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A HEMIVARIATIONAL-LIKE INEQUALITY WITH APPLICATIONS

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Abstract. In this work, we study the following *hemivariational-like inequality* (HV-L-I) for finding $x = (x_1, \dots, x_n) \in K$ such that for all $u = (u_1, \dots, u_n) \in K$,

$$A_i^o(Tx;\ \eta_i(T_iu_i, T_ix_i)) + \langle B_ix,\ \mu_i(u_i, x_i)\rangle_i + J_i^o(Sx;\ \xi_i(S_iu_i, S_ix_i)) \geq 0\ (i = \overline{1,n}).$$

The solution of the hemivariational-like inequality gives the position of the state equilibrium of the structure. As applications, we study the existence of a Nash-type derivative equilibrium point for a family $\{A_i^o : i = \overline{1,n}\}$ of the partial Clarke derivatives of a family $\{A_i : i = \overline{1,n}\}$ of locally Lipschitz continuous functions $A_i : Y \to \mathbb{R}$ $(i = \overline{1,n})$. Our results generalize and improve the corresponding results in [5, 9, 15].

1. Introduction

Generally, the existence theory of the solutions to variational inequalities is based on monotonicity, convexity, continuity, differentiability, or coercivity conditions of the given functions. If the corresponding energy functions involved are nonconvex, another type of inequality expressions arises as variational formulations of the problem which are called hemivariational inequalities. Their derivations are based on the mathematical notions of the generalized gradient of Clarke. In 1983, Panagiotopoulos [10] firstly introduced the hemivariational inequality and investigated its properties by using the mathematical notion of the generalized gradient of Clarke (that is, generalized directional derivative) for nonconvex and nondifferentiable functions in [2]. Since then, many kinds of hemivariational inequalities have been introduced and studied with their applications in infinite-dimensional spaces [1, 3, 5, 8, 11, 13–15, 17]. Xiao and Huang [11] introduced a generalized quasi-variational-like inequality with multivalued-pseudomonotone operarors and studied the existence of solutions to it in Banach spaces. Panagiotopoulos [12] studied the coercive and semicoercive hemivariational inequalities. Liu [7] considered the existence of solutions to the quasi-variational hemi-variational inequalities with multivalued, discontinuous pseudomonotone operators. Carl et al. [1] considered quasi-linear elliptic variational hemivariational inequalities involving convex, lower semi-continuous and locally Lipschitz functionals by developing the sub-supersolution method for variational-hemivariational inequalities including existence, comparison, compactness and extremality results. Noor [9] analyzed some iterative method for solving hemivariational inequalities by using auxiliary principles. He showed a

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variety of techniques to suggest and analyze various iterative algorithms for solving hemivariational inequalities and equilibrium problems. In 2003, Molnar and Vas [9] introduced nonlinear hemivariational-like inequality systems and considered the solvability of the systems with the existence of Nash generalized derivative points as applications.

Inspired and motivated by recent works [5, 6, 8, 9, 11, 14, 15, 17], we study the following *hemivariational-like inequality* (HV-L-I) for finding $x = (x_1, \ldots, x_n) \in K$ such that for all $u = (u_1, \ldots, u_n) \in K$,

$$A_i^o(Tx;\ \eta_i(T_iu_i, T_ix_i)) + \langle B_ix,\ \mu_i(u_i, x_i)\rangle_i + J_i^o(Sx;\ \xi_i(S_iu_i, S_ix_i)) \geq 0\ (i = \overline{1, n}).$$

The solution of the hemivariational-like inequalities gives the position of the state equilibrium of the structure. As applications, we study the existence of a Nash-type derivative equilibrium point for a family $\{A_i^o : i = \overline{1, n}\}$ of the partial Clarke derivatives of a family $\{A_i : i = \overline{1, n}\}$ of locally Lipschitz continuous functions $A_i : Y \to \mathbb{R}$ $(i = \overline{1, n})$. Our results generalize and improve the corresponding results in [5, 9, 15].

2. Preliminaries with basic assumptions

The following FKKM Theorem due to Ky Fan and the equivalent Tarafdar fixed point theorem are very useful in our main result.

Definition 2.1. Let X be a vector space and $E \subset X$. A set valued mapping $G : E \to 2^X$ is called a KKM mapping if for any finite subset $\{x_1, x_2, \ldots, x_n\}$ of E the following holds

$$conv\,(\{x_1, x_2, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} G(x_i).$$

Theorem 2.2 ([4, FKKM Theorem])**.** *Suppose that X is a locally convex Hausdorff topological space, $E \subset X$ and $G : E \to 2^X$ is a closed set-valued KKM mapping. If there exists $x_0 \in E$ such that $G(x_0)$ is compact, then*

$$\bigcap_{x \in E} G(x) \neq \emptyset.$$

The following theorem is called Tarafdar fixed point theorem [14].

Theorem 2.3. *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Let $G : K \to 2^K$ be a set-valued mapping such that*

(a) *for each $x \in K, G(x)$ is a nonempty convex subset of K,*
(b) *for each $y \in K, G^{-1}(y) = \{x \in K : y \in G(x)\}$ contains an open set O_y which may be empty,*
(c) $\bigcup_{y \in K} O_y = K$ *and*
(d) *there exists a nonempty set K_0 contained in a compact convex subset K_1 of K such that*

$$D = \bigcap_{y \in K_0} O_y^c$$

is either empty or compact (O_y^c is the complement of O_y in K).

Then there exists a point $x_0 \in K$ such that

$$x_0 \in G(x_0).$$

In this paper, K_i is a subset of a reflexive Banach space X_i and Y_i , Z_i are two Banach spaces, $T_i : X_i \to Y_i, S_i : X_i \to Z_i$, $\eta_i : Y_i \times Y_i \to Y_i$, $\xi_i : Z_i \times Z_i \to Z_i$ and $\mu_i : K_i \times K_i \to X_i$ are mappings, X_i^* is the topological dual of X_i, $\langle \cdot, \cdot \rangle$ the associated duality mapping between X_i and X_i^* $(i = \overline{1, n})$,

$K = \prod_{i=1}^{n} K_i$, $X = \prod_{i=1}^{n} X_i$, $Y = \prod_{i=1}^{n} Y_i$ and $Z = \prod_{i=1}^{n} Z_i$, unless specifically noted.

Let $A_i : X \times X_i \to \mathbb{R}$ be a continuous function, which is locally Lipschitz continuous in the ith variable with the partial Clarke directional derivative at the point $x_i \in X_i$ in the direction of $u_i \in X_i$ as

$$\begin{aligned} &A_i^o(x_1, x_2, \ldots, x_i, \ldots, x_n; u_i) \\ &= \limsup_{z \to x_i, \tau \searrow 0} \frac{A_i(x_1, \ldots, z + \tau u_i, \ldots, x_n) - A_i(x_1, \ldots, z, \ldots, x_n)}{\tau}. \end{aligned}$$

Needed Notations

$x = (x_1, x_2, \ldots, x_n),\ u = (u_1, u_2, \ldots, u_n) \in X,$
$y = (y_1, y_2, \ldots, y_n),\ v = (v_1, v_2, \ldots, v_n) \in Y,$
$z = (z_1, z_2, \ldots, z_n),\ w = (w_1, w_2, \ldots, w_n) \in Z,$
$Tx = (T_1x_1, \ldots, T_nx_n)$ and $Sx = (S_1x_1, \ldots, S_nx_n),$
$\eta(y, v) = (\eta_1(y_1, v_1), \eta_2(y_2, v_2), \ldots, \eta_n(y_n, v_n)),$
$\xi(z, w) = (\xi_1(z_1, w_1), \xi_2(z_2, w_2), \ldots, \xi_n(z_n, w_n)),$
$\mu(x, u) = (\mu_1(x_1, u_1), \mu_2(x_2, u_2), \ldots, \mu_n(x_n, u_n)),$
$A(Tx,\ \eta(Tu, Tx)) = \sum_{i=1}^{n} A_i^o(Tx;\ \eta_i(T_iu_i, T_ix_i)),$
in particular, $A(Tx,\ Tu - Tx) = \sum_{i=1}^{n} A_i^o(Tx,\ T_iu_i - T_ix_i),$
$B(x,\ \mu(u, x)) = \sum_{i=1}^{n} \langle B_ix,\ \mu_i(u_i, x_i)\rangle_i,$
in particular, $B(x,\ u - x) = \sum_{i=1}^{n} \langle B_ix,\ u_i - x_i\rangle_i$, where $B_i : K \to X_i^*$ is a mapping.

Basic assumptions

For all $i = \overline{1, n}$,

(i) $T_i : X_i \to Y_i$ and $S_i : X_i \to Z_i$ are compact,
(ii) $A_i^o : X \times X_i \to \mathbb{R}$ is upper semi-continuous,
(iii) the function $x \mapsto \langle B_ix, x_i\rangle_i$ is weakly upper semi-continuous for every $x_i \in X_i$ $(i = \overline{1, n})$, where $B_i : K \to X_i^*$,
(iv) the mappings $\eta_i : Y_i \times Y_i \to X_i$, $\xi_i : Z_i \times Z_i \to Z_i$ and $\mu_i : K_i \times K_i \to X_i$ are continuous, positive homogeneous and linear in the first variable satisfying $\eta_i(y_i, v_i) = -\eta_i(v_i, y_i)$, $\xi_i(z_i, w_i) = -\xi_i(w_i, z_i)$ and $\mu_i(x_i, u_i) = -\mu_i(u_i, x_i)$,
(v) η_i and ξ_i are compact.

Definition 2.4 ([2])**.** Let X be a Banach space and $j : X \to \mathbb{R}$ a locally Lipschitz continuous function. We say that j is regular at $x \in X$ if for all $u \in X$ the one sided directional derivative $j'(x; u)$ exists and $j'(x; u) = j^o(x; u)$. If j is regular at every point $x \in X$, we say that j is regular, where $j^o(x;u) = \lim\sup_{\substack{z\to x \\ t\to 0^+}} \frac{j(z+ut)-j(z)}{t}$ is the Clarke directional derivative of f at x in the direction of u.

Proposition 2.5 ([2]). *Let $J : Z \to \mathbb{R}$ be a regular, locally Lipschitz continuous function, then the followings hold:*

(a) $\partial J(z_1, z_2, \ldots, z_n) \subseteq \partial_1 J(z_1, z_2, \ldots, z_n) \times \cdots \times \partial_n J(z_1, z_2, \ldots, z_n)$, *where $\partial_i J$ denotes the Clarke subdifferential in the ith variable* $(i = \overline{1, n})$,

(b) $J^o(z_1, z_2, \ldots, z_n; w_1, w_2, \ldots, w_n) \leq \sum_{i=1}^n J_i^o(z_1, z_2, \ldots, z_n; w_i)$, *where J_i^o denotes the Clarke derivative in the ith variable and*

(c) $J^o(z_1, z_2, \ldots, z_n; 0, \ldots, w_i, \ldots, 0) \leq J_i^o(z_1, z_2, \ldots, z_n; w_i)$.

3. Main results

Theorem 3.1. *Let K_i be nonempty bounded convex and A_i satisfy the assumption (ii). Assume that T_i and S_i satisfy the assumption (i) and B_i satisfies the assumption (iii). Assume that η_i, ξ_i and μ_i satisfy the assumption (iv) and positive homogeneous, linear in the first variable. Assume that η_i and ξ_i satisfy the assumption (v) $(i = \overline{1, n})$. Let $J : Z \to \mathbb{R}$ be a regular, locally Lipschitz continuous function, then* (HV-L-I) *is solvable.*

Proof. From the assumptions (i), (iv) and (v), T_i and η_i are compact and $\eta(\cdot, \cdot)$ is semi-continuous, so $A(Tx, \eta(Tu, Tx))$ is weakly upper semi-continuous. From the assumptions (iii) and (iv), $B(x, \mu(u, x))$ is also weakly upper semi-continuous. Since $J^o(\cdot, \cdot)$ and $\xi(\cdot, \cdot)$ are upper semi-continuous, from the assumptions (i) and (v), $J^o(Sx; \xi(Su, Sx))$ is also weakly upper semi-continuous. Thus a function $H : K \to \mathbb{R}$ defined by

$$H(x) = A(Tx, \eta(Tu, Tx)) + B(x, \eta(u, x)) + J^o(Sx; \xi(Su, Sx))$$

is weakly upper semi-continuous.

Now we continue our proof by using two methods; one is established by FKKM theorem and the other is by Tarafdar fixed point theorem.

One Proof

Proof. Let $G : K \to 2^K$ be a set-valued mapping defined by, for $u \in K$,

$$G(u) = \{x \in K : A(Tx, \eta(Tu, Tx)) + B(x, \mu(u, x)) + J^o(Sx; \xi(Su, Sx)) \geq 0\}.$$

For every $u \in K$, obviously $G(u) \neq \emptyset$. Taking into account that the function

$$x \mapsto A(Tx,\ \eta(Tu, Tx)) + B(x,\ \mu(u, x)) + J^o(Sx;\ \xi(Su, Sx))$$

is weakly upper semi-continuous, it follows that $G(u)$ is weakly closed. Now we prove that G is a KKM-mapping by the contradiction. Let $u^1, u^2, \ldots, u^i, \ldots, u^n \in K$ and $x \in conv\ (\{u^1, u^2, \ldots, u^n\})$ be such that

$$x \notin \bigcup_{i=1}^n G(u^i),$$

where $u^i = (u_1^i, u_2^i, \ldots, u_n^i) \in K$ $(i = \overline{1, n})$.

Then we have

$$A(Tx,\ \eta(Tu^i,Tx)) + B(x,\ \mu(u^i,x)) + J^o(Sx;\ \xi(Su^i,Sx)) < 0 \quad \text{for } i=\overline{1,n}. \tag{3.1}$$

Since $x=\sum_{i=1}^n t_iu^i$ for $t_1,t_2,\dots,t_n \in [0,1]$ with $\sum_{i=1}^n t_i = 1$, by multiplying the inequality (3.1) with t_i ($i=\overline{1,n}$) and then, adding the following n inequalities;

$$t_i[A(Tx,\eta(Tu^i,Tx)) + B(x,\ \mu(u^i,x)) + J^o(Sx;\xi(Su^i,Sx))] < 0,$$
$$\text{for } i=\overline{1,n},$$

we obtain

$$\begin{aligned} &A(Tx,\eta(Tx,Tx)) + B(x,\mu(x,x)) + J^o(Sx;\xi(Sx,Sx)) \\ &\quad = A\Big(Tx,\eta\Big(T\Big(\sum_{i=1}^n t_iu^i\Big),Tx\Big)\Big) + B\Big(x,\mu\Big(\sum_{i=1}^n t_iu^i,x\Big)\Big) \\ &\qquad + J^o\Big(Sx;\xi\Big(S\Big(\sum_{i=1}^n t_iu^i\Big),Sx\Big)\Big) \\ &\quad < 0 \end{aligned} \tag{3.2}$$

from the fact that $A(\cdot,\cdot), B(\cdot,\cdot)$ and $J^o(\cdot,\cdot)$ are continuous, positive homogeneous and affine in the second variable and η,μ and ξ are positive homogeneous and linear in the first variable.

Consequently, (3.2) contradicts the fact that

$$\begin{aligned} &A(Tx,\eta(Tx,Tx)) + B(x,\mu(x,x)) + J^o(Sx;\xi(Sx,Sx)) \\ &\quad = A(Tx,0) + B(x,0) + J^o(Sx;0) = 0 \end{aligned}$$

by the assumption (iv). On the other hand, the set K is bounded convex and closed, by Eberlein-Smulian Theorem it is weakly compact, therefore it is weakly closed. Since $G(u) \subset K$ is weakly closed, $G(u)$ is also weakly compact.

Now from Theorem 2.2, we have

$$\bigcap_{u\in K} G(u) \neq \emptyset.$$

Hence there exists $x \in K$ such that for every $u \in K$

$$A(Tx,\ \eta(Tu,Tx)) + B(x,\ \mu(u,x)) + J^o(Sx;\ \xi(Su,Sx)) \geq 0,$$

which implies that

$$\sum_{i=1}^n A_i^o\big(Tx;\eta_i(T_iu_i,T_ix_i)\big) + \sum_{i=1}^n \langle B_ix,\mu_i(u_i,x_i)\rangle_i + \sum_{i=1}^n J_i^o\big(Sx;\xi_i(S_iu_i,S_ix_i)\big) \geq 0. \tag{3.3}$$

Fixing an $i=\overline{1,n}$ and putting $u_j = x_j$ for $j \neq i$ in (3.3), by the assumption (iv) and Proposition 2.5(c) we have

$$A_i^o(Tx;\ \eta_i(T_iu_i,T_ix_i)) + \big\langle B_ix,\ \mu_i(u_i,x_i)\big\rangle_i + J_i^o(Sx;\ \xi_i(S_iu_i,S_ix_i)) \geq 0 \quad \text{for } i=\overline{1,n}.$$

□

Another Proof

Proof. Assume that the conclusion fails, then for each $x \in K$, there exists $u \in K$ such that

$$A(Tx, \eta(Tu, Tx)) + B(x, \mu(u, x)) + J^o(Sx; \xi(Su, Sx)) < 0.$$

Define a set-valued mapping $G : K \to 2^K$ by, for $x \in K$

$$G(x) = \{u \in K : A(Tx, \eta(Tu, Tx)) + B(x, \mu(u, x)) + J^o(Sx; \xi(Su, Sx)) < 0\}.$$

By assumption, the set

$$G(x) \neq \emptyset \text{ for every } x \in K.$$

Since the function $A(\cdot, \cdot) + B(\cdot, \cdot) + J^o(\cdot, \cdot)$ is convex in the second variable, $G(x)$ is a convex set. Now the function

$$x \mapsto A(Tx,\ \eta(Tu, Tx)) + B(x,\ \mu(u, x)) + J^o(Sx;\ \xi(Su, Sx))$$

is weakly upper semi-continuous due to the weak upper semi-continuities of η, μ and ξ

$$[G^{-1}(u)]^c = \{x \in K : A(Tx, \eta(Tu, Tx)) + B(x, \mu(u, x)) + J^o(Sx; \xi(Su, Sx)) \geq 0\}$$

is weakly closed, hence $G^{-1}(u) := \{x \in K : u \in G(x)\}$ is weakly open.

Now we verify that

$$\bigcup_{u \in K} G^{-1}(u) = K.$$

For every $u \in K$ we have

$$G^{-1}(u) \subset K,$$

therefore

$$\bigcup_{u \in K} G^{-1}(u) \subset K.$$

Conversely, let $x \in K$ be fixed. Since $G(x) \neq \emptyset$ there exists $u_0 \in K$ such that $u_0 \in G(x)$. Hence $x \in G^{-1}(u_0)$, which shows that $K \subset G^{-1}(u_0) \subset \bigcup_{u \in K} G^{-1}(u)$.

Now we show that

$$D := \bigcap_{u \in K} [G^{-1}(u)]^c$$

is empty or weakly compact.

Indeed, if $D \neq \emptyset$ then D is a weakly closed subset of K, since it is the intersection of weakly closed sets. By the weak compactness of K, D is also weakly compact. Taking $O_u = G^{-1}(u)$ and $K_0 = K_1 = K$ we can apply Theorem 2.3 to conclude that there exists $x_0 \in K$ such that

$$x_0 \in G(x_0).$$

Hence

$$A(Tx_0, \eta(Tx_0, Tx_0)) + B(x_0, \mu(x_0, x_0)) + J^o(Sx_0; \xi(Sx_0, Sx_0)) < 0,$$

which is a contradiction to the assumption (iv).

Thus there exists a $x \in K$ such that for every $u \in K$

$$A(Tx, \eta(Tu, Tx)) + B(x, \mu(u, x)) + J^o(Sx; \xi(Su, Sx)) \geq 0.$$

By the same method shown in the first proof, we obtain the existence of solution to (HV-L-I). $\square$

In 2013, Molnar and Vas [9] considered the existence of solutions to the following nonlinear hemivariational-like inequality systems :

Finding $x = (x_1, \dots, x_n) \in K$ such that

$$A_i(x;\ \delta_i(x_i, u_i)) + I_i^o(Tx;\ T(\delta(x, u))) \geq 0 \tag{3.4}$$

for all $u = (u, \dots, u_n) \in K$, where K_i is bounded, closed and convex, a function $A_i : X \times X_i \to \mathbb{R}$ is nonlinear, a function $I : Y \to \mathbb{R}$ is regular locally Lipschitz and $\delta_i : X_i \times X_i \to X_i$ is a mapping $(i = \overline{1, n})$.

They showed the existence of solutions to the system (3.4) by using Lin's fixed point theorem for set-valued mappings in [7] under the following conditions ; for $i = \overline{1, n}$,

1. $\delta_i(x_i, x_i) = 0$ for all $x_i \in X_i$,
2. $\delta_i(x_i, 0)$ is linear for all $x_i \in X_i$,
3. for each $u_i \in X_i$, $\delta_i(x_i^m, u_i) \rightharpoonup \delta_i(x_i, u_i)$, whenever $x_i^m \rightharpoonup x_i$,
4. $A_i(x; 0) = 0$ for all $x_i \in X_i$,
5. for all $u_i \in X_i$, the mapping $x \longmapsto A_i(x; \delta_i(x_i, u_i))$ is weakly upper semicontinues,
6. the mapping $u_i \longmapsto \sum_{i=1}^{n} A_i(x; \delta_i(x_i, u_i))$ is convex for each $x \in X$.

Remark 3.2. Suppose that K_i $(i = \overline{1, n})$ is unbounded in Theorem 3.1. By imposing the following coercivity condition;
there exist a compact set D_i in K_i and $u_i \in D_i$ $(i = \overline{1, n})$ such that

$$A(Tx,\ \eta(Tu, Tx)) + B(x,\ \mu(u, x)) + J^o(Sx;\ \xi(Su, Sx)) < 0$$

for all $x = (x_1, x_2, \dots, x_i, \dots, x_n) \in K \setminus D$, where $D := \prod_{i=1}^{n} D_i$ and $u = (u_1, u_2, \dots, u_i, \dots, u_n)$, we obtain the solvability for (HV-L-I).

Remark 3.3. Putting $y_i = (T_i u_i, T_i x_i) = T_i u_i - T_i x_i, \mu_i(u_i, x_i) = u_i - x_i$ and $\xi_i(S_i u_i, S_i x_i) = S_i u_i - S_i x_i$ for $i = \overline{1, n}$, we obtain corresponding results in [15] as corollaries of Theorem 3.1 and Remark 3.2, respectively.

4. Applications

The solution of (HV-L-I) gives the position of the state equilibrium of the structure. As applications, we study the existence of a Nash-type derivative equilibrium point for a family $\{A_i^o : i = \overline{1, n}\}$ of the partial Clarke derivatives of a family $\{A_i : i = \overline{1, n}\}$ of locally Lipschitz continuous functions $A_i : Y \to \mathbb{R}$ $(i = \overline{1, n})$.

Definition 4.1. A point $u = (u_1, u_2, \dots, u_i, \dots, u_n) \in K$ is called a Nash-type derivative equilibrium point of a family $\{f_i : K \to \mathbb{R} : i = \overline{1, n}\}$ of functions with respect to $\{\eta_i : i = \overline{1, n}\}$ if

$$f_i^o(u;\ \eta_i(x_i, u_i)) \geq 0 \ (i = \overline{1, n})$$

$$\text{for } x = (x_1, x_2, \dots, x_i, \dots, x_n) \in K.$$

Theorem 4.2. *Let X_i, Y_i be reflexive Banach spaces, $K_i \subset X_i$ bounded closed and convex sets, $T_i : K_i \to Y_i$ compact and linear mappings and $A_i : Y \to \mathbb{R}$ locally Lipschitz continuous function in the ith variable ($i = \overline{1,n}$).*

Assume that A_i satisfies the assumption (ii) and $\eta_i : K_i \times K_i \to K_i$ the assumption (iv) ($i = \overline{1,n}$). Then there exists $u = (u_1, u_2, \ldots, u_i, \ldots, u_n) \in K$ such that

$$A_i^o(T_1u_1, \ldots, T_iu_i \ldots, T_nu_n; \eta_i(T_ix_i, T_iu_i)) \geq 0$$

$$\textit{for all } x = (x_1, x_2, \ldots, x_i, \ldots, x_n) \in K,$$

which means that $Tu := (T_1u_1, \ldots, T_iu_i, \ldots, T_nu_n)$ is a Nash-type derivative equilibrium point for $\{A_i : i = \overline{1,n}\}$ with respect to η_i ($i = \overline{1,n}$).

Proof. Putting $B_i = 0$ ($i = \overline{1,n}$) and $J = 0$ in Theorem 3.1, we have the desired result. □

Remark 4.3. Furthemore, by putting $K_i = Y_i$ and $T_i = I$ the identitiy ($i = \overline{1,n}$) in Theorem 4.1, we also obtain a Nash-type derivative equilibrium point $u = (u_1, \ldots, u_i, \ldots, u_n) \in K$, as shown in Theorem 4.1 [9] under suitable weaker conditions as a corollary.

Theorem 4.4. *Let K_i be closed and convex ($i = \overline{1,n}$). Assume that there exist a bounded and closed set $D_i \subset K_i$ and $u_i \in D_i$ ($i = \overline{1,n}$) such that*

$$A(Tx; \eta(Tu, Tx)) < 0$$

$$\textit{for all } x = (x_1, x_2, \ldots, x_i, \ldots, x_n) \in K \setminus D,$$

where $D := \prod_{i=1}^{n} D_i$ and $u = (u_1, u_2, \ldots, u_i, \ldots, u_n) \in D$. Then there exists $v = (v_1, v_2, \ldots, v_i, \ldots, v_n) \in K$ such that $A_i^o(Tv; \eta_i(T_ix_i, T_iv_i)) \geq 0$ for all $x = (x_1, x_2, \ldots, x_i, \ldots, x_n) \in K$, which implies that $v = (v_1, v_2, \ldots, v_i, \ldots, v_n)$ is a Nash-type derivative equilibrium point for $\{A_i : i = \overline{1,n}\}$ with respect to η_i ($i = \overline{1,n}$).

We consider the existence of solutions to (HV-L-I) for a differentiable function $A_i : Y \to \mathbb{R}$.

Let $Y_i = Z_i$, $A_i : Y \to \mathbb{R}$ be differentiable in the ith variable and $\eta_i, \xi_i : Y_i \times Y_i \to Y_i$ be mappings ($i = \overline{1,n}$). Assume that the function $A_i^{'} : Y \times Y_i \to \mathbb{R}$ is continuous and $J : Y \to \mathbb{R}$ is regular locally Lipschitz. Under these assumptions we have the following result.

Corollary 4.5. *Let $J, A_i : Y \to \mathbb{R}$ be the functions mentioned above. Suppose that the assumption (i) holds and $\xi_i, \eta_i : Y_i \times Y_i \to Y_i$ satisfy the assumption (iv). Let $K_i \subset X_i$ ($i = \overline{1,n}$) be a bounded closed and convex set. Then there exists an element $u = (u_1, u_2, \ldots, u_i, \ldots, u_n) \in K$ such that*

$$A_i^{'}(Tu;\ \eta_i(T_ix_i, T_iu_i)) + J_i^o(Tu;\ \xi_i(T_ix_i, T_iu_i)) \geq 0$$

$$\textit{for all } x = (x_1, x_2, \ldots, x_i, \ldots, x_n) \in K \ (i = \overline{1,n}).$$

If $A_i = 0$ in Theorem 3.1, then we also have the following existence result for (HV-L-I).

Corollary 4.6. *Let K_i be a bounded closed and convex subset of a reflexive Banach spaces X_i $(i = \overline{1,n})$. Assume that $B_i : K \to X_i^*$ satisfies the assumption (iii) and $\mu_i, \xi_i : K_i \times K_i \to K_i$ satisfy the assumption (iv). Let $J : Z \to \mathbb{R}$ be a regular locally Lipschitz function and assumption (i) holds. Then there exists $u = (u_1, u_2, \ldots, u_i, \ldots, u_n) \in K$ such that*

$$\langle B_i u,\ \mu_i(x_i, u_i)\rangle_i + J_i^o(Su;\ \xi_i(S_i x_i, S_i u_i)) \geq 0$$

$$\text{for all } x = (x_1, x_2, \ldots, x_i, \ldots, x_n) \in K \ (i = \overline{1,n}).$$

The above result generalizes the main result in Kristaly [5].

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