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TRIPLED FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIVE MAPPINGS IN PARTIALLY ORDERED G-METRIC SPACES

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Abstract. In this paper, we prove some tripled coincidence point and tripled common fixed point theorems for nonlinear contractive mappings having the mixed g-monotone property in partially ordered G-matric spaces. The results on fixed point theorems are generalizations of the results of Aydi and Karapinar in [1], extending Wangkeeree and Bantaojai's [30] and some others in the related topics.

1. Introduction and preliminaries

One of the most well-known and most useful results in the fixed point theory is the Banach Caccioppoli contraction mapping principle [4], a powerful tool in analysis. This principle has been generalized in different directions in different spaces by mathematicians over the years (see e.g. [4], [15], [27]–[29] and references mentioned therein). On the other hand, fixed point theory has received much attention in metric spaces endowed with a partial ordering. Existence and uniqueness of a fixed point for contraction type mappings in partially ordered metric spaces were first discussed by Ran and Reurings [23] in 2004. Later so many results were reported on existence and uniqueness of a fixed point and its applications in partially ordered metric space (see [2]–[30]).

In 1987, the notion of coupled fixed point was introduced by Guo and Lakshmikantham [12]. And in 2006, Bhaskar and Lakshmikantham [6] reconsidered the concept of coupled fixed point in partially ordered metric spaces by introducing the notion of a mixed monotone mapping. They proved some coupled fixed point theorems for mixed monotone mapping and considered the existence and uniqueness of solution for periodic boundary value problem.

Very recently, Berinde and Borcut [5] introduced the concept of tripled fixed point theorems by virtue of mixed monotone mappings. Their contributions generalized and extended Bhaskar and Lakshmikantham's work for nonlinear mappings.

Mustafa and Sims ([21], [22]) introduce a new structure of generalized metric spaces which are called G-metric spaces, as a generalization of metric spaces to develop and introduce a new fixed point theory for various mappings in the new structure. Later, several fixed point theorems in G-metric spaces were obtained (see e.g. [2], [3], [8], [14], [18], [20], [21], [24], [30]).

The notion of fixed point of order $N \geq 3$ was first introduced by Samet and Vetro [25]. Very recently, Aydi and Karapinar [1] used the concept of tripled fixed

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point introduced by Berinde and Borcut [5] to prove some new tripled fixed point theorems in partially ordered metric spaces depended on another function.

Let's recall some basic definitions from [5] and [21].

Definition 1.1 ([5]). Let X be a nonempty set and let $F : X \times X \times X \to X$ be a mapping. An element $(x, y, z) \in X \times X \times X$ is said to be a *tripled fixed point* of F if

$$F(x, y, z) = x\ , \quad F(y, x, y) = y\ , \quad F(z, y, x) = z.$$

From now on, we shall denote $X \times X \times X$ by X^3 and $g(x)$ by gx.

Definition 1.2 ([5]). Let $(X, \preceq)$ be a parially ordered set and let $F : X^3 \to X$ be a mapping. We say that F has the *mixed monotone property* if $F(x, y, z)$ is monotone non-decreasing in x and$z$ and is monotone non-increasing in y, that is, for any $x, y, z \in X$

$$\begin{aligned}
x_1, x_2 \in X, x_1 \preceq x_2 &\implies F(x_1, y, z) \preceq F(x_2, y, z),\\
y_1, y_2 \in X, y_1 \preceq y_2 &\implies F(x, y_1, z) \succeq F(x, y_2, z),\\
z_1, z_2 \in X, z_1 \preceq z_2 &\implies F(x, y, z_1) \preceq F(x, y, z_2).
\end{aligned}$$

Definition 1.3. Let $(X, \preceq)$ be a parially ordered set. Let $F : X^3 \to X$ and $g : X \to X$ be two mappings. We say that F has the *mixed g-monotone property* if for any $x, y, z \in X$

$$\begin{aligned}
x_1, x_2 \in X, gx_1 \preceq gx_2 &\implies F(x_1, y, z) \preceq F(x_2, y, z),\\
y_1, y_2 \in X, gy_1 \preceq gy_2 &\implies F(x, y_1, z) \succeq F(x, y_2, z),\\
z_1, z_2 \in X, gz_1 \preceq gz_2 &\implies F(x, y, z_1) \preceq F(x, y, z_2).
\end{aligned}$$

Definition 1.4 ([5]). Let $F : X^3 \to X$ and $g : X \to X$ be two mappings. An element $(x, y, z) \in X^3$ is said to be:

(1) a *tripled coincidence point* of F and g if $F(x, y, z) = gx, F(y, x, y) = gy$ and $F(z, y, x) = gz$

(2) a *tripled common fixed point* of F if $F(x, y, z) = gx = x, F(y, x, y) = gy = y$ and $F(z, y, x) = gz = z$

Definition 1.5 ([5]). Let X be a nonempty set, then we say that the mapping $F : X^3 \to X$ and $g : X \to X$ are *commutative* if for any $x, y, z \in X$

$$g\left(F(x, y, z)\right) \;=\; F(gx, gy, gz)$$

Definition 1.6 ([21]). Let X be a nonempty set. Let $G : X \times X \times X \to \mathbb{R}$ be a function satisfying the following properties :

(G_1) $G(x, y, z) = 0$ if $x = y = z$;
(G_2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$;
(G_3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$
(symmetry in all three variables) ;
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$
(rectangular inequality).

Then the function G is called a *G-metric* on X and (X, G) is called a *G-metric space.*

Definition 1.7 ([21])**.** Let X be a G-metric space, and let $\{x_n\}$ be a sequence of points of X, a point $x \in X$ is said to be the *limit* of a sequence $\{x_n\}$ if $G(x, x_n, x_m) \to 0$ as $n, m \to \infty$ and sequence $\{x_n\}$ is said to be *G-convergent to x.*

Definition 1.8 ([21])**.** Let X be a G-metric space, a sequence $\{x_n\}$ is called *G-Cauchy* if for every $\varepsilon > 0$, there is a positive integer N such that $G(x_n, x_m, x_\ell) < \varepsilon$ for all $n, m, \ell \geq N$, that is, if $G(x_n, x_m, x_\ell) \to 0$, as $n, m, l \to \infty$.

We state the following lemmas.

Lemma 1.9 ([21])**.** *If X is a G-metric space, then the following are equivalent:*
(1) *$\{x_n\}$ is G-convergent to x.*
(2) *$G(x_n, x_n, x) \to 0$ as $n \to \infty$.*
(3) *$G(x_n, x, x) \to 0$ as $n \to \infty$.*
(4) *$G(x_m, x_n, x) \to 0$ as $n, m \to \infty$.*

Lemma 1.10 ([21])**.** *If X is a G-metric space, then the following are equivalent:*
(1) *$\{x_n\}$ is G-Cauchy.*
(2) *For every $\varepsilon > 0$, there exists a positive integer N such that*

$$G(x_n, x_m, x_m) < \varepsilon \ ; \ \text{for all } n, m \geq N.$$

Lemma 1.11 ([21])**.** *If X is a G-metric space, then*

$$G(x, y, y) \leq 2G(y, x, x) \ ; \quad \text{for all } x, y \in X.$$

Lemma 1.12 ([21])**.** *If X is a G-metric space, then*

$$G(x, x, y) \leq G(x, x, z) + G(z, z, y) \ ; \quad \text{for all } x, y, z \in X.$$

Lemma 1.13 ([21])**.** *Let $(X, G), (X', G')$ be two G-metric spaces. A mapping $f : X \to X'$ is G-continuous at $x \in X$ if and only if it is G sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G'-convergent to $f(x)$.*

Definition 1.14 ([21])**.** A G-metric space X is called a *symmetric* G-metric space if

$$G(x, y, y) = G(y, x, x) \ ; \quad \text{for all } \ x, y \in X.$$

Definition 1.15 ([21])**.** A G-metric space X is said to be *G-complete* (or *complete G-metric space*) if every G-Cauchy sequence in X is convergent in X.

Definition 1.16. Let X be a G-metric space. A mapping $F : X^3 \to X$ is said to be *continuous* if for any three G-convergent sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converging to x, y and z respectively, $\{F(x_n, y_n, z_n)\}$ is G-convergent to $F(x, y, z)$.

Let Ψ denote the class of all functions $\varphi : [0, \infty) \times [0, \infty) \to [0, \infty)$ satisfying the following condition :

$$\lim_{\substack{t_1 \to r_1 \\ t_2 \to r_2}} \varphi(t_1, t_2) > 0$$

for all $(r_1, r_2) \in [0, \infty) \times [0, \infty)$ with $r_1 + r_2 > 0$.

Recently, Wangkeeree and Bantaojai [30] proved the following coupled fixed point and coupled coincidence point theorems for generalized contractive mappings in partially ordered G-metric spaces.

Theorem 1.17 ([30]). *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Let $g : X \to X$ be a mapping and $F : X \times X \to X$ be a mapping having the mixed g-monotone property on X. Suppose that there exists $\varphi \in \Psi$ such that*

$$\begin{aligned} M_F^G(x,u,w,y,v,z) \ &\leq \ [G(gx,gu,gw) + G(gy,gv,gz)] \\ &\quad - 2\varphi\left(G(gx,gu,gw), G(gy,gv,gz)\right), \end{aligned} \tag{1.1}$$

for all $x, y, z, u, v, w \in X$, for which $gx \succeq gu \succeq gw$ and $gy \preceq gv \preceq gz$ where

$$M_F^G(x,u,w,y,v,z) \ = \ G(F(x,y), F(u,v), F(w,z)) + G(F(y,x), F(v,u), F(z,w)).$$

If there exists $x_0, y_0 \in X$ such that

$$gx_0 \ \preceq F(x_0, y_0) \quad \text{and} \quad gy_0 \succeq F(y_0, x_0),$$

and suppose $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F, and also suppose either

(a) *F is continuous, or*

(b) *X has the following property:*

  (i) *if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*

  (ii) *if a non-increasing sequence $y_n \to y$, then $y_n \succeq y$ for all n,*

  then F and g have a coupled coincidence point, that is, there exists $(x, y) \in X \times X$ such that $gx = F(x, y)$ and $gy = F(y, x)$.

Theorem 1.18 ([30]). *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Suppose that there exists $\varphi \in \Psi$ such that*

$$\begin{aligned} &G\left(F(x,y), F(u,v), F(w,z)\right) + G\left(F(y,x), F(v,u), F(z,w)\right) \\ &\qquad \leq \ [G(x,u,w) + G(y,v,z)] - 2\varphi\left(G(x,u,w), G(y,v,z)\right), \end{aligned} \tag{1.2}$$

for all $x \succeq u \succeq w$ and $y \preceq v \preceq z$. Suppose that either

(a) *F is continuous or,*

(b) *X has the following property:*

  (i) *if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*

  (ii) *if a non-increasing sequence $y_n \to y$, then $y_n \succeq y$ for all n.*

  If there exists $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point in X.

On the other hand, Aydi and Karapinar [1] proved tripled fixed point theorems in partially ordered metric spaces depended on another function which generalized the theorem of Berinde and Borcut [5]. They proved the following results.

Definition 1.19. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be ICS if T is injective, continuous and has the property : for every sequence $\{x_n\}$ in X, if $\{Tx_n\}$ is convergent then $\{x_n\}$ is also convergent.

Let Φ be the class of all functions $\phi : [0, \infty) \to [0, \infty)$ such that

(1) ϕ is non-decreasing,
(2) $\phi t < t$ for all $t > 0$,
(3) $\lim_{r \to t^+} \phi r < t$ for all $t > 0$.

Theorem 1.20 ([1]). *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X which that (X, d) is a complete metric space. Suppose that $T : X \to X$ is an ICS and $F : X^3 \to X$ is such that F has the mixed monotone property. Assume that there exists $\phi \in \Phi$ such that*

$$d(TF(x, y, z), TF(u, v, w)) \leq \phi\left(max\{d(Tx, Tu), d(Ty, Tv), d(Tz, Tw)\}\right) \tag{1.3}$$

for any $x, y, z \in X$, for which $x \preceq u, v \preceq y$ and $z \preceq w$, Suppose either

(a) *F is continuous, or*
(b) *X has the following property :*
  (i) *if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*
  (ii) *if a non-increasing sequence $y_n \to y$, then $y_n \succeq y$ for all n,*
  (iii) *if a non-decreasing sequence $z_n \to z$, then $z_n \preceq z$ for all n.*

Suppose also that there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0, z_0), y_0 \succeq F(y_0, x_0, y_0) \quad and \quad z_0 \preceq F(z_0, y_0, x_0),$$

then F has a tripled fixed point, that is, there exists $(x, y, z) \in X^3$ such that

$$x = F(x, y, z)\ ,\ y = F(y, x, y) \quad and \quad z = F(z, y, x),$$

In this paper, inspired by Wangkeeree and Bantaojai [30] and Aydi and Karapinar [1], we prove some tripled fixed point and tripled coincidence point theorems for generalized contractive mappings in partially ordered G-metric space which are generalization of Aydi and Karapinar [1], extending Wankeeree and Bantaojai [30] and many others in the related topics.

2. Main results

We start with a tripled coincidence point theorem. Let Θ be the class of all functions $\psi : [0, \infty) \times [0, \infty) \times [0, \infty) \to [0, \infty)$ satisfying condition :

$$\lim_{\substack{t_1 \to r_1 \\ t_2 \to r_2 \\ t_3 \to r_3}} \psi(t_1, t_2, t_3) > 0\ ;\ \text{for all } (r_1, r_2, r_3) \in [0, \infty)^3 \text{ with } r_1 + r_2 + r_3 > 0$$

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and (X, G) be a complete G-metric space. Let $F : X^3 \to X$ and $g : X \to X$ be two mappings such that F has the mixed g-monotone property on X. Suppose that there exists $\psi \in \Theta$ such that*

$$\begin{aligned} & G\left(F(x, y, z), F(u, v, w), F(h, k, \ell)\right) \\ & \qquad + G\left(F(y, z, x), F(v, w, u), F(k, \ell, h)\right) \\ & \qquad + G\left(F(z, x, y), F(w, u, v), F(\ell, h, k)\right) \\ \leq\ & [G(gx, gu, gh) + G(gy, gv, gk) + G(gz, gw, g\ell)] \\ & - 3\psi\left(G(gx, gu, gh), G(gy, gv, gk), G(gz, gw, g\ell)\right) \end{aligned} \tag{2.1}$$

for all $x, y, z, u, v, w, h, k, \ell \in X$, for which $gx \succeq gu \succeq gh$ and $gy \preceq gv \preceq gk$ and $gz \succeq gw \succeq g\ell$.

Assume that $F(X \times X \times X) \subseteq g(X)$, g is continuous and commute with F, and also suppose that either

(a) *F is continuous, or*

(b) *X has the following property :*

  (i) *if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*

  (ii) *if a non-increasing sequence $y_n \to y$, then $y_n \succeq y$ for all n,*

  (iii) *if a non-decreasing sequence $z_n \to z$, then $z_n \preceq z$ for all n.*

Suppose also that there exist $x_0, y_0, z_0 \in X$ such that

$$gx_0 \preceq F(x_0, y_0, z_0), gy_0 \succeq F(y_0, x_0, y_0) \quad \text{and} \quad gz_0 \preceq F(z_0, y_0, x_0),$$

then F and g have tripled coincidence point, that is, there exists $(x, y, z) \in X^3$ such that

$$gx = F(x, y, z)\ ,\ gy = F(y, x, y) \quad \text{and} \quad gz = F(z, y, x).$$

Proof. Let $x_0, y_0, z_0 \in X$ such that

$$gx_0 \preceq F(x_0, y_0, z_0)\ ,\ gy_0 \succeq F(y_0, x_0, y_0) \text{ and } gz_0 \preceq F(z_0, y_0, x_0).$$

Since $F(X \times X \times X) \subseteq g(X)$, we can choose $x_1, y_1, z_1 \in X$ such that

$$gx_1 = F(x_0, y_0, z_0)\ ,\ gy_1 = F(y_0, x_0, y_0) \text{ and } gz_1 = F(z_0, y_0, x_0).$$

Again, since $F(X \times X \times X) \subseteq g(X)$, we can choose $x_2, y_2, z_2 \in X$ such that

$$gx_2 = F(x_1, y_1, z_1)\ ,\ gy_2 = F(y_1, x_1, y_1) \text{ and } gz_2 = F(z_1, y_1, x_1).$$

Continuing this process, we can construct sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X such that

$$\begin{aligned} gx_{n+1} = F(x_n, y_n, z_n)\ ,\ gy_{n+1} = F(y_n, x_n, y_n) \quad &\text{and} \\ gz_{n+1} = F(z_n, y_n, x_n).& \end{aligned} \tag{2.2}$$

Since F has the mixed g-monotone property, then by using a mathematical induction, one can show that

$$\begin{aligned} gx_n \preceq gx_{n+1}\ ,\ gy_n &\succeq gy_{n+1} \quad \text{and} \\ gz_n &\preceq gz_{n+1} \quad \text{for all } n \geq 0. \end{aligned} \tag{2.3}$$

Since $gx_n \preceq gx_{n+1}, gy_n \succeq gy_{n+1}$ and $gz_n \preceq gz_{n+1}$ for all $n \geq 0$, so from (2.1), we have

$$\begin{aligned} &G(gx_{n+1}, gx_{n+1}, gx_n) + G(gy_{n+1}, gy_{n+1}, gy_n) + G(gz_{n+1}, gz_{n+1}, gz_n) \\ &\quad = G\left(F(x_n, y_n, z_n), F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1})\right) \\ &\qquad + G\left(F(y_n, x_n, y_n), F(y_n, x_n, y_n), F(y_{n-1}, x_{n-1}, y_{n-1})\right) \\ &\qquad + G\left(F(z_n, y_n, x_n), F(z_n, y_n, x_n), F(z_{n-1}, y_{n-1}, x_{n-1})\right) \\ &\quad \leq [G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1}) + G(gz_n, gz_n, gz_{n-1})] \\ &\qquad - 3\psi\left(G(gx_n, gx_n, gx_{n-1}), G(gy_n, gy_n, gy_{n-1}), G(gz_n, gz_n, gz_{n-1})\right). \end{aligned} \tag{2.4}$$

Setting

$$\omega_{n+1}^x := G(gx_{n+1}, gx_{n+1}, gx_n),$$

$$\begin{aligned}\omega_{n+1}^y &:= G(gy_{n+1}, gy_{n+1}, gy_n),\\ \omega_{n+1}^z &:= G(gz_{n+1}, gz_{n+1}, gz_n) \quad \text{for all } n \geq 0,\end{aligned}$$

we have, by (2.4), that

$$\omega_{n+1}^x + \omega_{n+1}^y + \omega_{n+1}^z \;\leq\; \omega_n^x + \omega_n^y + \omega_n^z - 3\psi\left(\omega_n^x, \omega_n^y, \omega_n^z\right) \quad \text{for all } n \geq 0. \tag{2.5}$$

As $\psi(t_1, t_2, t_3) \geq 0$ for all $(t_1, t_2, t_3) \in [0, \infty)^3$, from (2.5) we have

$$\omega_{n+1}^x + \omega_{n+1}^y + \omega_{n+1}^z \;\leq\; \omega_n^x + \omega_n^y + \omega_n^z \quad \text{for all } n \geq 0.$$

Then the sequence $\{\omega_n^x + \omega_n^y + \omega_n^z\}$ is decreasing.

Therefore, there exists $\omega \geq 0$ such that

$$\begin{aligned}\lim_{n\to\infty} (\omega_n^x + \omega_n^y + \omega_n^z) &= \lim_{n\to\infty} \Big(G(gx_{n+1}, gx_{n+1}, gx_n)\\ &\quad + G(gy_{n+1}, gy_{n+1}, gy_n) + G(gz_{n+1}, gz_{n+1}, gz_n)\Big)\\ &= \omega.\end{aligned} \tag{2.6}$$

Now, we show that $\omega = 0$. Suppose, to contrary, that $\omega > 0$. From (2.6), the sequences $\{G(gx_{n+1}, gx_{n+1}, gx_n)\}$, $\{G(gy_{n+1}, gy_{n+1}, gy_n)\}$ and $\{G(gz_{n+1}, gz_{n+1}, gz_n)\}$ have convergent subsequences $\big\{G(gx_{n(j)+1}, gx_{n(j)+1}, gx_{n(j)}\big\}$, $\big\{G(gy_{n(j)+1}, gy_{n(j)+1}, gy_{n(j)}\big\}$ and $\big\{G(gz_{n(j)+1}, gz_{n(j)+1}, gz_{n(j)}\big\}$ respectively. Suppose that

$$\begin{aligned}\lim_{j\to\infty} \omega_{n(j)+1}^x &= \lim_{j\to\infty} G(gx_{n(j)+1}, gx_{n(j)+1}, gx_{n(j)}) = \omega_1,\\ \lim_{j\to\infty} \omega_{n(j)+1}^y &= \lim_{j\to\infty} G(gy_{n(j)+1}, gy_{n(j)+1}, gy_{n(j)}) = \omega_2,\\ \lim_{j\to\infty} \omega_{n(j)+1}^z &= \lim_{j\to\infty} G(gz_{n(j)+1}, gz_{n(j)+1}, gz_{n(j)}) = \omega_3,\end{aligned}$$

for which $\omega_1 + \omega_2 + \omega_3 = \omega$. From (2.5), we have

$$\omega_{n(j)+1}^x + \omega_{n(j)+1}^y + \omega_{n(j)+1}^z \;\leq\; \omega_{n(j)}^x + \omega_{n(j)}^y + \omega_{n(j)}^z - 3\psi(\omega_{n(j)}^x, \omega_{n(j)}^y, \omega_{n(j)}^z). \tag{2.7}$$

Taking the limit as $j \to \infty$ in the above inequality, we obtain

$$\omega \;\leq\; \omega - 3 \lim_{j\to\infty} \psi\left(\omega_{n(j)}^x, \omega_{n(j)}^y, \omega_{n(j)}^z\right) \;<\; \omega,$$

which is a contradiction. Therefore $\omega = 0$; that is

$$\begin{aligned}&\lim_{n\to\infty} \Big[G(gx_{n+1}, gx_{n+1}, gx_n) + G(gy_{n+1}, gy_{n+1}, gy_n) + G(gz_{n+1}, gz_{n+1} gz_n)\Big]\\ &\qquad = \lim_{n\to\infty} \left(\omega_{n+1}^x + \omega_{n+1}^y + \omega_{n+1}^z\right) \;=\; 0.\end{aligned} \tag{2.8}$$

We next show that $\{gx_n\}, \{gy_n\}$ and $\{gz_n\}$ are G-Cauchy sequences. On the contrary, assume that at least one of $\{gx_n\}, \{gy_n\}$ and $\{gz_n\}$ is not a G-Cauchy sequence. By lemma 1.10, thare is an $\varepsilon > 0$ for which we can find subsequence $\big\{gx_{n(k)}\big\}, \big\{gx_{m(k)}\big\}$ of $\{gx_n\}$, $\big\{gy_{n(k)}\big\}, \big\{gy_{m(k)}\big\}$ of $\{gy_n\}$ and $\big\{gz_{n(k)}\big\}, \big\{gz_{m(k)}\big\}$ of $\{gz_n\}$ with $n(k) > m(k) \geq k$ such that

$$\begin{aligned}&G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)})\\ &\qquad + G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) \;\geq\; \varepsilon.\end{aligned} \tag{2.9}$$

Corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfying (2.9). Then

$$\begin{aligned} G(gx_{n(k)-1}, gx_{n(k)-1}, gx_{m(k)}) + G(gy_{n(k)-1}, gy_{n(k)-1}, gy_{m(k)}) \\ + G(gz_{n(k)-1}, gz_{n(k)-1}, gz_{m(k)}) \ < \ \varepsilon. \end{aligned} \tag{2.10}$$

By Lemma 1.12, we have

$$\begin{aligned} G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) \ \leq \ & G(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}) \\ & + G(gx_{n(k)-1}, gx_{n(k)-1}, gx_{m(k)}), \end{aligned} \tag{2.11}$$

$$\begin{aligned} G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) \ \leq \ & G(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}) \\ & + G(gy_{n(k)-1}, gy_{n(k)-1}, gy_{m(k)}), \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) \ \leq \ & G(gz_{n(k)}, gz_{n(k)}, gz_{n(k)-1}) \\ & + G(gz_{n(k)-1}, gz_{n(k)-1}, gz_{m(k)}). \end{aligned} \tag{2.13}$$

Using (2.9)-(2.13), we obtain

$$\begin{aligned} \varepsilon \ & \leq G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) \\ & \quad + G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) \\ & \leq G(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}) + G(gx_{n(k)-1}, gx_{n(k)-1}, gx_{m(k)}) \\ & \quad + G(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}) + G(gy_{n(k)-1}, gy_{n(k)-1}, gy_{m(k)}) \\ & \quad + G(gz_{n(k)}, gz_{n(k)}, gz_{n(k)-1}) + G(gz_{n(k)-1}, gz_{n(k)-1}, gz_{m(k)}) \\ & < G(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}) + G(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}) \\ & \quad + G(gz_{n(k)}, gz_{n(k)}, gz_{n(k)-1}) + \varepsilon. \end{aligned}$$

Letting $k \to \infty$ in the last inequality and using (2.8), we have

$$\begin{aligned} \lim_{k\to\infty} \Big[& G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) \\ & + G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) \Big] \ = \ \varepsilon. \end{aligned} \tag{2.14}$$

By Lemma 1.11 and Lemma 1.12, we have

$$\begin{aligned} G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) & \leq G(gx_{n(k)}, gx_{n(k)}, gx_{n(k)+1}) \\ & \quad + G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)}) \\ & \leq 2G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{n(k)}) \\ & \quad + G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}) \\ & \quad + G(gx_{m(k)+1}, gx_{m(k)+1}, gx_{m(k)}), \end{aligned} \tag{2.15}$$

$$\begin{aligned} G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) & \leq 2G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{n(k)}) \\ & \quad + G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}) \\ & \quad + G(gy_{m(k)+1}, gy_{m(k)+1}, gy_{m(k)}), \end{aligned} \tag{2.16}$$

and

$$
\begin{aligned}
G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) \leq\ & 2G(gz_{n(k)+1}, gz_{n(k)+1}, gz_{n(k)}) \\
&+ G(gz_{n(k)+1}, gz_{n(k)+1}, gz_{m(k)+1}) \\
&+ G(gz_{m(k)+1}, gz_{m(k)+1}, gz_{m(k)}).
\end{aligned} \tag{2.17}
$$

By (2.15), (2.16) and (2.17), we obtain

$$
\begin{aligned}
G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) &+ G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) \\
&+ G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) \\
&\leq 2\left(\omega^x_{n(k)+1} + \omega^y_{n(k)+1} + \omega^z_{n(k)+1}\right) \\
&\quad + \left(\omega^x_{m(k)+1} + \omega^y_{m(k)+1} + \omega^z_{m(k)+1}\right) \\
&\quad + G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}) \\
&\quad + G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}) \\
&\quad + G(gz_{n(k)+1}, gz_{n(k)+1}, gz_{m(k)+1}).
\end{aligned} \tag{2.18}
$$

Since $n(k) > m(k)$, we have that

$$g(x_{n(k)}) \succeq g(x_{m(k)}) \ , \ g(y_{n(k)}) \preceq g(y_{m(k)}) \ \text{ and } \ g(z_{n(k)}) \succeq g(z_{m(k)}),$$

and also, from (2.1),

$$
\begin{aligned}
&G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}) + G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}) \\
&\quad + G(gz_{n(k)+1}, gz_{n(k)+1}, gz_{m(k)+1}) \\
&= G\left(F(x_{n(k)}, y_{n(k)}, z_{n(k)}), F(x_{n(k)}, y_{n(k)}, z_{n(k)}), F(x_{m(k)}, y_{m(k)}, z_{m(k)})\right) \\
&\quad + G\left(F(y_{n(k)}, x_{n(k)}, y_{n(k)}), F(y_{n(k)}, x_{n(k)}, y_{n(k)}), F(y_{m(k)}, x_{m(k)}, y_{m(k)})\right) \\
&\quad + G\left(F(z_{n(k)}, y_{n(k)}, x_{n(k)}), F(z_{n(k)}, y_{n(k)}, x_{n(k)}), F(z_{m(k)}, y_{m(k)}, x_{m(k)})\right) \\
&\leq \big[G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) \\
&\quad + G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)})\big] - 3\psi\big(G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}), \\
&\qquad\qquad G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}), G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)})\big).
\end{aligned} \tag{2.19}
$$

From (2.18) and (2.19), we have

$$
\begin{aligned}
&2\left(\omega^x_{n(k)+1} + \omega^y_{n(k)+1} + \omega^z_{n(k)+1}\right) + \left(\omega^x_{m(k)+1} + \omega^y_{m(k)+1} + \omega^z_{m(k)+1}\right) \\
&\geq G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) + G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) \\
&\quad - G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}) - G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}) \\
&\quad - G(gz_{n(k)+1}, gz_{n(k)+1}, gz_{m(k)+1}) \\
&\geq G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) + G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) \\
&\quad - \left[G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) + G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)})\right] \\
&\quad + 3\psi\left(G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}), G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}), G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)})\right) \\
&= 3\psi\left(G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}), G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}), G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)})\right).
\end{aligned} \tag{2.20}
$$

This implies that

$$\begin{aligned}
&2\left(\omega^x_{n(k)+1}+\omega^y_{n(k)+1}+\omega^z_{n(k)+1}\right)+\left(\omega^x_{m(k)+1}+\omega^y_{m(k)+1}+\omega^z_{m(k)+1}\right)\\
&\quad\geq 3\psi\left(G(gx_{n(k)},gx_{n(k)},gx_{m(k)}),G(gy_{n(k)},gy_{n(k)},gy_{m(k)}),\right.\\
&\qquad\left.G(gz_{n(k)},gz_{n(k)},gz_{m(k)})\right).
\end{aligned}\tag{2.21}$$

From (2.14), the sequence $\left\{G(gx_{n(k)},gx_{n(k)},gx_{m(k)})\right\},\left\{G(gy_{n(k)},gy_{n(k)},gy_{m(k)})\right\}$ and $\left\{G(gz_{n(k)},gz_{n(k)},gz_{m(k)})\right\}$ have subsequences converging to, say $\varepsilon_1,\varepsilon_2$ and ε_3 respectively, and $\varepsilon_1+\varepsilon_2+\varepsilon_3=\varepsilon>0$.

We can write

$$\begin{aligned}
\lim_{k\to\infty}G(gx_{n(k)},gx_{n(k)},gx_{m(k)}) &= \varepsilon_1,\\
\lim_{k\to\infty}G(gy_{n(k)},gy_{n(k)},gy_{m(k)}) &= \varepsilon_2\\
\lim_{k\to\infty}G(gz_{n(k)},gz_{n(k)},gz_{m(k)}) &= \varepsilon_3.
\end{aligned}$$

Letting $k\to\infty$ in (2.21) and using (2.8), we have

$$\begin{aligned}
0 &= \lim_{k\to\infty}\left[2\left(\omega^x_{n(k)+1}+\omega^y_{n(k)+1}+\omega^z_{n(k)+1}\right)+\left(\omega^x_{m(k)+1}+\omega^y_{m(k)+1}+\omega^z_{m(k)+1}\right)\right]\\
&\geq \lim_{k\to\infty}3\psi\left(G(gx_{n(k)},gx_{n(k)},gx_{m(k)}),G(gy_{n(k)},gy_{n(k)},gy_{m(k)}),\right.\\
&\qquad\left.G(gz_{n(k)},gz_{n(k)},gz_{m(k)})\right)\\
&= \lim_{\substack{t_1\to\varepsilon_1\\ t_2\to\varepsilon_2\\ t_3\to\varepsilon_3}}3\psi(t_1,t_2,t_3),\ \text{where}\left\{\begin{array}{l} t_1=G(gx_{n(k)},gx_{n(k)},gx_{m(k)}),\\ t_2=G(gy_{n(k)},gy_{n(k)},gy_{m(k)})\\ \text{and } t_3=G(gz_{n(k)},gz_{n(k)},gz_{m(k)})\end{array}\right.\\
&> 0,
\end{aligned}$$

which is a contradiction. Therefore, $\{gx_n\},\{gy_n\}$ and $\{gz_n\}$ are G-Cauchy.

Since X is G-complete, there exists $x,y,z\in X$ such that

$$\lim_{n\to\infty}gx_n=x\ ,\ \lim_{n\to\infty}gy_n=y\ \text{ and }\ \lim_{n\to\infty}gz_n=z.\tag{2.22}$$

The continuity of g implies that

$$\lim_{n\to\infty}gx_n=gx\ ,\ \lim_{n\to\infty}gy_n=gy\ \text{ and }\ \lim_{n\to\infty}gz_n=gz.\tag{2.23}$$

Now, suppose that assumption (a) holds. From (2.2) and the commutativity of F and g, we obtain

$$\begin{aligned}
gx = \lim_{n\to\infty}g(gx_{n+1}) &= \lim_{n\to\infty}g(F(x_n,y_n,z_n))\\
&= \lim_{n\to\infty}(F(gx_n,gy_n,gz_n))\\
&= F\left(\lim_{n\to\infty}gx_n,\lim_{n\to\infty}gy_n,\lim_{n\to\infty}gz_n\right)\\
&= F(x,y,z),
\end{aligned}$$

and

$$gy = \lim_{n\to\infty}g(gy_{n+1}) = \lim_{n\to\infty}g(F(y_n,x_n,y_n))$$

$$\begin{aligned} &= \lim_{n\to\infty}(F(gy_n, gx_n, gy_n)) \\ &= F\left(\lim_{n\to\infty} gy_n, \lim_{n\to\infty} gx_n, \lim_{n\to\infty} gy_n\right) \\ &= F(y, x, y). \end{aligned}$$

Similarly, we have

$$\begin{aligned} gz = \lim_{n\to\infty} g(gz_{n+1}) &= \lim_{n\to\infty} g(F(z_n, y_n, x_n)) \\ &= \lim_{n\to\infty}(F(gz_n, gy_n, gx_n)) \\ &= F\left(\lim_{n\to\infty} gy_n, \lim_{n\to\infty} gx_n, \lim_{n\to\infty} gy_n\right) \\ &= F(z, y, x). \end{aligned}$$

Hence, (x, y, z) is a tripled coincidence point of F and g.

On the other hand, suppose that assumption (b) holds. Since $\{gx_n\}$ is non-decreasing satisfying $gx_n \to x$, and $\{gy_n\}$ is non-increasing satisfying $gy_n \to y$, and $\{gz_n\}$ is non-decreasing satisfying $gz_n \to z$, we have

$$g(gy_n) \preceq gx\ ,\ g(gy_n) \succeq gy \ \text{ and } \ g(gz_n) \preceq gz\ ; \quad \text{for all } n \geq 0.$$

Using the rectangle inequality and (2.1), we obtain

$$\begin{aligned} &G(F(x,y,z), gx, gx) + G(F(y,x,y), gy, gy) + G(F(z,y,x), gz, gz) \\ &\quad \leq G(F(x,y,z), g(gx_{n+1}), g(gx_{n+1})) + G(g(gx_{n+1}), gx, gx) \\ &\qquad + G(F(y,x,y), g(gy_{n+1}), g(gy_{n+1})) + G(g(gy_{n+1}), gy, gy) \\ &\qquad + G(F(z,y,x), g(gz_{n+1}), g(gz_{n+1})) + G(g(gz_{n+1}), gz, gz) \\ &\quad = G\left(F(x,y,z), F(gx_n, gy_n, gz_n), F(gx_n, gy_n, gz_n)\right) + G(g(gx_{n+1}), gx, gx) \\ &\qquad + G\left(F(y,x,y), F(gy_n, gx_n, gy_n), F(gy_n, gx_n, gy_n)\right) + G(g(gy_{n+1}), gy, gy) \\ &\qquad + G\left(F(z,y,x), F(gz_n, gy_n, gx_n), F(gz_n, gy_n, gx_n)\right) + G(g(gz_{n+1}), gz, gz) \\ &\quad \leq \Big[G(gx, g(gx_n), g(gx_n)) + G(gy, g(gy_n), g(gy_n)) + G(gz, g(gz_n), g(gz_n))\Big] \\ &\qquad - 3\psi\Big(G(gx, g(gx_n), g(gx_n)), G(gy, g(gy_n), g(gy_n)), G(gz, g(gz_n), g(gz_n))\Big) \\ &\qquad\ + G(g(gx_{n+1}), gx, gx) + G(g(gy_{n+1}), gy, gy) + G(g(gz_{n+1}), gz, gz). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$G(F(x,y,z), gx, gx) + G(F(y,x,y), gy, gy) + G(F(z,y,x), gz, gz) = 0.$$

This gives that

$$G(F(x,y,z), gx, gx) = G(F(y,x,y), gy, gy) = G(F(z,y,x), gz, gz) = 0,$$

this means, $F(x,y,z) = gx, F(y,x,y) = gy$ and $F(z,y,x) = gz$. Therefore, (x, y, z) is a tripled coincidence point of F and g. This completes our proof. □

If we set $g(x) = x,\ \forall x \in X$, in Theorem 2.1, we obtain the following new tripled fixed point theorem which is a generalization of Theorem 1.20, the main result of Aydi and Karapinar [1].

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X,G) is a complete G-metric space. Let $g : X \to X$ be a mapping and $F : X \times X \times X \to X$ be a mapping having the mixed monotone property on X. Suppose that there exists $\psi \in \Theta$ such that*

$$\begin{aligned} &G(F(x,y,z),F(u,v,w),F(h,k,\ell)) + G(F(y,z,x),F(v,w,u),\\ &\qquad F(k,\ell,h)) + G(F(z,x,y),F(w,u,v),F(\ell,h,k))\\ &\qquad \leq \Big[G(x,u,h) + G(y,v,k) + G(z,w,\ell)\Big]\\ &\qquad\quad - 3\psi\Big(G(x,u,h),G(y,v,k),G(z,w,\ell)\Big), \end{aligned} \tag{2.24}$$

for all $x,y,z,w,u,v,h,k,\ell \in X$ for which $x \succeq u \succeq h, y \preceq v \preceq k$ and $z \succeq w \succeq \ell$. Suppose either that

(a) *F is continuous, or*

(b) *X has the following property :*
  - (i) *if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*
  - (ii) *if a non-increasing sequence $y_n \to y$, then $y_n \succeq y$ for all n,*
  - (iii) *if a non-decreasing sequence $z_n \to z$, then $z_n \preceq z$ for all n.*

Suppose also that there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \preceq F(x_0,y_0,z_0), y_0 \succeq F(y_0,x_0,y_0) \quad \textit{and} \quad z_0 \preceq F(z_0,y_0,x_0),$$

then F has a tripled fixed point in X.

Remark 2.3. Theorem 2.2 is more general than Theorem 1.20 [1], since the contractive condition (2.24) is weaker than (1.3). This fact is clearly illustrated by the following example.

Example 2.4. Let $X = \mathbb{R}^+$ be a set with uasual ordering, i.e. a set endowed with order $x \preceq y \Leftrightarrow x \leq y$. Let the mapping $G : X \times X \times X \to \mathbb{R}$ be defined by $G(x,y,z) = |x-y| + |y-z| + |z-x|$, for all $x,y,z \in X$. Then G is a G-metric on X. Define the mapping $F : X \times X \times X \to X$ by $F(x,y,z) = \frac{1}{9}(x-y+z)$, for all $x,y,z \in X$. Then the following properties hold :

(1) F is mixed monotone;
(2) F satisfies condition (2.24), but F does not satisfy condition (1.3).

We first show that F satisfies condition (2.24). Indeed , we have

$$\begin{aligned} &G(F(x,y,z),F(u,v,w),F(h,k,\ell)) + G(F(y,z,x),F(v,w,u),F(k,\ell,h))\\ &\qquad + G(F(z,x,y),F(w,u,v),F(\ell,h,k))\\ &\qquad = \frac{1}{9}\Big[|(x-u)+(v-y)+(z-w)| + |(u-h)+(k-v)+(w-\ell)|\\ &\qquad\qquad + |(h-x)+(y-k)+(\ell-z)|\Big]\\ &\qquad + \frac{1}{9}\Big[|(y-v)+(w-z)+(x-u)| + |(v-k)+(\ell-w)+(u-h)|\\ &\qquad\qquad + |(k-y)+(z-\ell)+(h-x)|\Big]\\ &\qquad + \frac{1}{9}\Big[|(z-w)+(u-x)+(y-v)| + |(w-\ell)+(h-u)+(v-k)| \end{aligned}$$

$$
\begin{aligned}
&\quad + |(\ell - z) + (x - h) + (k - y)|\Big] \\
&\le \frac{2}{3}\Big[(|x-u| + |u-h| + |h-x|) + (|y-v| + |v-k| + |k-y|) \\
&\quad + (|z-w| + |w-\ell| + |\ell - z|)\Big] \\
&= \Big[(|x-u| + |u-h| + |h-x|) + (|y-v| + |v-k| + |k-y|) \\
&\quad + (|z-w| + |w-\ell| + |\ell - z|)\Big] \\
&- \frac{1}{3}\Big[(|x-u| + |u-h| + |h-x|) + (|y-v| + |v-k| + |k-y|) \\
&\quad + (|z-w| + |w-\ell| + |\ell - z|)\Big] \\
&= \Big[G(x,u,h) + G(y,v,k) + G(z,w,\ell)\Big] \\
&\quad - 3\psi\Big(G(x,u,h), G(y,v,k), G(z,w,\ell)\Big),
\end{aligned}
$$

which is exactly the condition (2.24) with $\psi(t_1, t_2, t_3) = \frac{1}{9}(t_1 + t_2 + t_3)$. Moreover, taking $x_0 = -1, y_0 = 1$ and $z_0 = -1$, we have $x_0 \le F(x_0, y_0, z_0), y_0 \ge F(y_0, x_0, y_0)$ and $z_0 \le F(z_0, y_0, x_0)$. Therefore, all the conditions of Theorem 2.2 hold.

Now we show that F does not satisfy condition (1.3). Define $T : X \to X$ by $T(x) = \ln(x) + 1$. It is easy to see that T is an ICS mapping. Assume to the contrary that there exists $\phi \in \Phi$ such that (1.3) holds. This means

$$
\begin{aligned}
d(TF(x,y,z), TF(u,v,w)) &= |TF(x,y,z) - TF(u,v,w)| \\
&= \left| \ln\left(\frac{x - y + z}{u - v + w}\right)\right| \\
&\le \phi\left(max\{d(Tx,Tu), d(Ty,Tv), d(Tz,Tw)\}\right).
\end{aligned}
$$

Taking $x = u = y = 4, v = 5, w = 2$ and $z = 6$, we get that $\ln 6 \le \ln 3$. This is a contradiction.

Let Ω be the class of all functions $\eta : [0,\infty) \to [0,\infty)$ satisfying $\lim_{t\to r} \eta(t) > 0$ for all $r > 0$.

Corollary 2.5. *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X,G) is a complete G-metric space. Let $g : X \to X$ be a mapping and $F : X \times X \times X \to X$ be a mapping having the mixed g-monotone property on X. Suppose that there exists $\eta \in \Omega$ such that*

$$
\begin{aligned}
&G(F(x,y,z), F(u,v,w), F(h,k,\ell)) \\
&\quad + G(F(y,z,x), F(v,w,u), F(k,\ell,h)) \\
&\quad + G(F(z,x,y), F(w,u,v), F(\ell,h,k)) \qquad (2.25) \\
&\le [G(gx,gu,gh) + G(gy,gv,gk) + G(gz,gw,g\ell)] \\
&\quad - 3\eta\left(\max\{G(gx,gu,gh), G(gy,gv,gk), G(gz,gw,g\ell)\}\right),
\end{aligned}
$$

for all $x,y,z,w,u,v,h,k,\ell \in X$ for which $gx \succeq gu \succeq gh, gy \preceq gv \preceq gk$ and $gz \succeq gw \succeq g\ell$. Suppose either that

(a) *F is continuous or*
(b) *X has the following property :*
  (i) *if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*
  (ii) *if a non-increasing sequence $y_n \to y$, then $y_n \succeq y$ for all n,*
  (iii) *if a non-decreasing sequence $z_n \to z$, then $z_n \preceq z$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that

$$gx_0 \preceq F(x_0, y_0, z_0), gy_0 \succeq F(y_0, x_0, y_0) \quad \text{and} \quad gz_0 \preceq F(z_0, y_0, x_0),$$

then F and g have a tripled coincidence point.

Proof. In Theorem 2.1, taking $\psi(t_1, t_2, t_3) = \eta(\max\{t_1, t_2, t_3\})$ for all $(t_1, t_2, t_3) \in [0, \infty) \times [0, \infty) \times [0, \infty)$, we get the desired results. $\square$

Corollary 2.6. *In Corollary 2.5, if we replace inequality* (2.25) *by*

$$\begin{aligned} &G(F(x,y,z), F(u,v,w), F(h,k,\ell)) + G(F(y,z,x), F(v,w,u), \\ &\quad F(k,\ell,h)) + G(F(z,x,y), F(w,u,v), F(\ell,h,k)) \\ &\leq [G(gx,gu,gh) + G(gy,gv,gk) + G(gz,gw,g\ell)] \\ &\quad - 3\eta\left(G(gx,gu,gh) + G(gy,gv,gk) + G(gz,gw,g\ell)\right). \end{aligned} \tag{2.26}$$

Then F and g have a tripled coincidence point.

Proof. In Theorem 2.1, taking $\psi(t_1, t_2, t_3) = \eta(t_1 + t_2 + t_3)$ for all $(t_1, t_2, t_3) \in [0, \infty)^3$, then we get the desired result. $\square$

Remark 2.7. We conclude that

(1) Theorem 2.1 extends Theorem 1.17 of Wangkeeree and Bantaojai [30].
(2) Theorem 2.2 extends Theorem 1.18 of Wangkeeree and Bantaojai [30], and generalizes the result of Aydi and Karapinar [1] given by Theorem 1.20.
(3) We also see that Theorem 2.2 extends Theorem 1.20 of Aydi and Karapinar [1] to partially ordered G-metric spaces.
(4) Corrolary 2.5 and Corrolary 2.6 extend Corrolary 2.6 and Corrolary 2.7.of Wangkeeree and Bantaojai [30], respectively.

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