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NEW CONVERGENCE THEOREMS FOR SPLIT COMMON FIXED POINT PROBLEMS IN HILBERT SPACES

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Dedicated to Professor Sompong Dhompongsa on the occasion of his 65th birthday

Abstract. This paper deals with strong convergence theorems for solving the split common fixed-point problems for the class of demicontractive mappings in real Hilbert spaces. Furthermore, we apply our results to solving split variational inequality problems, split convex minimization problems and split common zeros problems in real Hilbert spaces.

1. Introduction

In this paper, we shall assume that H is a real Hilbert space with inner product $\langle .,. \rangle$ and norm $||.||$. Let I denote the identity operator on H. Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem (SFP) is to find a point

$$x \in C \text{ such that } Ax \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [5] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. The SFP attracts the attention of many authors due to its application in signal processing. Various algorithms have been invented to solve it (see, for example, [3, 13, 18–20, 22, 23, 26] and references therein).

Note that the split feasibility problem (1.1) can be formulated as a fixed-point equation by using the fact

$$P_C(I - \gamma A^*(I - P_Q)A)x^* = x^*; \tag{1.2}$$

that is, x^* solves the SFP (1.1) if and only if x^* solves the fixed point equation (1.2) (see [19] for the details). This implies that we can use fixed-point algorithms (see [24, 25, 27]) to solve SFP. A popular algorithm that solves the SFP (1.1) is due to Byrne's CQ algorithm [2] which is found to be a gradient-projection method (GPM) in convex minimization. Subsequently, Byrne [3] applied KM iteration to the CQ algorithm, and Zhao and Yang [29] applied KM iteration to the perturbed CQ algorithm to solve the SFP. It is well known that the CQ algorithm and the KM algorithm for a split feasibility problem do not necessarily converge strongly in the infinite-dimensional Hilbert spaces.

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Now let us first recall the definitions of some operators that are often used in fixed-point theory which are related to demicontractive operators and which appear naturally when using subgradient projection operator techniques in solving some feasibility problems.

Let $T : H \to H$ be a mapping. A point $x \in H$ is said to be a *fixed point* of T provided that $Tx = x$. In this paper, we use $F(T)$ to denote the fixed-points set and use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

Definition 1.1. The mapping $T : H \to H$ is said to be

(a) *nonexpansive* if

$$||Tx - Ty|| \le ||x - y||, \forall x, y \in H.$$

(b) *quasi-nonexpansive* if

$$||Tx - Tp|| \le ||x - p||, \forall x \in H, p \in F(T).$$

(c) *firmly nonexpansive* mapping if

$$||Tx - Ty||^2 \le ||x - y||^2 - ||(x - y) - (Tx - Ty)||^2, \forall x, y \in H.$$

(d) *quasi-firmly nonexpansive* mapping if

$$||Tx - Tp||^2 \le ||x - p||^2 - ||x - Tx||^2, \forall x \in H, p \in F(T).$$

(e) *strictly pseudocontractive* mapping if there exists a constant $k \in [0, 1)$ such that

$$||Tx - Ty||^2 \le ||x - y||^2 + k||(x - y) - (Tx - Ty)||^2, \forall x, y \in H.$$

(f) *pseudocontractive* mapping if

$$||Tx - Ty||^2 \le ||x - y||^2 + ||(x - y) - (Tx - Ty)||^2, \forall x, y \in H.$$

(g) *demicontractive (or k-demicontractive)* if there exists $k < 1$ such that

$$||Tx - Tp||^2 \le ||x - p||^2 + k||x - Tx||^2, \forall x \in H, p \in F(T).$$

Remark 1.2. Denoting by $\Im_N, \Im_{QN}, \Im_{FN}, \Im_{QF}, \Im_S, \Im_P, \Im_D$ (with $k \ge 0$) the classes of nonexpansive, quasi-nonexpansive, firmly-nonexpansive, quasi-firmly non-expansive, strictly pseudocontractive, pseudocontractive and demicontractive mappings, respectively, we easily observe that $\Im_{FN} \subsetneq \Im_N \subsetneq \Im_{QN} \subsetneq \Im_D$, $\Im_{FN} \subsetneq \Im_{QF} \subsetneq \Im_{QN} \subsetneq \Im_D$ and $\Im_{FN} \subsetneq \Im_N \subsetneq \Im_S \subsetneq \Im_D$ through the following examples.

Given below is an example of a demicontractive mapping which is not pseudo-contractive and hence not strictly pseudocontractive.

Example 1.3 ([11]). Let H be the real line and $C = [-1, 1]$. Define T on C by

$$Tx = \begin{cases} \frac{2}{3}x \sin(\frac{1}{x}), x \neq 0 \\ 0, x = 0. \end{cases} \tag{1.3}$$

An example of a demicontractive function which is not quasi-nonexpansive and is not pseudocontractive is given by

Example 1.4 ([9]). $f : [-2, 1] \to [-2, 1], f(x) := -x^2 - x.$

Furthermore, $\Im_{FN}$ is well known to include resolvents and projection operators, while $\Im_{QF}$ contains subgradient projection operators (see, e.g., [14] and the reference therein).

In this paper, we shall focus our attention on the following general two-operator split common fixed-point problem (SCFPP):

$$\text{find } x \in C \text{ such that } Ax \in Q, \tag{1.4}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $S : H_1 \to H_1$ and $T : H_2 \to H_2$ are two demicontractive operators with nonempty fixed-point sets $F(S) = C$ and $F(T) = Q$, and denote the solution set of the two-operator SCFP by

$$\Omega := \{y \in C : Ay \in Q\} = C \cap A^{-1}(Q). \tag{1.5}$$

Recall that $F(S)$ and $F(T)$ are nonempty closed convex subsets of H_1 and H_2, respectively. If $\Omega \neq \emptyset$, we have that Ω is closed and convex subset of H_1. The split common fixed-point problem (SCFP) is a generalization of the split feasibility problem (SFP) and the convex feasibility problem (CFP) (see [2, 7]).

To solve (1.4), Censor and Segal [7] proposed and proved, in finite-dimensional spaces, the convergence of the following algorithm:

$$x_{n+1} = S(x_n + \gamma A^t(T - I)Ax_n), n \geq 1, \tag{1.6}$$

where $\gamma \in \left(0, \frac{2}{\lambda}\right)$, with λ being the largest eigenvalue of the matrix A^tA (A^t stands for matrix transposition). In 2011, Moudafi [15] introduced the following relaxed algorithm:

$$x_{n+1} = (1 - \alpha_n)y_n + \alpha_n Sy_n, n \geq 1, \tag{1.7}$$

where $y_n = x_n + \gamma A^*(T - I)Ax_n, \beta \in (0, 1), \alpha_n \in (0, 1)$, and $\gamma \in \left(0, \frac{1}{\lambda\beta}\right)$, with λ being the spectral radius of the operator A^*A. Moudafi proved weak convergence result of the algorithm (1.7) in Hilbert spaces where S and T are quasi-nonexpansive operators. We observe that strong convergence result can be obtained in the results of Moudafi [15] if a compactness type condition like demicompactness is imposed on the operator S. Furthermore, we can also obtain strong convergence result by suitably modifying the algorithm (1.7). Recently, Zhao and He [28] introduced the following viscosity approximation algorithm.

$$\begin{cases} y_n = x_n + \gamma A^*(T - I)Ax_n \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)((1 - w_n)x_n + w_n Sy_n, \ n \geq 1, \end{cases} \tag{1.8}$$

where $f : H_1 \to H_1$ is a contraction of modulus $\rho > 0, w_n \in \left(0, \frac{1}{2}\right), \gamma \in \left(0, \frac{1}{\lambda}\right)$, with λ being the spectral radius of the operator A^*A and proved strong convergence results concerning (1.4) for quasi-nonexpansive operators S and T in real Hilbert spaces. Inspired by the work of Zhao and He [28], Moudafi [16] quite recently revisited the viscosity-type approximation method (1.8) above introduced in [28]. First, he proposed a simple proof of the strong convergence of the iterative sequence $\{x_n\}$ defined by (1.8) based on attracting operator properties and then proposed a modification of this algorithm (1.8) and proved its strong convergence (see Theorem

2.1 of [16]).

We remark here that it is quite usual to seek a particular solution of a given nonlinear problem, in particular, the minimum-norm solution. For instance, given a nonempty, closed and convex subset C of a Hilbert space H_1 and a bounded linear operator $A : H_1 \to H_2$, where H_2 is another Hilbert space. The C-constrained pseudoinverse of A, $A_C^\dagger$ is then defined as the minimum-norm solution of the constrained minimization problem

$$A_C^\dagger(b) := \text{argmin}_{x\in C}||Ax - b||$$

which is equivalent to the fixed point problem

$$x = P_C(x - \lambda A^*(Ax - b)),$$

where P_C is the metric projection from H_1 onto C, A^* is the adjoint of A, $\lambda > 0$ is a constant, and $b \in H_2$ is such that $P_{\overline{A(C)}}(b) \in A(C)$. It is therefore an interesting problem to invent iterative algorithms that can generate sequences which converge strongly to the minimum-norm solution of a given SCFPP (1.4).

So, our main purpose here is to consider the split common fixed point problems for *demicontractive operators* such that strong convergence is achieved for the iterative sequence without imposing compactness type condition on the operators or on the domains of the operators in the context of real Hilbert spaces. Our results are motivated by the results of Moudafi [15]. Thus, we modify algorithm (1.7) above and prove strong convergence results to the minimum-norm solution for split common fixed point problems concerning demicontractive operators in real Hilbert spaces.

2. Preliminaries

Definition 2.1. A mapping $T : H \to H$ is called *demiclosed at the origin* if any sequence $\{x_n\}$ weakly converges to x, and if the sequence $\{Tx_n\}$ strongly converges to 0, then $Tx = 0$.

Next, we state the following well-known lemmas which will be used in the sequel.

Lemma 2.2. *Let H be a real Hilbert space. Then there holds the following well-known results:*

(i) $||x + y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2, \forall x, y \in H.$
(ii) $||x + y||^2 \le ||x||^2 + 2\langle y, x + y\rangle, \forall x, y \in H.$

Lemma 2.3 (Xu, [21])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n,\ n \ge 1,$$

where

(i) $\{a_n\} \subset [0, 1],\ \sum \alpha_n = \infty;$
(ii) $\limsup \sigma_n \le 0;$

(iii) $\gamma_n \ge 0;\ (n \ge 1),\ \Sigma\gamma_n < \infty.$

Then, $a_n \to 0$ as $n \to \infty$.

3. Main results

In this section, we modify the algorithm (1.7) above so as to have strong convergence. Below we include such modification.

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be the adjoint of A. Let $S : H_1 \to H_1$ be a k_1-demicontractive mapping such that $S - I$ is demi-closed at 0, $C := F(S) \neq \emptyset$, $T : H_2 \to H_2$ be a k_2-demicontractive mapping such that such that $T - I$ is demi-closed at 0 and $Q := F(T) \neq \emptyset$. Suppose that SCFPP (1.4) has a nonempty solution set Ω. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$, $\{\beta_n\}$ a sequence in $(0,(1-k_1)(1-\alpha_n)) \subset (0,1)$ and $\gamma \in \left(0, \frac{1-k_2}{||A||^2}\right)$. Let sequences $\{y_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be generated by $x_1 \in H_1$,*

$$\begin{cases} y_n = x_n + \gamma A^*(T-I)Ax_n \\ x_{n+1} = (1-\alpha_n-\beta_n)y_n + \beta_n S y_n, \ n \geq 1. \end{cases} \tag{3.1}$$

Suppose the following conditions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and*

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1 - k_1$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to an element $x^* \in \Omega$ which is also the minimum-norm solution (i.e., $x^* \in \Omega$ and $||x^*|| = \min\{||x|| : x \in \Omega\}$).

Proof. Let $x^* \in \Omega$. From (3.1) and Lemma 2.2 (i), we obtain that

$$\begin{aligned} ||y_n - x^*||^2 &= ||x_n - x^* + \gamma A^*(T-I)Ax_n||^2 \\ &= ||x_n - x^*||^2 + 2\gamma\langle x_n - x^*, A^*(T-I)Ax_n\rangle + \gamma^2||A^*(T-I)Ax_n||^2. \end{aligned} \tag{3.2}$$

Since

$$\begin{aligned} \gamma^2||A^*(T-I)Ax_n||^2 &= \gamma^2\langle A^*(T-I)Ax_n, A^*(T-I)Ax_n\rangle \\ &= \gamma^2\langle AA^*(T-I)Ax_n, (T-I)Ax_n\rangle \\ &\leq \gamma^2||A||^2||(T-I)Ax_n||^2, \end{aligned} \tag{3.3}$$

$Ax^* \in Q = F(T)$ and T is a demicontractive mapping, then we obtain

$$\begin{aligned} \langle x_n - x^*, A^*(T-I)Ax_n\rangle &= \langle A(x_n - x^*), (T-I)Ax_n\rangle \\ &= \langle A(x_n - x^*) + (T-I)Ax_n - (T-I)Ax_n, (T-I)Ax_n\rangle \\ &= \langle TAx_n - Ax^*, (T-I)Ax_n\rangle - ||(T-I)Ax_n||^2 \\ &= \frac{1}{2}\Big[||TAx_n - Ax^*||^2 + ||(T-I)Ax_n||^2 - ||Ax_n - Ax^*||^2\Big] \\ &\quad - ||(T-I)Ax_n||^2 \\ &\leq \frac{1}{2}\Big[||Ax_n - Ax^*||^2 + k_2||(T-I)Ax_n||^2\Big] \\ &\quad + \frac{1}{2}\Big[||(T-I)Ax_n||^2 - ||Ax_n - Ax^*||^2\Big] - ||(T-I)Ax_n||^2 \end{aligned}$$

$$= \frac{k_2 - 1}{2}||(T - I)Ax_n||^2. \tag{3.4}$$

Substituting (3.4) and (3.3) into (3.2), we have

$$||y_n - x^*||^2 \leq ||x_n - x^*||^2 - \gamma(1 - k_2 - \gamma||A||^2)||(T - I)Ax_n||^2. \tag{3.5}$$

From (3.1), we obtain that

$$\begin{aligned} ||x_{n+1} - x^*|| &= ||(1 - \beta_n - \alpha_n)(y_n - x^*) + \beta_n(Sy_n - x^*) - \alpha_n x^*|| \\ &\leq ||(1 - \beta_n - \alpha_n)(y_n - x^*) + \beta_n(Sy_n - x^*)|| + \alpha_n||x^*||. \end{aligned} \tag{3.6}$$

Using the fact that S is demicontractive, we obtain that

$$\begin{aligned} & ||(1 - \beta_n - \alpha_n)(y_n - x^*) + \beta_n(Sy_n - x^*)||^2 \\ = \ & (1 - \beta_n - \alpha_n)^2||y_n - x^*||^2 + \beta_n^2||Sy_n - x^*||^2 \\ & +2(1 - \beta_n - \alpha_n)\beta_n\langle Sy_n - x^*, y_n - x^*\rangle \\ \leq \ & (1 - \beta_n - \alpha_n)^2||y_n - x^*||^2 \\ & +\beta_n^2[||y_n - x^*||^2 + k_1||y_n - Sy_n||^2] \\ & +2(1 - \beta_n - \alpha_n)\beta_n\Big[||y_n - x^*||^2 - \frac{1 - k_1}{2}||y_n - Sy_n||^2\Big] \\ = \ & (1 - \alpha_n)^2||y_n - x^*||^2 + [k_1\beta_n^2 - (1 - k_1)(1 - \beta_n - \alpha_n)\beta_n]||y_n - Sy_n||^2 \\ = \ & (1 - \alpha_n)^2||y_n - x^*||^2 + \beta_n[\beta_n - (1 - \alpha_n)(1 - k_1)]||y_n - Sy_n||^2 \\ \leq \ & (1 - \alpha_n)^2||y_n - x^*||^2, \end{aligned} \tag{3.7}$$

which implies

$$||(1 - \beta_n - \alpha_n)(y_n - x^*) + \beta_n(Sy_n - x^*)|| \leq (1 - \alpha_n)||y_n - x^*||.$$

Therefore it follows from (3.5), (3.6) and the last inequality above that

$$\begin{aligned} ||x_{n+1} - x^*|| &\leq (1 - \alpha_n)||y_n - x^*|| + \alpha_n||x^*|| \\ &\leq (1 - \alpha_n)||x_n - x^*|| + \alpha_n||x^*|| \\ &\leq \max\{||x_n - x^*||, ||x^*||\} \\ &\leq \ \vdots \\ &\leq \max\{||x_1 - x^*||, ||x^*||\}. \end{aligned} \tag{3.8}$$

Therefore, $\{x_n\}$ and $\{y_n\}$ are bounded.
Now, for any $x \in H_1$, we have

$$\begin{aligned} ||Sx - x^*||^2 &\leq ||x - x^*||^2 + k_1||x - Sx||^2 \\ &\Rightarrow \langle Sx - x^*, Sx - x^*\rangle \leq \langle x - x^*, x - Sx\rangle \\ &\quad +\langle x - x^*, Sx - x^*\rangle + k_1||x - Sx||^2 \\ &\Rightarrow \langle Sx - x^*, Sx - x\rangle \leq \langle x - x^*, x - Sx\rangle \\ &\quad +k_1||x - Sx||^2 \\ &\Rightarrow \langle Sx - x, Sx - x\rangle + \langle x - x^*, Sx - x\rangle \\ &\leq \langle x - x^*, x - Sx\rangle + k_1||x - Sx||^2 \\ &\Rightarrow (1 - k_1)||x - Sx||^2 \leq 2\langle x - x^*, x - Sx\rangle. \end{aligned} \tag{3.9}$$

Furthermore, from (3.5) we have

$$\begin{aligned}
||y_{n+1}-x^*||^2 &\leq ||x_{n+1}-x^*||^2 = ||(1-\beta_n)y_n+\beta_n Sy_n-\alpha_n y_n-x^*||^2\\
&= ||(y_n-x^*)-\beta_n(y_n-Sy_n)-\alpha_n y_n||^2\\
&\leq ||(y_n-x^*)-\beta_n(y_n-Sy_n)||^2-2\alpha_n\langle y_n,x_{n+1}-x^*\rangle\\
&= ||y_n-x^*||^2-2\beta_n\langle y_n-Sy_n,y_n-x^*\rangle+\beta_n^2||y_n-Sy_n||^2\\
&\quad -2\alpha_n\langle y_n,x_{n+1}-x^*\rangle\\
&\leq ||y_n-x^*||^2-\beta_n(1-k_1)||y_n-Sy_n||^2+\beta_n^2||y_n-Sy_n||^2\\
&\quad -2\alpha_n\langle y_n,x_{n+1}-x^*\rangle\\
&= ||y_n-x^*||^2-\beta_n[(1-k_1)-\beta_n]||y_n-Sy_n||^2\\
&\quad -2\alpha_n\langle y_n,x_{n+1}-x^*\rangle\\
&\leq ||x_n-x^*||^2-\beta_n[(1-k_1)-\beta_n]||y_n-Sy_n||^2\\
&\quad -2\alpha_n\langle y_n,x_{n+1}-x^*\rangle. \qquad (3.10)
\end{aligned}$$

Since $\{y_n\}$ and $\{x_n\}$ are bounded, $\exists M>0$ such that $-2\langle y_n,x_{n+1}-x^*\rangle \leq M$ for all $n\geq 0$. Therefore,

$$\begin{aligned}
\beta_n[(1-k_1)-\beta_n]||y_n-Sy_n||^2 &\leq ||y_n-x^*||^2-||y_{n+1}-x^*||^2\\
&\quad +\alpha_n M. \qquad (3.11)
\end{aligned}$$

The rest of the proof will be divided into two parts.

<u>Case 1</u>. Suppose that there exists $n_0\in\mathbb{N}$ such that $\{||y_n-x^*||\}_{n=n_0}^{\infty}$ is nonincreasing. Then $\{||y_n-x^*||\}_{n=1}^{\infty}$ converges and $||y_n-x^*||^2-||y_{n+1}-x^*||^2\to 0,\ n\to\infty$. This together with (3.11) and the condition that $\alpha_n\to 0$ imply that,

$$||y_n-Sy_n||\to 0,\ n\to\infty. \qquad (3.12)$$

From (3.5) and (3.8), we have that

$$\begin{aligned}
&\gamma(1-k_2-\gamma||A||^2)||(T-I)Ax_n||^2 \leq ||x_n-x^*||^2-||y_n-x^*||^2\\
\leq\ & \Big(||y_{n-1}-x^*||+\alpha_{n-1}||x^*||\Big)^2-||y_n-x^*||^2\\
=\ & ||y_{n-1}-x^*||^2-||y_n-x^*||^2+2\alpha_{n-1}||x^*||||y_{n-1}-x^*||+\alpha_{n-1}^2||x^*||^2.
\end{aligned}$$

Using condition (a) above implies that

$$\gamma(1-k_2-\gamma||A||^2)||(T-I)Ax_n||^2\to 0,\ n\to\infty.$$

Hence, we obtain

$$||(T-I)Ax_n||\to 0,\ n\to\infty. \qquad (3.13)$$

Also, we observe that

$$||y_n-x_n||=\gamma||A^*(T-I)Ax_n||\to 0,\ n\to\infty.$$

Since $\{x_n\}$ is bounded, there exists $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j}\rightharpoonup z\in H_1$. Using the fact that $x_{n_j}\rightharpoonup z\in H_1$ and $||y_n-x_n||\to 0,\ n\to\infty$, we have that $y_{n_j}\rightharpoonup z\in H_1$. By the demiclosedness principle of $S-I$ at zero and (3.12), we have that

$z \in F(S) = C$. On the other hand, Since A is a linear bounded operator, it follows from $x_{n_j} \rightharpoonup z \in H_1$ that $Ax_{n_j} \rightharpoonup Az \in H_2$. Hence, from (3.13), we have that

$$||TAx_{n_j} - Ax_{n_j}|| = ||TAx_{n_j} - Ax_{n_j}|| \to 0, \ j \to \infty.$$

Since $T - I$ is demiclosed at zero, we have that $Az \in F(T) = Q$. Hence, $z \in \Omega$.

Now, set $w_n = (1 - \beta_n)y_n + \beta_n Sy_n, n \geq 1$. Then from (3.1) we have that

$$x_{n+1} = w_n - \alpha_n y_n.$$

It then follows that

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)w_n - \alpha_n(y_n - w_n) \\ &= (1-\alpha_n)w_n + \alpha_n\beta_n(y_n - Sy_n). \end{aligned} \tag{3.14}$$

Also

$$\begin{aligned} ||w_n - x^*||^2 &= ||y_n - x^* - \beta_n(y_n - Sy_n)||^2 \\ &= ||y_n - x^*||^2 - 2\beta_n\langle y_n - Sy_n, y_n - x^*\rangle + \beta_n^2||y_n - Sy_n||^2 \\ &\leq ||y_n - x^*||^2 - \beta_n[(1-k_1) - \beta_n]||y_n - Sy_n||^2 \\ &\leq ||y_n - x^*||^2. \end{aligned} \tag{3.15}$$

Applying Lemma 2.2 (ii) to (3.14), we have

$$\begin{aligned} ||y_{n+1} - x^*||^2 &\leq ||x_{n+1} - x^*||^2 \\ &= ||(1-\alpha_n)(w_n - x^*) + \alpha_n\beta_n(y_n - Sy_n) - \alpha_n x^*||^2 \\ &\leq (1-\alpha_n)^2||w_n - x^*||^2 + 2\alpha_n\langle\beta_n(y_n - Sy_n) - x^*, x_{n+1} - x^*\rangle \\ &= (1-\alpha_n)^2||w_n - x^*||^2 + 2\alpha_n\beta_n\langle y_n - Sy_n, x_{n+1} - x^*\rangle \\ &\quad -2\alpha_n\langle x^*, x_{n+1} - x^*\rangle \\ &\leq (1-\alpha_n)^2||y_n - x^*||^2 + 2\alpha_n\beta_n\langle y_n - Sy_n, x_{n+1} - x^*\rangle \\ &\quad -2\alpha_n\langle x^*, x_{n+1} - x^*\rangle \\ &\leq (1-\alpha_n)||y_n - x^*||^2 \\ &\quad +\alpha_n[2\beta_n\langle y_n - Sy_n, x_{n+1} - x^*\rangle - 2\langle x^*, x_{n+1} - x^*\rangle]. \end{aligned}$$

We observe that $\limsup_{n\to\infty}\Big\{ - 2\langle x^*, x_{n+1} - x^*\rangle\Big\} \leq -2\langle x^*, z - x^*\rangle \leq 0$ (since $x^* = P_\Omega 0$) and $2\beta_n\langle y_n - Sy_n, x_{n+1} - x^*\rangle \to 0, \ n \to \infty$. Now, using Lemma 2.3 in the inequality above, we have $||y_n - x^*|| \to 0$ and consequently $||x_n - x^*|| \to 0$ by (3.8). That is, $x_n \to x^*, n \to \infty$.

Case 2

Assume that $\{||y_n - x^*||\}$ is not monotonically decreasing sequence. Set $\Gamma_n = ||y_n - x^*||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough)by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_n \leq \Gamma_{n+1}\}.$$

Clearly, τ is a non decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \ \forall n \geq n_0.$$

From (3.11), it is easy to see that

$$||y_{\tau(n)} - Sy_{\tau(n)}||^2 \leq \frac{t_{\tau(n)}M}{\beta_{\tau(n)}[(1-k_1) - \beta_{\tau(n)}]} \to 0, n \to \infty.$$

Thus,

$$||y_{\tau(n)} - Sy_{\tau(n)}|| \to 0, n \to \infty.$$

Furthermore, we can show that

$$||(T-I)Ax_{\tau(n)}|| \to 0, \; n \to \infty.$$

By similar argument as above in Case 1, we conclude immediately that $y_{\tau(n)}$ weakly converges to z as $\tau(n) \to \infty$. At the same time, we note that, for all $n \geq n_0$,

$$\begin{aligned} 0 &\leq ||y_{\tau(n)+1} - x^*||^2 - ||y_{\tau(n)} - x^*||^2 \\ &\leq \alpha_{\tau(n)}[2\langle \beta_{\tau(n)}(y_{\tau(n)} - Sy_{\tau(n)}), x_{\tau(n)+1} - x^* \rangle \\ &\quad -2\langle x^*, x_{\tau(n)+1} - x^* \rangle - ||y_{\tau(n)} - x^*||^2], \end{aligned}$$

which implies

$$||y_{\tau(n)} - x^*||^2 \leq 2\langle \beta_{\tau(n)}(y_{\tau(n)} - Sy_{\tau(n)}), x_{\tau(n)+1} - x^* \rangle - 2\langle x^*, x_{\tau(n)+1} - x^* \rangle.$$

Hence, we deduce that

$$\lim_{n\to\infty} ||y_{\tau(n)} - x^*|| = 0.$$

Therefore,

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$), because $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence, $\lim \Gamma_n = 0$, that is $\{y_n\}$ converges strongly to x^*. Hence, $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Corollary 3.2. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be the adjoint of A. Let $S : H_1 \to H_1$ be a quasi-nonexpansive mapping such that $S - I$ is demi-closed at 0, $C := F(S) \neq \emptyset$, $T : H_2 \to H_2$ be a quasi-nonexpansive mapping such that such that $T - I$ is demi-closed at 0 and $Q := F(T) \neq \emptyset$. Suppose that SCFPP (1.4) has a nonempty solution set Ω. Let $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ be two real sequences in $(0,1)$ and $\gamma \in \left(0, \frac{1}{||A||^2}\right)$. Let sequences $\{y_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ be generated by (3.1). Suppose the following conditions are satisfied:*

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and*

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to an element $x^* \in \Omega$ which is also the minimum-norm solution (i.e., $x^* \in \Omega$ and $||x^*|| = \min\{||x|| : x \in \Omega\}$).

4. Applications

4.1. **Split Variational Inequality Problem.** In this section, we apply our results to solving the split variational inequality problem in a real Hilbert space, newly introduced in [6].

Let H_1 and H_2 be two real Hilbert spaces. Given operators $f : H_1 \to H_1$ and $g : H_2 \to H_2$, a bounded linear operator $A : H_1 \to H_2$, and nonempty, closed and convex subsets $C \subseteq H_1$ and $Q \subseteq H_2$, the *split variational inequality problem* (SVIP) is formulated as follows:

Find a point $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0, \ \forall x \in C \tag{4.1}$$

and such that the point $y^* = Ax^* \in Q$ and solves

$$\langle g(y^*), y - y^* \rangle \geq 0, \ \forall y \in Q. \tag{4.2}$$

When looked at separately, (4.1) is the classical Variational Inequality Problem (VIP) and we denote its solution set by $SOL(C, f)$. The SVIP constitutes a pair of VIPs, which have to be solved so that the image $y^* = Ax^*$, under a given bounded linear operator A; of the solution x^* of the VIP in H_1, is a solution of another VIP in another space H_2. SVIP is quite general and should enable split minimization between two spaces so that the image of a solution point of one minimization problem, under a given bounded linear operator, is a solution point of another minimization problem.

Recalling that $SOL(C, f)$ and $SOL(Q, g)$ are the solution sets of (4.1) and (4.2), respectively, we see that the solution set of the SVIP is

$$\Omega := \Omega(C, Q, f, g, A) := \{z \in SOL(C, f) : Az \in SOL(Q, g)\}.$$

We now prove the following convergence theorem for split variational inequality problem.

Theorem 4.1. *Let H_1 and H_2 be real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Let $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be α_1-inverse strongly monotone and α_2-inverse strongly monotone operators on H_1 and H_2 respectively, and set $\alpha := \min\{\alpha_1, \alpha_2\}$. Assume that $\gamma \in \left(0, \frac{1}{||A||^2}\right)$. Consider the operators $S := P_C(I - \lambda f)$ and $T := P_Q(I - \lambda g)$ with $\lambda \in [0, 2\alpha]$. Assume further that $\Omega \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ be two real sequences in $(0, 1)$ satisfying:*

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^\infty \alpha_n = \infty$ *and*

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequences $\{y_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ generated by $x_1 \in H_1$,

$$\begin{cases} y_n = x_n + \gamma A^*(T - I)Ax_n \\ x_{n+1} = (1 - \alpha_n - \beta_n)y_n + \beta_n S y_n, \ n \geq 1 \end{cases} \tag{4.3}$$

converge strongly to an element $x^* \in \Omega$ which is also the minimum-norm solution (i.e., $x^* \in \Omega$ and $||x^*|| = \min\{||x|| : x \in \Omega\}$).

Proof. Let $S = P_C(I - \lambda f)$ and $T = P_Q(I - \lambda g)$. Then, for $x_1, x_2 \in C$, we obtain

$$\begin{aligned} ||Sx_1 - Sx_2||^2 &= ||P_C(I - \lambda f)x_1 - P_C(I - \lambda f)x_2||^2 \\ &\leq ||(I - \lambda f)x_1 - (I - \lambda f)x_2||^2 \\ &= ||x_1 - x_2||^2 - 2\lambda\langle f(x_1) - f(x_2), x_1 - x_2\rangle + \lambda^2||f(x_1) - f(x_2)||^2 \\ &\leq ||x_1 - x_2||^2 - 2\lambda\alpha_1||f(x_1) - f(x_2)||^2 + \lambda^2||f(x_1) - f(x_2)||^2 \\ &\leq ||x_1 - x_2||^2 - \lambda(2\alpha - \lambda)||f(x_1) - f(x_2)||^2 \\ &\leq ||x_1 - x_2||^2 \end{aligned}$$

and for $y_1, y_2 \in Q$, we have

$$\begin{aligned} ||Ty_1 - Ty_2||^2 &= ||P_Q(I - \lambda g)y_1 - P_Q(I - \lambda g)y_2||^2 \leq ||(I - \lambda g)y_1 - (I - \lambda g)y_2||^2 \\ &= ||y_1 - y_2||^2 - 2\lambda\langle g(y_1) - g(y_2), y_1 - y_2\rangle + \lambda^2||g(y_1) - g(y_2)||^2 \\ &\leq ||y_1 - y_2||^2 - 2\lambda\alpha_2||g(y_1) - f(y_2)||^2 + \lambda^2||g(y_1) - g(y_2)||^2 \\ &\leq ||y_1 - y_2||^2 - \lambda(2\alpha - \lambda)||g(y_1) - g(y_2)||^2 \\ &\leq ||y_1 - y_2||^2. \end{aligned}$$

This implies that S and T are nonexpansive and hence quasi-nonexpansive (0-demicontractive) mappings. We obtain the desired conclusion by following the line of arguments of proof of Theorem 3.1. □

4.2. **Split Convex Minimization Problem.** Consider the following constrained convex minimization problem:

$$minimize\{F(x) : x \in C\}, \tag{4.4}$$

where $F : C \to \mathbb{R}$ is a real-valued convex function. We say that the minimization problem (4.4) is consistent if the minimization problem (4.4) has a solution. In the sequel, we shall denote the solutions set of problem (4.4) by Ω. If F is (Fréchet) differentiable, then $x^* \in \Omega$ if and only if

$$\langle \nabla F(x^*), x - x^*\rangle \geq 0, \forall x \in C, \tag{4.5}$$

where ∇F is the gradient of f. Since (4.5) is a VIP, we make the following observation. If $F : H_1 \to H_1$ and $G : H_2 \to H_2$ are Fréchet differentiable convex functions on closed and convex subsets $C \subset H_1$ and $Q \subset H_2$ respectively, and if in the SVIP we take $f = \nabla F$ and $g = \nabla G$, then we obtain the following Split Minimization Problem (SMP):
find a point $x^* \in C$ such that

$$x^* = argmin\{f(x) : x \in C\} \tag{4.6}$$

and such that the point $y^* = Ax^* \in Q$ and solves

$$y^* = argmin\{g(y) : y \in Q\}. \tag{4.7}$$

Following the line of proof of Theorem 3.1 and Theorem 4.1, we can prove the following theorem.

Theorem 4.2. *Let H_1 and H_2 be real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Let $F : H_1 \to H_1$ and $G : H_2 \to H_2$ be Fréchet differentiable convex functions on closed and convex subsets $C \subset H_1$ and $Q \subset H_2$ respectively. Let ∇F be α_1-inverse strongly monotone and ∇G be α_2-inverse strongly monotone operators on H_1 and H_2 respectively, and set $\alpha := \min\{\alpha_1, \alpha_2\}$. Assume that $\gamma \in \left(0, \frac{1}{||A||^2}\right)$. Consider the operators $S := P_C(I - \lambda\nabla F)$ and $T := P_Q(I - \lambda\nabla G)$ with $\lambda \in [0, 2\alpha]$. Assume further that $\Omega \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ be two real sequences in $(0,1)$ satisfying:*

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and*

(c) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then the sequences $\{y_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ generated by $x_1 \in H_1$,

$$\begin{cases} y_n = x_n + \gamma A^*(T - I)Ax_n \\ x_{n+1} = (1 - \alpha_n - \beta_n)y_n + \beta_n S y_n, \ n \ge 1 \end{cases} \tag{4.8}$$

converge strongly to an element $x^* \in \Omega$ which is also the minimum-norm solution (i.e., $x^* \in \Omega$ and $||x^*|| = \min\{||x|| : x \in \Omega\}$).

4.3. **Split Common Zeros Problem.** The *Split Zeros Problem* (SZP), newly introduced in [6], is defined as: Let H_1 and H_2 be two Hilbert spaces. Given operators $B_1 : H_1 \to H_1$ and $B_2 : H_2 \to H_2$, and a bounded linear operator $A : H_1 \to H_2$, the SZP is formulated as follows:
find a point $x \in H_1$ such that

$$B_1(x) = 0 \text{ and } B_2(Ax) = 0. \tag{4.9}$$

This problem is a special case of the SVIP if A is a surjective operator. To see this, take in (4.1) - (4.2) $C = H_1, Q = H_2, f = B_1$ and $g = B_2$, and choose $x := x^* - B_1(x^*) \in H_1$ in (4.1) and $x \in H_1$ such that $Ax := Ax^* - B_2(Ax^*) \in H_2$ in (4.2).

The next lemma proved in [6] shows when the only solution of an SVIP is a solution of an SZP.

Lemma 4.3. *Let H_1 and H_2 be real Hilbert spaces, and $C \subseteq H_1$ and $Q \subseteq H_2$ nonempty, closed and convex subsets. Let $B_1 : H_1 \to H_1$ and $B_2 : H_2 \to H_2$ be α-ISM operators and let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $C \cap \{x \in H_1 : B_1(x) = 0\} \neq 0$, and that $Q \cap \{y \in H_2 : B_2(y) = 0\} \neq 0$, and denote*

$$\Omega := \Omega(C, Q, B_1, B_2, A) := \{z \in SOL(C, B_1) : Az \in SOL(Q, B_2)\}.$$

Then, for any $x^* \in C$ with $Ax^* \in Q$, x^* solves (4.9) if and only if $x^* \in \Omega$.

In view of Lemma 4.3 and using Theorem 3.1 and Theorem 4.1, we can prove the following convergence theorem for Split Common Zeros Problem.

Theorem 4.4. *Let H_1 and H_2 be real Hilbert spaces, and $C \subseteq H_1$ and $Q \subseteq H_2$ nonempty, closed and convex subsets. Let $B_1 : H_1 \to H_1$ and $B_2 : H_2 \to H_2$ be*

α-inverse strongly monotone operators (ISM) operators and let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $C \cap \{x \in H_1 : B_1(x) = 0\} \neq 0$, and that $Q \cap \{y \in H_2 : B_2(y) = 0\} \neq 0$. Assume that $\gamma \in \left(0, \frac{1}{||A||^2}\right)$. Consider the operators $S := P_C(I - \lambda B_1)$ and $T := P_Q(I - \lambda B_2)$ with $\lambda \in [0, 2\alpha]$. Assume further that $\Omega \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two real sequences in $(0, 1)$ satisfying:

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and*

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequences $\{y_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ generated by $x_1 \in H_1$,

$$\left\{ \begin{array}{l} y_n = x_n + \gamma A^*(T - I)Ax_n \\ x_{n+1} = (1 - \alpha_n - \beta_n)y_n + \beta_n S y_n, \ n \geq 1 \end{array} \right. \tag{4.10}$$

converge strongly to an element $x^* \in \Omega$ which is also the minimum-norm solution (i.e., $x^* \in \Omega$ and $||x^*|| = \min\{||x|| : x \in \Omega\}$).

Remark 4.5. The condition

$$\langle f(x), P_C(I - \lambda f)(x) - x^* \rangle \geq 0, \forall x \in H_1, x^* \in SOL(C, f).$$

imposed in Theorem 6.3 of [6] is dispensed with in our results

Remark 4.6. We make the following remarks concerning our results.

(1) In order to obtain strong convergence results, in this paper, we consider convergence analysis of split common fixed problem using a modified Mann-type iteration. In other words, the assumption of the condition "semi-compactness" as assumed in Chang *et al.* [8] or "demi-compactness" as assumed in Boonchari and Saejung [4] are dispensed with in our results. In addition, our iterative method is easy to implement.

(2) Our results extend the class of operators for split common fixed point problem considered in the results of Moudafi [15, 16] and Zhao and He [28] to a wider class of operators:- demicontractive mappings. Furthermore, our results extend the results of Li and Yao [12].

(3) In the results of Moudafi [17], weak convergence results were given concerning split common fixed point problem for demicontractive mappings while in this paper, we give *strong convergence* results for split common fixed point problem for demicontractive mappings.

(4) Since demicontractive operators include directed operators (an operator $T : H \to H$ is called directed if $\langle z - Tx, x - Tx \rangle \leq 0, \forall z \in F(T), x \in H$), then all the results in this paper hold if S and T are directed operators. Please see, for example, Cui *et al.* [10] and Bauschke and Combettes [1] for more details.

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