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FIXED POINT PROPERTIES ON BANACH SPACES ASSOCIATED TO LOCALLY COMPACT GROUPS AND RELATED GEOMETRIC PROPERTIES

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Dedicated to Professor Sompong Dhompongsa on the occasion of his 65^{th} birthday with admiration and respect

Abstract. It is the purpose of this paper to report on some recent progress on common fixed point properties for semigroup of non-expansive mappings on bounded closed convex subsets of Banach spaces associated to locally compact groups and their relation with other geometric properties.

1. Introduction

A central problem in the fixed point theory of nonexpansive maps is to determine those subsets of a Banach space which have the fixed point property for nonexpansive self-maps. A long standing problem was whether a weakly compact convex subset of a Banach space has the fixed point property for nonexpansive self-maps. With Alspach's example [1] we know that there is a weakly compact convex set in $L_1[0,1]$ which need not have the fixed point property for nonexpansive self-maps. On the other hand, we know a weakly compact (weak* compact) convex subset in a (dual) Banach space does have the fixed point property for nonexpansive self-maps if the set has normal structure (see [27] and [44]).

It is the purpose of this paper to report on some recent progress on common fixed point properties for semigroup of nonexpansive mappings on bounded convex subset in Banach spaces associated to locally compact groups and their relation with various other geometric properties. Some open problems will be posted in Section 5.

2. Some preliminaries

Let E be a Banach space and K be a nonempty bounded closed convex subset of E. We say that K has the *fixed point property* if every nonexpansive mapping $T : K \to K$ (i.e. $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$) has a fixed point. We say that E has the *weak fixed point property* if every weakly compact convex subset of E has the fixed point property. A dual Banach space E is said to have the weak*

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fixed point property if each weak* compact convex subset of E has the fixed point property.

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$, the mappings $s \to as$ and $s \to sa$ from S into S are continuous. S is called *left reversible* if $\overline{aS} \cap \overline{bS} \neq \emptyset$ for any $a, b \in S$, where, in general, $\overline{K}$ denotes the closure of the set K. Clearly abelian semigroups and groups are left reversible. Let $CB(S)$ be the C^*-algebra of bounded continuous complex-valued functions on S and for $a \in S$, let ℓ_a be the left translation operator on $CB(S)$ be defined by $(\ell_a f)(t) = f(at)$ for all $f \in CB(S)$ and for all $t \in S$. Then S is *left amenable* if there is an $m \in CB(S)^*$ such that $\|m\| = m(1) = 1$ and $m(\ell_a f) = m(f)$ for all $f \in CB(S)$ and $a \in S$. If the topology on S is normal and S is left amenable, then S is left reversible. In particular, if S is left amenable as a discrete semigroup , then S is left reversible. Left reversible semigroups have played an important role in the study of common fixed point theorems and ergodic type theorems for semigroups of nonexpansive mappings (see [9], [10], [13], [25], [31], [32], [38], [39], [41], [44], [47]).

Let S be a semitopological semigroup, and K be a topological space. An action of S on K is a map ψ from $S \times K$ to K, denoted by $\psi(s,k) = sk, \quad s \in S$, $k \in K$, such that $s_1 s_2(k) = s_1(s_2 k)$, for all $s_1, s_2 \in S$, and $k \in K$. The action is separately continuous if ψ is continuous in each of the variables when the other is kept fixed. Lau showed in [29, Corollary 3.3] that if E is a Banach space and $\mathcal{S} = \{T_s : s \in S\}$ is a separately continuous representation of a left reversible semitopological semigroup S as nonexpansive self-maps on a compact convex subset K of E, then K contains a common fixed point for $\mathcal{S}$. We say a Banach space E has the *weak fixed point property for left reversible semigroups* if whenever S is a left reversible semitopological semigroup and K is a nonempty weakly compact convex subset of E for which the action of S on K (with the norm topology) is separately continuous and nonexpansive, then K has a common fixed point for S. Similarly a dual Banach space E has the *weak* fixed point property for left reversible semigroups* if whenever S is a left reversible semitopological semigroup and K is a nonempty weak* compact convex subset of E for which the action of S on K is separately continuous and nonexpansive, then K has a common fixed point for S. In general, a weakly compact convex set of a Banach space need not have the fixed point property for left reversible semigroups, not even commutative semigroups. Indeed, Alspach [1] (see also [9, Theorem 4.2], [10], [13]) showed there is a weakly compact convex subset K in $L^1[0,1]$ and an isometry $T : K \to K$ without a fixed point. Hence if $S = (\mathbb{N}, +)$ and $\mathcal{S} = \{T^n : n \in \mathbb{N}\}$, then K does not have a common fixed point for $\mathcal{S}$. However, Bruck showed in [5] that a Banach space E having the weak fixed point property has the weak fixed point property for commutative semigroups, and Lim showed in [43] that a Banach space with weak normal structure has the weak fixed point property for left reversible semigroups. For dual Banach spaces, it is known (see [43], [44]) that ℓ_1 and any uniformly convex Banach space have the weak* fixed point property for left reversible semigroups.

Let K be a bounded closed convex subset of a Banach space E. A point x in K is called a *diametral point* if

$$\sup\{\|x - y\| : y \in K\} = \operatorname{diam}(K),$$

where diam(K) denotes the diameter of K. The set K is said to have *normal structure* if every nontrivial (i.e., contains at least two points) convex subset H of K contains a non-diametral point of H (see [20] and [27]).

A Banach space E has *weak normal structure* if every nontrivial weakly compact convex subset of E has normal structure. A dual Banach space E has *weak* normal structure* if every nontrivial weak* compact convex subset of E has normal structure.

Kirk [27] proved that if E has the weak normal structure, then E has the weak fixed point property.

A Banach space E is said to have property UKK *(uniformly Kadec-Klee property)* if for any $\varepsilon > 0$ there is a $0 < \delta < 1$ such that whenever (x_n) is a sequence in the unit ball of E converging weakly to x and satisfying $\operatorname{sep}\big((x_n)\big) \equiv \inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon$, then $\|x\| \leq \delta$. A dual Banach space E is said to have property UKK* *(weak* uniformly Kadec-Klee property)* if for any $\varepsilon > 0$ there is a $0 < \delta < 1$ such that whenever A is a subset of the closed unit ball of E containing a sequence (x_n) with $\operatorname{sep}\big((x_n)\big) > \varepsilon$, then there is an x in the weak* closure of A such that $\|x\| \leq \delta$. The property UKK* was introduced by van Dulst and Sims [14]. They proved that if E has property UKK*, then E has weak* normal structure and hence has the weak* fixed point property.

The following types of Kadec-Klee properties satisfied by a norm will be considered.

(a) A Banach space E is said to have the *Kadec-Klee property* (KK) if whenever (x_n) is a sequence in the unit ball of E that converges weakly to x, and $\operatorname{sep}\big((x_n)\big) > 0$, where
$$\operatorname{sep}\big((x_n)\big) \equiv \inf\{\|x_n - x_m\| : n \neq m\},$$
then $\|x\| < 1$. This property, given in different form, is known as property (H) in Day [8], property (A) in [7]. The definition as given above is due to Huff [26].

(b) A Banach space E is said to have the *uniformly Kadec-Klee property* (UKK) if for every $\varepsilon > 0$ there is a $0 < \delta < 1$ such that whenever (x_n) is a sequence in the unit ball of E converging weakly to x and $\operatorname{sep}\big((x_n)\big) > \varepsilon$ then $\|x\| \leq \delta$. This property was introduced by Huff [26], who showed that property UKK is strictly stronger than property KK. Van Dulst and Sims showed that a Banach space with property UKK has property FPP [14].

Corresponding to each of the above properties there is a stronger property defined by replacing the sequence in the above definitions by a net. We call these properties the *strong Kadec-Klee property* (SKK) and the *strong uniformly Kadec-Klee property* (SUKK). For dual Banach spaces we have the corresponding properties involving the weak* topology, which we denote by KK*, SKK*, UKK* and SUKK*. First, properties KK* and SKK* are defined by replacing weak convergence by weak* convergence in the definition of properties KK and SKK, respectively. It is known that a dual space which is locally uniformly convex has property SKK* and that a space with property SKK* has the Radon-Nikodym property.

It is natural to define a property similar to UKK by replacing the weak convergence by weak* convergence in UKK and calling it UKK*. However, van Dulst and Sims [14] found the following definition is more useful.

(c) A dual Banach space E has property UKK* if for every $\varepsilon > 0$ there is a $0 < \delta < 1$ such that whenever A is a subset of the closed unit ball of E containing a sequence (x_n) with $\operatorname{sep}\big((x_n)\big) > \varepsilon$ then there is an x in weak*-closure (A) such that $\|x\| \leq \delta$.

They proved that a dual Banach space with property UKK* has property FPP* [14]. Moreover, they observed that if the dual unit ball is weak* sequentially compact then property UKK*, as defined above, is equivalent to the condition obtained from UKK by replacing weak convergence by weak* convergence. Finally, we define property SUKK* as follows:

(d) A dual Banach space E has property SUKK* if for every $\varepsilon > 0$ there is a $0 < \delta < 1$ such that whenever (x_α) is a net is the unit ball of E that converges to x in the weak* topology and $\operatorname{sep}\big((x_\alpha)\big) > \varepsilon$ then $\|x\| \leq \delta$. It is easy to see that property SUKK* implies property UKK*.

We summarize the relationships among the various concepts in the following diagram. We shall denote weak- (weak*-) normal structure by w.n.s. (w*.n.s.) and quasi-weak- (weak*-) normal structure by q.w.n.s. (q.w*.n.s.).(See [35] for the definitions.) It is understood that for some comparison to make sense we assume that we are in a dual space.

$$
\begin{array}{ccccccccc}
\text{SKK} & \Leftarrow & \text{SUKK} & \Leftarrow & \text{SUKK}^* & \Rightarrow & \text{SKK}^* & \Rightarrow & \text{q.w.n.s.} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
\text{KK} & \Leftarrow & \text{UKK} & \Leftarrow & \text{UKK}^* & \Rightarrow & \text{KK}^* & & \\
\Downarrow & & \Downarrow & & \Downarrow & & & & \\
\text{q.w.n.s} & \Leftarrow & \text{w.n.s.} & \Leftarrow & \text{w}^*\text{.n.s.} & \Rightarrow & \text{q.w}^*\text{.n.s.} & \Rightarrow & \text{q.w.n.s} \\
 & & \Downarrow & & \Downarrow & & & & \\
 & & \text{FPP} & \Leftarrow & \text{FPP}^* & & & &
\end{array}
$$

We now mention some examples which show that, generally, some of the above implications cannot be reversed.

(1) Let $X = (\ell_2 \oplus \ell_3 \oplus \cdots \oplus \ell_n \oplus \cdots)_2$. Then, as noted by Huff [26], X is reflexive, has property KK but not UKK. Moreover, we observe that (see also [7], [28] and [49]) X is separable, and being reflexive, its unit ball is weakly (or weak*) metrizable. Thus in X we have SKK* $\Leftrightarrow$ KK* $\Leftrightarrow$ SKK $\Leftrightarrow$ KK, and SUKK* $\Leftrightarrow$ UKK* $\Leftrightarrow$ SUKK $\Leftrightarrow$ UKK. So this same example shows that SKK* or SKK does not imply SUKK or SUKK*.

(2) Let ℓ_2 be renormed according to [50]. Then, as proved by Smith and Turett [50], ℓ_2 with this new norm is reflexive and locally uniformly rotund but does not have normal structure. Since the space is reflexive, normal structure is equivalent to weak-normal structure as well as weak*-normal structure. On the other hand, since the space is locally uniformly rotund, it has property SKK*. Consequently we have a space that has property SKK* (or SKK) but not weak*- (or weak-) normal structure. Note also we have an example of a space with quasi-weak*-normal structure but not weak*-normal structure.

Let $\mathcal{H}$ be a Hilbert space, and $\mathcal{T}(\mathcal{H})$ be the space of trace-class operators on $\mathcal{H}$. Lennard [42] showed that $\mathcal{T}(\mathcal{H})$ has property UKK*. Consequently, $\mathcal{T}(\mathcal{H})$ has weak*-normal structure. This answers affirmatively a question raised in [35]. See also [21] for related results.

3. The measure algebra $M(G)$ and other dual spaces

Let X be a locally compact Hausdorff space. Let $C_0(X)$ be the subspace of $CB(X)$ consisting of functions "vanishing at infinity," and $M(X)$ be the space of bounded regular Borel measure on X, with the variation norm. Let $M_d(X)$ be the subspace of $M(X)$ consisting of the discrete measures on X. It is well known that the dual of $C_0(X)$ can be identified with $M(X)$, and that $M_d(X)$ is isometrically isomorphic to $\ell_1(X)$. See [24].

Recall that a topological space X is scattered (or dispersed) if X does not contain any perfect subset. For more information on scattered spaces we refer the reader to [39].

In what follows let G be a locally compact group. For $f \in CB(G)$ we shall also denote $\ell_x f$ $(r_x f)$ by ${}_x f$ (f_x). The left orbit of f is defined by $LO(f) = \{{}_x f : x \in G\}$. Similarly we can define the right orbit $RO(f)$ of f. We recall the definitions of the following function spaces defined on G :

(a) $LUC(G) = \{f \in CB(G)$: the map $x \to {}_x f,\ x \in G$, is continuous when $CB(G)$ has the norm topology$\}$. This is the space of left uniformly continuous functions (or the right uniformly continuous functions in the language of [24]) on G.
(b) $WAP(G) = \{f \in CB(G) : LO(f)$ is relatively weakly compact in $CB(G)\}$. As is well known $f \in WAP(G)$ if and only if $RO(f)$ is relatively weakly compact.
(c) $AP(G) = \{f \in CB(G) : LO(f)$ is relatively norm compact$\}$. It is known that $f \in AP(G)$ if and only if $RO(f)$ is relatively norm compact. Functions in $WAP(G)\big(AP(G)\big)$ are called weakly almost periodic (almost periodic). For further information we refer the reader to [6].

Theorem 3.1. *Let G be a locally compact group. Then the following statements are equivalent.*

(1) *G is discrete.*
(2) *$M(G)$ is isometrically isomorphic to $\ell_1(G)$.*
(3) *$M(G)$ has property SUKK*.*
(4) *$M(G)$ has property* UKK*.
(5) *$M(G)$ has property SKK*.*
(6) *$M(G)$ has property KK*.*
(7) *Weak* convergence and weak convergence of sequences agree on the unit sphere of $M(G)$.*
(8) *$M(G)$ has weak* normal structure.*
(9) *$M(G)$ has weak*-fixed point property*
(10) *$M(G)$ has weak* fixed point property for left reversible semigroup.*

Theorem 3.2. *Let G be a locally compact group. Let N be a C^*-subalgebra of $WAP(G)$ containing $C_0(G)$ and the constants. Then the following statements are equivalent:*

(1) *G is finite.*
(2) *N^* has property SUKK*.*
(3) *N^* has property SKK*.*
(4) *N^* has property UKK*.*
(5) *N^* has property* KK*.
(6) *Weak* convergence and weak convergence for sequences agree on the unit sphere of N^*.*
(7) *N^* has weak*-normal structure.*

Remark 3.3. Let G be an infinite discrete group, and let N be the closed C^*-subalgebra generated by $C_0(G)$ and the constants. Then N is isometrically isomorphic to $C(\Delta)$, where Δ is the one-point compactification of G. Since Δ is scattered, it follows from Theorem 3.1 that N^* has property SUKK. However N^* does not have property KK* by Theorem 3.2. Thus SUKK does not imply KK*.

Theorem 3.4. *Let G be a locally compact group. Then*
(1) *Weak* convergence and weak convergence for sequences agree on the unit sphere of $LUC(G)^*$ if and only if G is discrete.*
(2) *$LUC(G)^*$ has weak*-normal structure if and only if G is finite.*

Theorem 3.5. *Let G be a locally compact group. Then $AP(G)^*$ has weak*-normal structure if and only if $AP(G)$ is finite dimensional.*

Remark 3.6. Results in this section are contained in [35], [22] and [23].
It should be noted that the implication of Theorem 3.1 (1) $\Rightarrow$ (4) follows from [44].

4. Fourier and Fourier-Stieltjes algebra

Let G be a locally compact group with a fixed left Haar measure λ. Let $L^1(G)$ be the group algebra of G with convolution product.

A function $\phi : G \to \mathbf{C}$ is *positive definite* if for any $x_1, x_2, \dots, x_n \in G$ and $\lambda_1, \dots, \lambda_n \in \mathbf{C}$,

$$\sum_{i,j=1}^{n} \lambda_i \overline{\lambda}_j \phi(x_i x_j^{-1}) \geq 0.$$

Denote the set of continuous positive definite functions on G by $P(G)$, and the set of continuous functions on G with compact support by $C_{00}(G)$. We define the Fourier-Stieltjes algebra of G, denoted by $B(G)$, to be the linear span of $P(G)$. Then $B(G)$ is a Banach algebra with the norm of each $\phi \in B(G)$ defined by

$$\|\phi\| = \sup_{f \in L^1(G), \|f\|_* \leq 1} \Big| \int f(t)\phi(t)d\lambda(t) \Big|.$$

where $\|f\|_* = \sup\{\|\pi(f)\|;$ where π is a *-representation of $L^1(G)\}$.

We define $C^*(G)$, the group C^*-algebra of G, to be the completion of $L^1(G)$ with respect to the norm

$$\|f\|_* = \sup \|\pi(f)\|,$$

where the supremum is taken over all nondegenerate $*$-representations π of $L^1(G)$ as a $*$-algebra of bounded operators on a Hilbert space. Let $\mathcal{B}\big(L^2(G)\big)$ be the set of all bounded operators on the Hilbert space $L^2(G)$ and ρ be the left regular representation of G, i.e., for each $f \in L^1(G)$, $\rho(f)$ is the bounded operator in $\mathcal{B}(L^2(G))$ defined by $\rho(f)(h) = f * h$, the convolution of f and h in $L^2(G)$. Denote by $C^*_\rho(G)$ the completion of $L^1(G)$ with the norm $\|\rho(f)\|$, $f \in L^1(G)$, and denote by $VN(G)$ the closure of $\{\rho(f) : f \in L^1(G)\}$ in the weak operator topology in $\mathcal{B}(L^2(G))$. In the case when G is left amenable, which is the case when G is compact, then $C^*(G)$ is isometric isomorphic to $C^*_\rho(G)$.

The Fourier algebra of G, denoted by $A(G)$, is defined to be the closed linear span of $P(G) \cap C_{00}(G)$. Clearly, $A(G) = B(G)$ when G is compact. It is known that $C^*(G)^* = B(G)$, where the duality is given by $\langle f, \phi\rangle = \int f(t)\phi(t)d\lambda(t)$, $f \in L^1(G)$, $\phi \in B(G)$, and $A(G)^* = VN(G)$.

For more properties of the Fourier algebra $A(G)$, the Fourier-Stieltjes algebra $B(G)$, the group C^*-algebra $C^*(G)$ and the von Neumann algebra $VN(G)$ we refer the reader to Eymard's paper [16] (see also [30] and [40]).

The group G is said to be an $[AU]$-group if the von Neumann algebra generated by every continuous unitary representation of G is atomic. It is said to be an $[AR]$-group if the von Neumann algebra $VN(G)$ is atomic. We have the following inclusions

$$[\text{compact}] \subseteq [AU] \subseteq [AR],$$

where [compact] denote the class of compact groups, etc. Moreover, these inclusions are proper [52] (see also [2]). The group G is called an $[IN]$-group if there is a compact neighbourhood of the identity e in G which is invariant under inner automorphisms; G is a $[SIN]$-group if there is a base for the neighbourhood system of e consisting of compact sets invariant under inner automorphisms (or equivalently, the left and right uniformities on G are the same). Obviously [compact] $\subseteq [SIN] \subseteq [IN]$ and the inclusions are also proper. Furthermore, all $[IN]$-groups are unimodular (see [17]).

For the undefined notations we refer the reader to the book [24].

Theorem 4.1. *Let G be a locally compact group. The following are equivalent:*

(a) *G is compact.*
(b) *$B(G)$ has the* UKK* *property.*
(c) *$B(G)$ has weak* normal structure.*
(d) *$B(G)$ has the weak* fixed point property for nonexpansive mappings.*

Let C be a non-empty subset of a Banach space X and $\{D_\alpha | \alpha \in \Lambda\}$ be a decreasing net of bounded non-empty subsets of X. For each $x \in C$, and $\alpha \in \Lambda$, let

$$r_\alpha(x) = \sup\{\|x - y\| \,\big|\, y \in D_\alpha\},$$
$$r(x) = \lim_\alpha r_\alpha(x) = \inf_\alpha r_\alpha(x),$$
$$r = \inf\{r(x) | x \in C\}.$$

The set (possibly empty)

$$\mathcal{AC}(\{D_\alpha \,|\, \alpha \in \Lambda\}) = \{x \in C \,|\, r(x) = r\}$$

is called the *asymptotic centre* of $\{D_\alpha \,|\, \alpha \in \Lambda\}$ with respect to C and r is the *asymptotic radius* of $\{D_\alpha \,|\, \alpha \in \Lambda\}$ with respect to C.

The notion of asymptotic centre is due to M. Edelstein [15]. See also [43].

Definition 4.2. Let E be a dual Banach space. We say that E has the *lim-sup property* for decreasing nets of bounded sets if the following property holds for any decreasing net $\{D_\alpha \,|\, \alpha \in \Lambda\}$ of bounded subsets of E, and any weak* convergent bounded nets $(\varphi_\mu)_{u\in I}$ with weak* limit φ :

$$r(\varphi) + \limsup_\mu \|\varphi_\mu - \varphi\| = \limsup_\mu r(\varphi_\mu), \tag{4.1}$$

i.e.:

$$\begin{aligned} &\lim_\alpha \sup\{\|\varphi - \psi\| \,\big|\psi \in D_\alpha\} + \limsup_\mu \|\varphi_\mu - \varphi\| \\ &\qquad = \limsup_\mu \lim_\alpha \sup\{\|\varphi_\mu - \psi\| \,\big|\psi \in D_\alpha\}. \end{aligned} \tag{4.2}$$

We say that E has the *asymptotic centre property* for decreasing nets of bounded subsets if for any non-empty weak* closed convex subset C in E and any decreasing net $\{D_\alpha \,|\, \alpha \in \Lambda\}$ of bounded non-empty subsets of C, the asymptotic centre of $\{D_\alpha \,|\, \alpha \in \Lambda\}$ with respect to C is a non-empty norm compact convex subset of C.

The "lim-sup property" for sequences was introduced by T.C. Lim in [44]. It was called "Lim's condition" in [34] in honour of T.C. Lim.

Theorem 4.3. *Let G be a locally compact group. The following are equivalent:*

(a) *G is compact.*
(b) *$B(G)$ has the lim-sup property.*
(c) *$B(G)$ has the asymptotic centre property.*
(d) *$B(G)$ has the weak* fixed point property for left reversible semigroups.*
(e) *$B(G)$ has the weak* fixed point property for nonexpansive mappings.*
(f) $\|\varphi\| + \limsup_\mu \|\varphi_\mu - \varphi\| = \limsup_\mu \|\varphi_\mu\|$ *for any bounded net (φ_μ) in $B(G)$ which converges to $\varphi \in B(G)$ in the weak* topology.*
(g) *For any net (φ_μ) in $B(G)$ and any $\varphi \in B(G)$ we have that $\|\varphi_\mu - \varphi\| \to 0$ if and only if $\varphi_\mu \to \varphi$ in the weak* topology and $\|\varphi_\mu\| \to \|\varphi\|$.*
(h) *On the unit sphere of $B(G)$ the weak* and the norm topology coincide.*

We recall that a Banach space E is said to have the *Radon-Nikodym property* (=RNP) if each closed convex subset D of E is dentable, i.e., for any $\varepsilon > 0$ there exists an x in D such that $x \notin \overline{\text{co}}\big(D \backslash B_\varepsilon(x)\big)$, where $B_\varepsilon(x) = \{y \in X : \|x - y\| < \varepsilon\}$ and $\overline{\text{co}}\, K$ is the closed convex hull of a set $K \subseteq E$. In general, there exists no connection between the RNP and the fpp, given that the RNP is an "isomorphic property" whereas the fpp is an "isometric property". However, for the preduals of von Neumann algebras we have the following. See also Lemma 1 in [35].

Theorem 4.4. *Let M be a von Neumann algebra, and M_* be its unique predual. The following are equivalent:*

(a) *M is atomic (i.e. generated by its minimal projections (see* [51]*)*
(b) *M_* has the Radon Nikodym property*
(c) *M_* has the weak fixed point property*

Theorem 4.5. *Let G be a locally compact group. The following are equivalent:*

(a) *$A(G)$ has the weak fixed point property*
(b) *$A(G)$ has the Radon Nikodym property*

(c) *The left regular representation of G is atomic (i.e. direct sum of irreducible unitary representations).*

In this case $A(G)$ has the weak fixed point property for left reversible semigroup.

Theorem 4.6. *Let G be a locally compact group. The following are equivalent:*

(a) *$B(G)$ has the weak fixed point property*
(b) *$B(G)$ has the Radon Nikodym property.*

A Banach space E is said to have the *fixed point property* if for every bounded closed convex subset K of E and every non-expansive mapping $T : K \to K$ has a fixed point. A well known result of Browder [4] shows that uniformly convex space has the fixed point property.

Theorem 4.7. *If G is a separable compact group, then $A(G)$ can be renormed to have the fixed point property.*

Theorem 4.8. *Let G be a locally compact group. If there is a non-zero closed ideal of $A(G)$ with the fixed point property, then G is discrete. In particular if $A(G)$ has the fixed point property, then G must be finite.*

Theorem 4.9. *The group C^*-algebra $C^*(G)$ of a locally compact group G has the fixed point property if and only if G is finite.*

Theorem 4.10. *Let G be a locally compact group. Then:*

(a) *$VN(G)$ has the weak fixed point property if and only if G is finite*
(b) *The group C^*-algebra $C^*(G)$ has the weak normal structure (i.e. every non-trivial weakly compact subset of $C^*(G)$ has normal structure) if and only if G is finite.*

5. Historical remarks and open problems

Remark 5.1.

(1) Theorem 4.1 (a) $\Rightarrow$ (b) was proved in [35], (b) $\Rightarrow$ (c) was proved in [14], (c) $\Rightarrow$ (d) was proved in [35], and (d) $\Rightarrow$ (a) follows from Theorem 4.5 in [18].
(2) Theorem 4.3 (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), (c) $\Rightarrow$ (d), and (e) $\Rightarrow$ (a) are proved in [18] and (h) $\Rightarrow$ (a) is in [3, Theorem 3.9]. For the case of separable $[IN]$ group, Theorem 5.3 (a) $\Rightarrow$ (d) $\Rightarrow$ (e) is in [36]. The equivalence of (a) $\Rightarrow$ (g) is in [3] (see also [33] and [35]).
(3) Theorem 4.4 (a) $\Rightarrow$ (b) is proved by [52]; (b) $\Rightarrow$ (c) is proved in [37] and (c) $\Rightarrow$ (b) is proved very recently in [19].
(4) Theorem 4.5 (b) $\Rightarrow$ (a) is proved in [37], (b) $\Rightarrow$ (a) in [48], (b) $\Rightarrow$ (c) in [52].
(5) Theorem 4.6 (b) $\Rightarrow$ (a) follows from [37] and (b) $\Rightarrow$ (a) is proved in [48].
(6) Theorem 4.7 is proved in [46]. It is a generalization of a recent result of P.K. Lin [45], that ℓ^1 can be renormed to have the fixed point property, answering a long standing open problem. Note that this result is not true for non-separable groups (see [13]).
(7) Theorem 4.8 is proved in [33].

Open problem 1. Let G be a locally compact group. Let $B_\rho(G)$ denote the reduced Fourier-Stieltjes algebra of $B(G)$ i.e. $B_\rho(G)$ is the weak* closure of $C_{00}(G) \cap B(G)$. Then $B_\rho(G) = C_\rho(G)^*$. Does the weak* fixed point property on $B_\rho(G)$ imply G is compact? This is true when G is amenable by Theorem 4.3, since $B(G) = B_\rho(G)$ is this case.

Open problem 2. Let G be a locally compact group. Does the asymptotic centre property on $B_\rho(G)$ imply that G is compact?

Open problem 3. It is point out in [18] that if E be a dual Banach space. Then the following implications hold:

$$\text{(a)} \Rightarrow \text{(b)} \Rightarrow \text{(c)} \quad \text{and} \quad \text{(a)} \Rightarrow \text{(d)} \Rightarrow \text{(e)} \Rightarrow \text{(f)}$$

where

(a) E has the lim-sup property.
(b) E has the asymptotic centre property.
(c) E has the weak* fixed point property for left reversible semigroups.
(d) $\|\varphi\| + \limsup_\mu \|\varphi_\mu - \varphi\| = \limsup_\mu \|\varphi_\mu\|$ for any bounded net (φ_μ) in E converging weak* to $\varphi \in E$.
(e) For any net (φ_μ) in E and any $\varphi \in E$ we have that $\|\varphi_\mu - \varphi\| \to 0$ if and only if $\varphi_\mu \to \varphi$ in the weak* topology and $\|\varphi_\mu\| \to \|\varphi\|$.
(f) The weak* topology and the norm topology coincide on the unit sphere S of E.

For which dual Banach space E are the conditions equivalent? Theorem 4.3 shows that this is the case when $E = B(G)$ of a compact group. In particular, this is the case when $E = \ell^1$ (which is isometrically isomorphic to $B(\pi)$, the Fourier-Stieltjes algebra of the circle group π) regarded as the dual of c_0.

Theorem 4.9 was proved recently by Dhompongsa, Fupingwong and Lawton [12] (see also [11]). Theorem 4.1 was proved in [37].

Open problem 4 (R. Bruck). If E is a Banach space with the weak fixed point property, does E have the weak fixed point property for left reversible semigroups. R. Bruck shows that [5] this is the case when the semigroup is commutative.

Open problem 5. If E is a dual Banach space with the weak* fixed point property, does E have the weak* fixed point property for commutative semigroups, or more general for left reversible semigroup?

References

[1] D. Alspach, *A fixed point free nonexpansive map*, Proc. Amer. Math. Soc. **82** (1981), 423–424.

[2] L. Baggett, *A separable group having a discrete dual space is compact*, J. Funct. Anal. **10** (1972), 131–148.

[3] M. B. Bekka, E. Kaniuth, A. T.-M. Lau and G. Schlichting, *Weak***-closedness of subspaces of Fourier-Stieltjes algebras and weak***-continuity of the restriction map*, Trans. Amer. Math. Soc. **350** (1998), 2277–2296.

[4] F. E. Browder, *Nonexpansive nonlinear operators in Banach spaces*, Proc. Nat. Acad. Sci. U.S.A. **54** (1965), 1041–1044.

[5] R. E. Bruck, *A common fixed point theorem for a commutative family of nonexpansive mappings*, Pacific J. Math. **53** (1974), 59–71.

[6] R. B. Burckel, *Weakly Almost Periodic Functions on Semigroups*, Gordon and Breach, New York, 1970.

[7] D. F. Cudia, *Rotundity*, Proc. Sympos. Pure Math. **7**, Convexity, Amer. Math. Soc., Providence, R.I., 1963, pp. 73–97.

[8] M. M. Day, *Normed Linear Spaces*, Springer-Verlag, New York, 1973.

[9] T. D. Benavides and M. A. Japon Pineda, *Fixed points of nonexpansive mappings in spaces of continuous functions*, Proc. Amer. Math. Soc. **133** (2005), 3037–3046.

[10] T. D. Benavides, M. A. Japon Pineda and S. Prus, *Weak compactness and fixed point property for affine maps*, J. Funct. Anal. **209** (2004), 1–15.

[11] S. Dhompongsa and W. Fupingwong, *The fixed point property of unital Banach algebras*, Fixed Point Theory and Applications **2010** (2010), Article ID 36259, 12 pp.

[12] S. Dhompongsa, W. Fupingwong and W. Lawton, *Fixed point properties for* C^**-algebras*, J. Math. Anal. Appl. **374** (2011), 22–28.

[13] P. N. Dowling, C. J. Lennard and B. Turett, *The fixed point property for subsets of some classical Banach spaces*, Nonlinear Anal. **49** (2002), 141–145.

[14] D. van Dulst and B. Sims, *Fixed points of nonexpansive mappings and Chebyshev centers in Banach spaces with norms of type* (K, K), in: Banach Space Theory and Its Applications, Proceedings Bucharest, 1981, in Lecture Notes in Math., **991**, Springer-Verlag, 1983.

[15] M. Edelstein, *The construction of asymptotic centre with a fixed point property*, Bull. Amer. Math. Soc. **78** (1972), 206–208.

[16] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181–236.

[17] G. B. Folland, *A Course in Abstract Harmonic Analysis*, Stud. Adv. Math., CRC Press, Boca Raton, FL., 1995.

[18] G. Fendler, A. T.-M. Lau and M. Leinert, *Weak** *fixed point property and asymptotic centre for the Fourier-Stieltjes algebra of a locally compact group*, J. Funct. Anal. **264** (2013), 288–302.

[19] G. Fendler and M. Leinert, *Separable* C^**-algebras and weak** *fixed point property*, Probability and Mathematical Statistics (to appear).

[20] K. Goebel and W.A. Kirk, *Topics in Metric, Fixed Point Theory*, University Press, Cambridge, 1990.

[21] J. Garcia Falset and B. Sims,*Property (M) and weak fixed point property*, Proc. Amer. Math. Soc. (1997), 2891–2986.

[22] C. C. Graham, A. T.-M. Lau and M. Leinert, *Separable translation-invariant subspaces of* $M(G)$ *and other dual spaces on locally compact groups*, Colloq. Math. **55** (1988), 131–145.

[23] E. E. Granirer and M. Leinert, *On some topologies which coincide on the unit sphere of the Fourier-Stieltjes algebra* $B(G)$ *and of the measure algebra* $M(G)$, Rocky Mountain J. Math. **11** (1981), 459–472.

[24] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. Vol. I*, Structure of Topological Groups, Integration Theory, Group Representations, Grundlehren Math. Wiss. **115** Academic Press Inc., New York 1963.

[25] R. D. Holmes and A. T.-M. Lau, *Nonexpansive actions of topological semigroups and fixed points*, J. London Math. Soc. **5** (1972), 330–336.

[26] R. Huff, *Banach spaces which are nearly uniformly convex*, Rocky Mountain J. Math. **10** (1980), 743–750.

[27] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), 1004–1006.

[28] H. E. Lacey, *The Isometric Theory of Classical Banach Spaces*, Springer-Verlag, New York, 1974.

[29] A. T.-M. Lau, *Invariant means on almost periodic functions and fixed point properties*, Rocky Mountain J. Math. **3** (1973), 69–76.

[30] A. T.-M. Lau, *Uniformly continuous functionals on the Fourier algebra of any locally compact group*, Trans. Amer. Math. Soc. **251** (1979), 39–59.

[31] A. T.-M. Lau, *Semigroup of nonexpansive mappings on a Hilbert space*, J. Math. Anal. Appl. **105** (1985), 514–522.

[32] A. T.-M. Lau, *Amenability and fixed point property for semigroup of nonexpansive mappings*, in: Fixed Point Theory and Applications, Marseille, 1989, Pitman Res. Notes Math. Ser., **252**, Longman Sci. Tech., Harlow, 1991, pp. 303–313.

[33] A. T.-M. Lau and M. Leinert, *Fixed point property and Fourier algebra of a locally compact group*, Trans. Amer. Math. Soc. **360** (2008), 6389–6402.

[34] A. T.-M. Lau and P. F. Mah, *Quasi-normal structures for certain spaces of operators on a Hilbert space*, Pacific J. Math. **121** (1986), 109–118.

[35] A. T.-M. Lau and P. F. Mah, *Normal structure in dual Banach spaces associated with a locally compact group*, Trans. Amer. Math. Soc. **310** (1988), 341–353.

[36] A. T.-M. Lau and P. F. Mah, *Fixed point property for Banach algebras associated to locally compact groups*, J. Funct. Anal. **258** (2010), 357–372.

[37] A. T.-M. Lau, P. F. Mah and A. Ülger, *Fixed point property and normal structure for Banach spaces associated to locally compact groups*, Proc. Amer. Math. Soc. **125** (1997), 2021–2027.

[38] A. T.-M. Lau and W. Takahashi, *Weak convergence and nonlinear ergodic theorems for reversible semigroups of nonexpansive mappings*, Pacific J. Math. **126** (1987), 277–294.

[39] A. T.-M. Lau and W. Takahashi, *Invariant submeans and semigroups of non-expansive mappings on Banach spaces with normal structure*, J. Funct. Anal. **25** (1996), 79–88.

[40] A. T.-M. Lau and A. Ülger, *Some geometric properties on the Fourier and Fourier-Stieltjes algebras of locally compact groups, Arens regularity and related problems*, Trans. Amer. Math. Soc. **337** (1993), 321–359.

[41] A. T.-M. Lau and Y. Zhang. *Fixed point properties of semigroups of non-expansive mappings*, J. Funct. Anal. **254** (2008), 2534–2554.

[42] C. Lennard, C_1 *is uniformly Kadec-Klee*, Proc. Amer. Math. Soc. **109** (1990), 71–77.

[43] T. C. Lim, *Characterization of normal structures*, Proc. Amer. Math. Soc. **43** (1974), 313–319.

[44] T. C. Lim, *Asymptotic centres and nonexpansive mappings in conjugate Banach spaces*, Pacific J. Math. **90** (1980), 135–143.

[45] P. K. Lin, *There is an equivalent norm on* ℓ_2 *that has the fixed point property*, Nonlinear Analysis **68** (2008), 2303–2308.

[46] C. A. H. Linares and M. A. Japon, *A renorming in some Banach spaces with applications to fixed point theory*, J. Funct. Anal. **258** (2010), 3452–3468.

[47] T. Mitchell, *Fixed points of left reversible semigroups of non-expansive mappings*, Kodai Math. Sem. Rep. **22** (1970), 322–323.

[48] N. Randrianantoanina, *Fixed point properties for semigroups of nonexpansive mappings*, J. Funct. Anal. **258** (2010), 3801–3817.

[49] Z. Semandeni, *Banach Spaces of Continuous Functions*, PNW, Warsaw, 1971.

[50] M. A. Smith and B. Turret, *A reflexive LUR Banach space that lacks normal structure*, Canad. Math. Bull. **28** (1985), 492–494.

[51] M. Takesaki, *Theory of Operator Algebras I*, Springer-Verlag, New York, 1979.

[52] K. Taylor, *Geometry of the Fourier algebras and locally compact groups with atomic representations*, Math. Ann. **262** (1983), 183–190.

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