag{2.1}$$

Then f has a fixed point.

Among others, Bae [3] and Suzuki [16], also studied some Caristi's fixed point theorem.

A partial metric on nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ (non-negative real numbers) such that for all $x, y, z \in X$:
(a) $p(x, y) = p(y, x)$
(b) if $p(x, x) = p(x, y) = p(y, y)$ then $x = y$
(c) $p(x, x) \leq p(x, y)$
(d) $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$.

The set X with partial metric p is called partial metric space (in short, PMS) and is denoted by (X, p). Note that if $p(x, y) = 0$ then it follows from (a) and (b) that $x = y$. But, if $x = y$ then $p(x, y)$ may be not 0.

Let $X = \mathbb{R}^+$, define $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then (X, p) is a PMS . Let X be the collection of all closed bounded intervals of reals. Define $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$. Then (X, p) is a PMS. Further, interesting examples of PMS can be formed in [11].

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base of the family of open p-balls $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ defined by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on X.

A sequence $\{x_n\}$ in a PMS (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$. A sequence $\{x_n\}$ in a PMS (X, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m) < \infty$. A PMS (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ with respect to τ_p such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Lemma 2.2 ([1, 7, 12])**.** *Let (X, p) be a PMS.*

(i) *A sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).*

(ii) *A PMS (X,p) is complete if and only if (X,p^s) is complete. Furthermore,* $\lim_{n\to\infty} p^s(x_n,x) = 0$ *if and only if* $p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Romaguera [15] proposed the following two alternatives to give an appropriate notion of a Caristi mapping in partial metric spaces.

(i) A self mapping f of a partial metric space (X,p) is called a p-Caristi mapping on X if there is a function $\mu : X \to \mathbb{R}^+$ which is lower semicontinuous for (X,p) and satisfies

$$p(x,fx) \le \mu(x) - \mu(fx) \tag{2.2}$$

for all $x \in X$.

(ii) A self mapping f of a partial metric space (X,p) is called a p^s-Caristi mapping on X if there is a function $\mu : X \to \mathbb{R}^+$ which is lower semicontinuous for (X,p^s) and satisfies (2.1).

A sequence $\{x_n\}$ in a PMS (X,p) is called 0-Cauchy if $\lim_{n,m\to\infty} p(x_n,x_m) = 0$ and (X,p) is called 0-complete if every 0-Cauchy sequence in X converges to a point $z \in X$ with respect to τ_p, such that $p(z,z) = 0$. It is clear that every complete PMS is 0-complete. Romaguera [15] proved that a partial metric space (X,p) is 0-complete if and only if every p^s-Caristi mapping f on X has a fixed point.

Note that, the identity mapping on X is neither p-Caristi nor p^s-Caristi mapping, although it is a Caristi mapping for metric space. In [1], a new notation of Caristi mapping has been introduced as follows.

A self mapping f of a partial metric space (X,p) is called a Caristi mapping on X if there is a function $\mu : X \to \mathbb{R}^+$ which is a lower semicontinuous function for (X,p^s) and satisfies $p(x,fx) \le p(x,x) + \mu(x) - \mu(fx)$ for all $x \in X$. The following theorem characterizes the completeness of partial metric spaces.

Theorem 2.3 ([1]). *A partial metric space (X,p) is complete if and only if every Caristi mapping on X has a fixed point.*

3. Main results

We prove multivalued version of the Caristi result in the setting of PMS.

Theorem 3.1. *Let (X,p) be a complete PMS, $\mu : X \to \mathbb{R}^+$ be a lower semicontinuous function for (X,p^s) such that for each $x,y \in X$,*

$$p(x,x) = p(x,y) \quad \textit{implies} \quad \mu(y) \le \mu(x) \tag{3.1}$$

and let $\eta : X \to \mathbb{R}^+$ be a function satisfying

$$\sup\left\{\eta(x) : \ x \in X, \ \mu(x) \le \inf_{z\in X} \mu(z) + k\right\} < \infty \tag{3.2}$$

for some $k > 0$. Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$ there exists $u_x \in T(x)$ satisfying

$$p(x, u_x) \leq p(x, x) + \eta(x)\{\mu(x) - \mu(u_x)\}. \tag{3.3}$$

Then T has a fixed point in X.

Proof. Define a function $f : X \to X$ by $f(x) = u_x \in T(x) \subseteq X$. Note that for each $x \in X$ we have

$$p(x, f(x)) \leq p(x, x) + \eta(x)\{\mu(x) - \mu(f(x))\}.$$

Now, if $\eta(x) > 0$, then it follows that,

$$\mu(f(x)) \leq \mu(x).$$

If $\eta(x) = 0$, then $p(x, f(x)) \leq p(x, x)$, but by the definition of PMS, it follows that $p(x, x) \leq p(x, f(x))$, so we get $p(x, x) = p(x, f(x))$. Due to hypotheses (3.2), it follows that

$$\mu(f(x)) \leq \mu(x). \tag{3.4}$$

Put

$$S = \{x \in X \ : \ \mu(x) \leq \inf_{z \in X} \mu(z) + k\}, \quad \beta = \sup_{z \in S} \eta(z) < \infty.$$

Note that S is nonempty set. Since (X, p) is complete, by Lemma 2.2 (X, p^s) is complete. By the lower semicontinuity of μ, it follows that S is closed subset of (X, p^s) and thus (X, p^s) is complete. Due to Lemma 2.2 (S, p) is complete PMS. Now we show that S is invariant under mapping f. Let $a \in S$, and $f(a) = b \in T(a)$, then we get

$$\mu(f(a)) \leq \mu(a) \leq \inf_{z \in X} \mu(z) + k,$$

which implies $f(a) \in S$, and thus $f(S) \subseteq S$. Define $\varphi(x) = \beta\mu(x)$ for all $x \in S$. Note that φ is lower semicontinuous in (S, p^s). Also,

$$p(x, f(x)) \leq p(x, x) + \beta\mu(x) - \beta\mu(f(x)).$$

So

$$p(x, f(x)) \leq p(x, x) + \varphi(x) - \varphi(f(x)).$$

Thus, f is a Caristi mapping on S. So, by Theorem 2.3 there exists $x_0 \in S$ such that $x_0 = f(x_0) \in T(x_0)$. □

Remark 3.2. If we consider p is an ordinary metric, then Theorem 3.1 reduces to Theorem 2 of Suzuki [15], which a generalization of the Caristi fixed theorem.

In the sequel, we assume X is complete PMS with partial metric p and $\mu : X \to \mathbb{R}^+$ is a lower semicontinuous function satisfying

$$p(x, x) = p(x, y) \quad \textit{implies} \quad \mu(y) \leq \mu(x).$$

Now, applying Theorem 3.1, we prove the following results.

Theorem 3.3. *Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$ there exists $u_x \in T(x)$ satisfying*

$$p\left(x, u_x\right) \leq p\left(x, x\right) + \max\{\psi\left(\mu\left(x\right)\right), \psi\left(\mu\left(u_x\right)\right)\}\{\mu\left(x\right) - \mu\left(u_x\right)\},$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is an upper semicontinuous function. Then T has a fixed point in X.

Proof. Put $t_0 = \inf_{x\in X} \mu(x)$. By the definition ψ, there exists some positive real numbers r, r_0 such that $\psi(t) \leq r_0$ for all $t \in [t_0, t_0 + r]$. Now, for all $x \in X$, we define

$$\mathrm{g}\left(\mathrm{x}\right) = \max\{\psi\left(\mu\left(x\right)\right), \psi\left(\mu\left(u_x\right)\right)\}.$$

Clearly, g maps from X into $\mathbb{R}^+$. We note that $\mu\left(u_x\right) \leq \mu\left(x\right)$ for all $x \in X$. Thus for any $x \in X$ with $\mu\left(x\right) \leq t_0 + r$, we have $\mu\left(u_x\right) \leq t_0 + r$. Now, clearly $g\left(x\right) \leq r_0 < \infty$ and hence we obtain

$$\sup\left\{\mathrm{g}\left(\mathrm{x}\right) : \ \mathrm{x} \in \mathrm{X}, \ \mu\left(\mathrm{x}\right) \leq \inf_{\mathrm{z}\in\mathrm{X}} \mu\left(\mathrm{z}\right) + \mathrm{r}\right\} < \infty \quad \text{for some} \quad \mathrm{r} > 0.$$

Thus, for any $x \in X$, there exists $u_x \in T(x)$ such that

$$p\left(x, u_x\right) \leq p\left(x, x\right) + \mathrm{g}(\mathrm{x})\{\mu\left(x\right) - \mu\left(u_x\right)\}.$$

By Theorem 3.1, there exists $x_0 \in X$ such that $x_0 \in T(x_0)$. $\square$

Theorem 3.4. *Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$ there exists $u_x \in T(x)$ satisfying*

$$p\left(x, u_x\right) \leq p\left(x, x\right) + \psi\left(\mu\left(x\right)\right)\{\mu\left(x\right) - \mu\left(u_x\right)\},$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing function. Then T has a fixed point in X.

Proof. For each $x \in X$, define $g\left(x\right) = \psi\left(\mu\left(x\right)\right)$. Clearly $g : X \to \mathbb{R}^+$. Since the function ψ is nondecreasing, for any real number $r > 0$ we have

$$\sup\left\{g\left(x\right) : \ x \in X, \ \mu\left(x\right) \leq \inf_{z\in X} \mu\left(z\right) + r\right\} \leq \psi(\inf_{z\in X} \mu\left(x\right) + r) \ < \infty$$

for some $r > 0$. Thus, by Theorem 3.1, the result follows. $\square$

Theorem 3.5. *Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$ there exists $u_x \in T(x)$ satisfying*

$$p\left(x, u_x\right) \leq p\left(x, x\right) + \psi\left(\mu\left(u_x\right)\right)\{\mu\left(x\right) - \mu\left(u_x\right)\},$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing function. Then T has a fixed point in X.

Proof. Since for each $x \in X$, there exists $u_x \in T(x)$ such that $\mu(u_x) \leq \mu(x)$. Since the function ψ is nondecreasing, we have

$$\psi\left(\mu\left(u_x\right)\right) \leq \psi\left(\mu\left(x\right)\right).$$

Thus the result follows from Theorem 3.4. $\square$

Theorem 3.6. *Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$ there exists $u_x \in T(x)$ satisfying $p(x, u_x) \leq \mu(x)$ for all $x \in X$ and*

$$p(x, u_x) \leq p(x, x) + \psi(p(x, u_x))\{\mu(x) - \mu(u_x)\},$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be an upper semicontinuous function. Then T has a fixed point in X.

Proof. Define a function $\tau(x) = \psi(p(x, u_x))$, where $\tau : X \to \mathbb{R}^+$. For $x \in X$ with $\mu(x) \leq \inf_{z \in X} \mu(z) + 1$, we have

$$\begin{aligned} \tau(x) &\leq \sup\{\psi(t) : 0 \leq t \leq p(x, u_x)\} \\ &\leq \sup\{\psi(t) : 0 \leq t \leq \mu(x)\} \\ &\leq \sup\{\psi(t) : 0 \leq t \leq \inf_{z \in X} \mu(z) + 1\}. \end{aligned}$$

Hence

$$\begin{aligned} \sup\left\{\psi(t) : x \in X, \mu(x) \leq \inf_{z \in X} \mu(z) + 1\right\} &\leq \max\left\{\psi(t) : 0 \leq t \leq \inf_{z \in X} \mu(z) + 1\right\} \\ &< \infty, \end{aligned}$$

because ψ is upper semicontinuous. So, by Theorem 3.1, we obtain the desired result. ☐

The following example shows that the condition (3.1) used in all our results, is natural.

Example 3.1. Let $X = \mathbb{R}^+$. Define $p : X \times X \to \mathbb{R}^+$ by

$$p(x, y) = \max\{x, y\}, \quad \text{for all } \ x, y \in X.$$

Then, (X, p) is a partial metric space, see; [1]. Define $\mu : X \to \mathbb{R}^+$ by $\mu(x) = 2x$. Then, the condition (3.1) holds, that is; $p(x, x) = p(x, y)$ implies $\mu(y) \leq \mu(x)$.

Now we give an example in support of our main Theorem 3.1 .

Example 3.2. Let (X, p) be a PMS as in Example 3.1. Define non-negative real valued functions μ and η on X by

$$\mu(x) = 2x, \quad \eta(x) = \frac{1}{1 + x} \quad \text{for all } \ x, y \in X.$$

Then, clearly the conditions (3.1) and (3.2) are satisfied. Define a multivalued map $T : X \to 2^X$ by $T(x) = [0, x)$. Then, T satisfies the condition (3.3) (e.g, take $u_x = \frac{x}{2}$). Thus, all the conditions of Theorem 3.1 are satisfied, it guarantees the existence of fixed point of T. Note that $x = 0$ is the required fixed point.

Acknowledgment

The authors thank the referee for his valuable comments and suggestions.

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Manuscript received September 18, 2013
revised May 15, 2014

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