Journal of Nonlinear and Convex Analysis
Volume 16, Number 1, 2015, 183–192

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL


# KOROVKIN TYPE APPROXIMATION THEOREMS FOR $\sigma$-CONVERGENCE OF DOUBLE SEQUENCES

A. ALOTAIBI, M. MURSALEEN, AND S. A. MOHIUDDINE

Abstract. Korovkin type approximation theorems exhibit a variety of test functions which assure that the approximation property holds on the whole space if it holds for them. In this paper, we prove Korovkin type approximation theorems for functions of two variables by using different sets of test functions through $\sigma$-convergence of double sequences. We also show that our results are stronger than some other Korovkin type approximation theorems.

## 1. Introduction and preliminaries

A double sequence $x = (x_{jk})$ of real or complex numbers is said to be *bounded* if $\|x\|_\infty = \sup_{j,k} |x_{jk}| < \infty$. The space of all bounded double sequences is denoted by $\mathcal{M}_u$. A double sequence $x = (x_{jk})$ is said to *converge to the limit $L$ in Pringsheim's sense* (shortly, *$P$-convergent to $L$*) (see [20]) if for every $\varepsilon > 0$ there exists an integer $N$ such that $|x_{jk} - L| < \varepsilon$ whenever $j, k > N$. In this case $L$ is called the $P$-limit of $x$. If in addition $x \in \mathcal{M}_u$, then $x$ is said to be *boundedly convergent to $L$ in Pringsheim's sense* (shortly, *$BP$-convergent to $L$*). A double sequence $x = (x_{jk})$ is said to *converge regularly to $L$* (shortly, *$R$-convergent to $L$*) if $x$ is $P$-convergent and the limits $x_j := \lim\limits_k x_{jk}$ $(j \in \mathbb{N})$ and $x^k := \lim\limits_j x_{jk}$ $(k \in \mathbb{N})$ exist. Note that in this case the limits $\lim\limits_j \lim\limits_k x_{jk}$ and $\lim\limits_k \lim\limits_j x_{jk}$ exist and are equal to the $P$-limit of $x$. In general, for any notion of convergence $\nu$, the space of all $\nu$-convergent double sequences will be denoted by $\mathcal{C}_\nu$ and the limit of a $\nu$-convergent double sequence $x$ by $\nu\text{-}\lim\limits_{j,k} x_{jk}$, where $\nu \in \{P, BP, R\}$.

Let $\sigma$ be a one-to-one mapping from the set $\mathbb{N}$ of natural numbers into itself. A continuous linear functional $\varphi$ on $\ell_\infty$ is said to be an *invariant mean* or a *$\sigma$-mean* if and only if (i) $\varphi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all $k$, (ii) $\varphi(e) = 1$, where $e = (1, 1, 1, \dots)$, and (iii) $\varphi(x) = \varphi(x_{\sigma(k)})$ for all $x \in \ell_\infty$.

Throughout this paper we consider the mapping $\sigma$ which has no finite orbits, that is, $\sigma^p(k) \neq k$ for all integer $k \geq 0$ and $p \geq 1$, where $\sigma^p(k)$ denotes the *pth* iterate of $\sigma$ at $k$. Note that, a $\sigma$-mean extends the limit functional on $c$ in the sense that $\varphi(x) = \lim x$ for all $x \in c$, (see [16]). Consequently, $c \subset V_\sigma$ the set of bounded sequences all of whose $\sigma$-means are equal. We say that a sequence $x = (x_k)$ is *$\sigma$-convergent* if and only if $x \in V_\sigma$.

---

2010 *Mathematics Subject Classification.* 41A10, 41A25, 41A36, 40A30, 40G15.

*Key words and phrases.* Double sequence, $\sigma$-convergence, positive linear operator, Korovkin type approximation theorem.

The authors gratefully acknowledge the financial support from King Abdulaziz University, Jeddah, Saudi Arabia.

The idea of $\sigma$-convergence for double sequences has been studied in [5] and further studied in [17], [18] and [19].

A double sequence $x = (x_{jk})$ of real numbers is said to be *$\sigma$-convergent* to a number $L$ if and only if $x \in \mathcal{V}_\sigma$, where

$$\mathcal{V}_\sigma = \{x \in \mathcal{M}_u : P\text{-}\lim_{p,q\to\infty} \tau_{pqst}(x) = L \text{ uniformly in } s,t; L = \sigma\text{–}\lim x\}$$

$$\tau_{pqst}(x) = \frac{1}{pq}\sum_{j=0}^{p-1}\sum_{k=0}^{q-1} x_{\sigma^j(s),\sigma^k(t)}$$

and $\tau_{-1,q,s,t} = \tau_{p,-1,s,t} = \tau_{-1,-1,s,t} = 0$.

For $\sigma(n) = n+1$, the set $\mathcal{V}_\sigma$ is reduced to the set $\mathcal{F}$ of almost convergent double sequences [15]. The concept of almost convergence for single sequences was introduced by Lorentz [9].

Note that a convergent double sequence need not be $\sigma$-convergent. However every bounded convergent double sequence is $\sigma$-convergent and every $\sigma$-convergent double sequence is bounded.

**Example 1.1.** Let $\sigma(n) = n+1$. The double sequence $z = (z_{mn})$ defined by

$$z_{mn} = \begin{cases} 1 \text{ if } m = n \text{ odd}, \\ \\ -1 \text{ if } m = n \text{ even}, \\ \\ 0 \ (m \neq n); \end{cases} \tag{1.1}$$

is $\sigma$-convergent to zero but not $P$-convergent.

Let $C[a,b]$ be the space of all functions $f$ continuous on $[a,b]$. We know that $C[a,b]$ is a Banach space with norm

$$\|f\|_\infty := \sup_{x\in[a,b]} |f(x)|, \ f \in C[a,b].$$

The classical Korovkin approximation theorem states as follows [5]:

Let $(T_n)$ be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then $\lim_n \|T_n(f,x) - f(x)\|_\infty = 0$, for all $f \in C[a,b]$ if and only if $\lim_n \|T_n(f_i,x) - f_i(x)\|_\infty = 0$, for $i = 0,1,2$, where $f_0(x) = 1, f_1(x) = x$ and $f_2(x) = x^2$.

Quite recently, such type of approximation theorems are proved in [1, 11] for almost convergence of single and double sequences. For more details on Korovkin's type approximation theorem, one can be referred to [2, 4, 7, 12, 13, 14]. In this paper, we use the notion of $\sigma$-convergence of double sequences to prove approximation theorems for functions of two variables by using different sets of test functions.

## 2. For test functions $1, \frac{x}{1-x}, \frac{y}{1-y}, (\frac{x}{1-x})^2 + (\frac{y}{1-y})^2$

Let $I = [0,A]$, $J = [0,B]$, $A, B \in (0,1)$ and $K = I \times J$. We denote by $C(K)$ the space of all continuous real valued functions on $K$. This space is a equipped with norm

$$\|f\|_{C(K)} := \sup_{(x,y)\in K} |f(x,y)|, \ f \in C(K).$$

Let $H_\omega(K)$ denote the space of all real valued functions $f$ on $K$ such that

$$| f(s,t) - f(x,y) | \leq \omega\left(f; \sqrt{\left(\frac{s}{1-s} - \frac{x}{1-x}\right)^2 + \left(\frac{t}{1-t} - \frac{y}{1-y}\right)^2}\right),$$

where $\omega$ is the modulus of continuity, i.e.

$$\omega(f;\delta) = \sup_{(s,t),(x,y)\in K} \left\{ |f(s,t) - f(x,y)| : \sqrt{(s-x)^2 + (t-y)^2} \leq \delta \right\}.$$

It is to be noted that any function $f \in H_\omega(K)$ is continuous and bounded on $K$.

The following result was given by Taşdelen and Erençin [21].

**Theorem A.** *Let $(T_{j,k})$ be a double sequence of positive linear operators from $H_\omega(K)$ into $C(K)$. Then for all $f \in H_\omega(K)$*

$$P\text{-}\lim_{j,k\to\infty} \left\| T_{j,k}(f;x,y) - f(x,y) \right\|_{C(K)} = 0$$

*if and only if*

$$P\text{-}\lim_{j,k\to\infty} \left\| T_{j,k}(f_i;x,y) - f_i \right\|_{C(K)} = 0 \ (i = 0,1,2,3),$$

*where*

$$f_0(x,y) = 1,$$
$$f_1(x,y) = \frac{x}{1-x},$$
$$f_2(x,y) = \frac{y}{1-y},$$

*and*

$$f_3(x,y) = \left(\frac{x}{1-x}\right)^2 + \left(\frac{y}{1-y}\right)^2.$$

We prove the following result:

**Theorem 2.1.** *Let $(T_{j,k})$ be a double sequence of positive linear operators from $H_\omega(K)$ into $C(K)$. Then for all $f \in H_\omega(K)$*

$$\sigma\text{-}\lim \left\| T_{j,k}(f;x,y) - f(x,y) \right\|_{C(K)} = 0. \tag{2.1}$$

*if and only if*

$$\sigma\text{-}\lim \left\| T_{j,k}(1;x,y) - 1 \right\|_{C(K)} = 0, \tag{2.2}$$

$$\sigma\text{-}\lim \left\| T_{j,k}\left(\frac{s}{1-s};x,y\right) - \frac{x}{1-x} \right\|_{C(K)} = 0, \tag{2.3}$$

$$\sigma\text{-}\lim \left\| T_{j,k}\left(\frac{t}{1-t};x,y\right) - \frac{y}{1-y} \right\|_{C(K)} = 0, \tag{2.4}$$

$$\sigma\text{-}\lim\left\|T_{j,k}\left(\left(\frac{s}{1-s}\right)^2+\left(\frac{t}{1-t}\right)^2;x,y\right)-\left(\left(\frac{x}{1-x}\right)^2+\left(\frac{y}{1-y}\right)^2\right)\right\|_{C(K)}=0. \tag{2.5}$$

*Proof.* Since each of $1, \frac{x}{1-x}, \frac{y}{1-y}, (\frac{x}{1-x})^2+(\frac{y}{1-y})^2$ belongs to $H_\omega(K)$, conditions (2.2)-(2.5) follow immediately from (2.1). Let $f\in H_\omega(K)$ and $(x,y)\in K$ be fixed. Then, after using the properties of $f$, a simple calculation gives

$$\begin{aligned}
&\mid T_{j,k}(f;x,y)-f(x,y)\mid\\
\leq& T_{j,k}(\mid f(s,t)-f(x,y)\mid;x,y)+\mid f(x,y)\mid\mid T_{j,k}(f_0;x,y)-f_0(x,y)\mid\\
\leq&\varepsilon+\left(\varepsilon+N+\frac{2N}{\delta^2}\right)\mid T_{j,k}(f_0;x,y)-f_0(x,y)\mid+\frac{4N}{\delta^2}\mid T_{j,k}(f_1;x,y)-f_1(x,y)\mid\\
&+\frac{4N}{\delta^2}\mid T_{j,k}(f_2;x,y)-f_2(x,y)\mid+\frac{2N}{\delta^2}\mid T_{j,k}(f_3;x,y)-f_3(x,y)\mid\\
\leq&\varepsilon+M\big\{\mid T_{j,k}(f_0;x,y)-f_0(x,y)\mid+\mid T_{j,k}(f_1;x,y)-f_1(x,y)\mid\\
&+\mid T_{j,k}(f_2;x,y)-f_2(x,y)\mid+\mid T_{j,k}(f_3;x,y)-f_3(x,y)\mid\big\},
\end{aligned}$$

where $N=\| f\|_{C(K)}$ and

$$M=\max\left\{\varepsilon+N+\frac{2N}{\delta^2}\left(\left(\frac{A}{1-A}\right)^2+\left(\frac{B}{1-B}\right)^2\right),\frac{4N}{\delta^2}\left(\frac{A}{1-A}\right),\frac{4N}{\delta^2}\left(\frac{B}{1-B}\right),\frac{2N}{\delta^2}\right\}.$$

Now replacing $T_{j,k}(f;x,y)$ by $\frac{1}{pq}\sum_{j=0}^{p-1}\sum_{k=0}^{q-1}T_{\sigma^j(s),\sigma^k(t)}(f;x,y)$ and taking $\sup_{(x,y)\in K}$, we get

$$\begin{aligned}
&\left\|\frac{1}{pq}\sum_{j=0}^{p-1}\sum_{k=0}^{q-1}T_{\sigma^j(s),\sigma^k(t)}(f;x,y)-f(x,y)\right\|_{C(K)}\\
\leq&\varepsilon+M\left(\left\|\frac{1}{pq}\sum_{j=0}^{p-1}\sum_{k=0}^{q-1}T_{\sigma^j(s),\sigma^k(t)}(f_0;x,y)-f_0(x,y)\right\|_{C(K)}\right.\\
&+\left\|\frac{1}{pq}\sum_{j=0}^{p-1}\sum_{k=0}^{q-1}T_{\sigma^j(s),\sigma^k(t)}(f_1;x,y)-f_1(x,y)\right\|_{C(K)}\\
&+\left\|\frac{1}{pq}\sum_{j=0}^{p-1}\sum_{k=0}^{q-1}T_{\sigma^j(s),\sigma^k(t)}(f_2;x,y)-f_2(x,y)\right\|_{C(K)}\\
&\left.+\left\|\frac{1}{pq}\sum_{j=0}^{p-1}\sum_{k=0}^{q-1}T_{\sigma^j(s),\sigma^k(t)}(f_3;x,y)-f_3(x,y)\right\|_{C(K)}\right)
\end{aligned}\tag{2.6}$$

Now taking $\lim_{p,q\to\infty}$ uniformly in $s,t$ on both sides in (2.6) and using the conditions (2.2)-(2.5), we get

$$P\text{-}\lim_{p,q\to\infty}\left\|\frac{1}{pq}\sum_{j=0}^{p-1}\sum_{k=0}^{q-1}T_{\sigma^j(s),\sigma^k(t)}(f;x,y)-f(x,y)\right\|_{C(K)},\ \text{uniformly in } s,t.$$

That is,

$$\sigma\text{-}\lim\left\|T_{j,k}(f;x,y)-f(x,y)\right\|_{C(K)}=0.$$

This completes the proof of the theorem. □

We show that the following double sequence of positive linear operators satisfies the conditions of Theorem 2.1 but does not satisfy the conditions of Theorem A.

**Example 2.2.** Consider the following Meyer-König and Zeller [10] (of two variables) operators:

$$(2.7)\quad B_{m,n}(f;x,y)$$
$$:= (1-x)^{m+1}(1-y)^{n+1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} f\left(\frac{j}{j+m+1},\frac{k}{k+n+1}\right)\binom{m+j}{j}\binom{n+k}{k}x^j y^k,$$

where $f \in H_\omega(K)$, and $K = [0,A]\times[0,B]$, $A, B \in (0,1)$.

Since, for $x \in [0,A]$, $A \in (0,1)$,

$$\frac{1}{(1-x)^{m+1}} = \sum_{j=0}^{\infty}\binom{m+j}{j}x^j,$$

it is easy to see that

$$B_{m,n}(f_0;x,y) = f_0(x,y).$$

Also, we obtain

$$\begin{aligned} B_{m,n}(f_1;x,y) &= (1-x)^{m+1}(1-y)^{n+1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{j}{m+1}\binom{m+j}{j}\binom{n+k}{k}x^j y^k \\ &= (1-x)^{m+1}(1-y)^{n+1}x\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{m+1}\frac{(m+j)!}{m!(j-1)!}\binom{n+k}{k}x^{j-1}y^k \\ &= (1-x)^{m+1}(1-y)^{n+1}x\frac{1}{(1-x)^{m+2}}\frac{1}{(1-y)^{n+1}} = \frac{x}{(1-x)}, \end{aligned}$$

and similarly

$$B_{m,n}(f_2;x,y) = \frac{y}{(1-y)}.$$

Finally, we get

$$\begin{aligned} &B_{m,n}(f_3;x,y) \\ &= (1-x)^{m+1}(1-y)^{n+1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left\{\left(\frac{j}{m+1}\right)^2+\left(\frac{k}{n+1}\right)^2\right\}\binom{m+j}{j}\binom{n+k}{k}x^j y^k \\ &= (1-x)^{m+1}(1-y)^{n+1}\frac{x}{m+1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{j}{m+1}\frac{(m+j)!}{m!(j-1)!}\binom{n+k}{k}x^{j-1}y^k \\ &+ (1-x)^{m+1}(1-y)^{n+1}\frac{y}{n+1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{k}{n+1}\binom{m+j}{j}\frac{(n+k)!}{n!(k-1)!}x^j y^{k-1} \\ &= (1-x)^{m+1}(1-y)^{n+1}\frac{x}{m+1}\left\{x\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(m+j+1)!}{(m+1)!(j-1)!}\binom{n+k}{k}x^{j-1}y^k\right. \end{aligned}$$

$$+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{m+j+1}{j}\binom{n+k}{k}x^jy^k\Big\}$$

$$+(1-x)^{m+1}(1-y)^{n+1}\frac{y}{n+1}\Big\{y\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(n+k+1)!}{(n+1)!(k-1)!}\binom{m+j}{j}x^jy^{k-1}$$

$$+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{n+k+1}{k}\binom{m+j}{j}x^jy^k\Big\}$$

$$=\frac{m+2}{m+1}\Big(\frac{x}{1-x}\Big)^2+\frac{1}{m+1}\frac{x}{1-x}+\frac{n+2}{n+1}\Big(\frac{y}{1-y}\Big)^2+\frac{1}{n+1}\frac{y}{1-y}$$

$$\to\Big(\frac{x}{1-x}\Big)^2+\Big(\frac{y}{1-y}\Big)^2.$$

Therefore the conditions of Theorem A are satisfied, and we get for all $f\in H_\omega(K)$ that

$$P\text{-}\lim_{j,k\to\infty}\Big\|T_{j,k}(f;x,y)-f(x,y)\Big\|=0.$$

Let $w=(z_{mn})$ be defined by (1.1) which is $\sigma$-convergent to 0 but not $P$-convergent.

Let $L_{m,n}:H_\omega(K)\to C(K)$ be defined by

$$L_{m,n}(f;x,y)=(1+z_{mn})B_{m,n}(f;x,y).$$

It is easy to see that the sequence $(L_{m,n})$ satisfies the conditions (2.2)-(2.5). Hence by Theorem 2.1, we have

$$\sigma\text{-}\lim\|L_{m,n}(f;x,y)-f(x,y)\|=0.$$

On the other hand, the sequence $(L_{m,n})$ does not satisfy the conditions of Theorem A, since $(L_{m,n})$ is not $P$-convergent. That is, Theorem A does not work for our operators $L_{m,n}$. Hence our Theorem 2.1 is stronger than Theorem A.

## 3. For test functions $1, \frac{x}{1+x}, \frac{y}{1+y}, (\frac{x}{1+x})^2+(\frac{y}{1+y})^2$

Let $K=[0,\infty)\times[0,\infty)$. We denote by $C_B(K)$ the space of all bounded and continuous real valued functions on $K$ equipped with norm

$$\|f\|_{C_B(K)}:=\sup_{(x,y)\in K}|f(x,y)|,\ f\in C_B(K).$$

Let $H_{\omega^*}(K)$ denote the space of all real valued functions $f$ on $K$ such that

$$|\, f(s,t)-f(x,y)\,|\le\omega^*\Bigg(f;\sqrt{\Big(\frac{s}{1+s}-\frac{x}{1+x}\Big)^2+\Big(\frac{t}{1+t}-\frac{y}{1+y}\Big)^2}\Bigg),$$

where $\omega^*$ is the modulus of continuity, i.e.

$$\omega^*(f;\delta)=\sup_{(s,t),(x,y)\in K}\Big\{|f(s,t)-f(x,y)|:\sqrt{(s-x)^2+(t-y)^2}\le\delta\Big\}.$$

It is to be noted that any function $f \in H_{\omega^*}(K)$ is bounded and continuous on $K$, and a necessary and sufficient condition for $f \in H_{\omega^*}(K)$ is that

$$\lim_{\delta \to 0} \omega^*(f;\delta) = 0.$$

The following is two-dimensional version of the Korovkin type theorem of Çakar and Gadjiev [6].

**Theorem B.** *Let $(T_{jk})$ be a sequence of positive linear operators from $H_{\omega^*}(K)$ into $C_B(K)$. Then for all $f \in H_{\omega^*}(K)$*

$$P\text{-}\lim_{j,k\to\infty} \left\| T_{jk}(f;x,y) - f(x,y) \right\|_{C_B(K)} = 0$$

*if and only if*

$$P\text{-}\lim_{j,k\to\infty} \left\| T_{jk}(f_i;x,y) - f_i \right\|_{C_B(K)} = 0 \ (i = 0,1,2,3),$$

*where*

$$f_0(x,y) = 1,$$
$$f_1(x,y) = \frac{x}{1+x},$$
$$f_2(x,y) = \frac{y}{1+y},$$

*and*

$$f_3(x,y) = \left(\frac{x}{1+x}\right)^2 + \left(\frac{y}{1+y}\right)^2.$$

We prove the following result:

**Theorem 3.1.** *Let $(T_{jk})$ be a double sequence of positive linear operators from $H_{\omega^*}(K)$ into $C_B(K)$. Then for all $f \in H_{\omega^*}(K)$*

$$\sigma\text{-}\lim \left\| T_{jk}(f;x,y) - f(x,y) \right\|_{C_B(K)} = 0 \tag{3.1}$$

*if and only if*

$$\sigma\text{-}\lim \left\| T_{jk}(1;x,y) - 1 \right\|_{C_B(K)} = 0 \tag{3.2}$$

$$\sigma\text{-}\lim \left\| T_{jk}\Big(\frac{s}{1+s};x,y\Big) - \frac{x}{1+x} \right\|_{C_B(K)} = 0 \tag{3.3}$$

$$\sigma\text{-}\lim \left\| T_{jk}\Big(\frac{t}{1+t};x,y\Big) - \frac{y}{1+y} \right\|_{C_B(K)} = 0 \tag{3.4}$$

$$\sigma\text{-}\lim \left\| T_{jk}\bigg(\Big(\frac{s}{1+s}\Big)^2 + \Big(\frac{t}{1+t}\Big)^2;x,y\bigg) - \bigg(\Big(\frac{x}{1+x}\Big)^2 + \Big(\frac{y}{1+y}\Big)^2\bigg) \right\|_{C_B(K)} = 0. \tag{3.5}$$

*Proof.* Since each of the functions $f_0(x,y)=1$, $f_1(x,y)=\frac{x}{1+x}$, $f_2(x,y)=\frac{y}{1+y}$, $f_3(x,y)=(\frac{x}{1+x})^2+(\frac{y}{1+y})^2$ belongs to $H_{\omega^*}(K)$, conditions (3.2)-(3.5) follow immediately from (3.1). Let $f\in H_{\omega^*}(K)$ and $(x,y)\in K$ be fixed. Then for $\varepsilon>0$ there exist $\delta_1,\delta_2>0$ such that $\mid f(s,t)-f(x,y)\mid<\varepsilon$ holds for all $(s,t)\in K$ satisfying $|\frac{s}{1+s}-\frac{x}{1+x}|<\delta_1$ and $|\frac{t}{1+t}-\frac{y}{1+y}|<\delta_2$. Let

$$K(\delta):=\left\{(s,t)\in K:\sqrt{\left(\frac{s}{1+s}-\frac{x}{1+x}\right)^2+\left(\frac{t}{1+t}-\frac{y}{1+y}\right)^2}<\delta=\min\{\delta_1,\delta_2\}\right\}.$$

Hence

$$\begin{aligned}\mid f(s,t)-f(x,y)\mid &=\mid f(s,t)-f(x,y)\mid\chi_{K(\delta)}(s,t)+\mid f(s,t)-f(x,y)\mid\chi_{K\setminus K(\delta)}(s,t)\\ &\leq\varepsilon+2N\chi_{K\setminus K(\delta)}(s,t),\end{aligned}\tag{3.6}$$

where $\chi_D$ denotes the characteristic function of the set $D$ and $N=\|f\|_{C_B(K)}$. Further we get

$$\chi_{K\setminus K(\delta)}(s,t)\leq\frac{1}{\delta_1^2}\left(\frac{s}{1+s}-\frac{x}{1+x}\right)^2+\frac{1}{\delta_2^2}\left(\frac{t}{1+t}-\frac{y}{1+y}\right)^2.\tag{3.7}$$

Combining (3.6) and (3.7), we get

$$\mid f(s,t)-f(x,y)\mid\leq\varepsilon+\frac{2N}{\delta^2}\left\{\left(\frac{s}{1+s}-\frac{x}{1+x}\right)^2+\left(\frac{t}{1+t}-\frac{y}{1+y}\right)^2\right\}.\tag{3.8}$$

After using the properties of $f$, a simple calculation gives that

$$\begin{aligned}\mid T_{jk}(f;x,y)-f(x,y)\mid\leq\varepsilon+M\{&\mid T_{jk}(f_0;x,y)-f_0(x,y)\mid\\ &+\mid T_{jk}(f_1;x,y)-f_1(x,y)\mid\\ &+\mid T_{jk}(f_2;x,y)-f_2(x,y)\mid\\ &+\mid T_{jk}(f_3;x,y)-f_3(x,y)\mid\},\end{aligned}\tag{3.9}$$

where

$$M:=\varepsilon+N+\frac{4N}{\delta^2}.$$

Now replacing $T_{j,k}(f;x,y)$ by $\frac{1}{pq}\sum_{j=0}^{p-1}\sum_{k=0}^{q-1}T_{\sigma^j(s),\sigma^k(t)}(f;x,y)$ and taking $\sup_{(x,y)\in K}$, we get

$$\begin{aligned}&\left\|\frac{1}{pq}\sum_{j=0}^{p-1}\sum_{k=0}^{q-1}T_{\sigma^j(s),\sigma^k(t)}(f;x,y)-f(x,y)\right\|_{C(K)}\\ &\qquad\leq\varepsilon+M\sum_{i=0}^{3}\left(\left\|\frac{1}{pq}\sum_{j=0}^{p-1}\sum_{k=0}^{q-1}T_{\sigma^j(s),\sigma^k(t)}(f_i;x,y)-f_i(x,y)\right\|_{C(K)}\right.,\end{aligned}\tag{3.10}$$

where the functions $f_i$ $(i=0,1,2,3)$ are same as in Theorem B. Now taking $\lim_{p,q\to\infty}$ uniformly in $s,t$ on both sides in (3.10) and using the conditions (3.2)-(3.5), we immediately get (3.1).

This completes the proof of the theorem. □

We show that the following double sequence of positive linear operators satisfies the conditions of Theorem 3.1 but does not satisfy the conditions of Theorem B.

**Example 3.2.** Consider the following Bleimann, Butzer and Hahn [3] (of two variables) operators:

$$B_{m,n}(f;x,y) := \frac{1}{(1+x)^m(1+y)^n}\sum_{j=0}^{m}\sum_{k=0}^{n} f\left(\frac{j}{m-j+1}, \frac{k}{n-k+1}\right)\binom{m}{j}\binom{n}{k}x^j y^k,$$

where $f \in H_\omega(K)$, $K = [0,\infty)\times[0,\infty)$ and $n \in \mathbb{N}$.

Since

$$(1+x)^m = \sum_{j=0}^{m}\binom{m}{j}x^j \text{ and } (1+y)^n = \sum_{k=0}^{n}\binom{n}{k}y^k,$$

it is easy to see that

$$B_{mn}(f_0;x,y) \to 1 = f_0(x,y).$$

Also by simple calculation, we obtain

$$B_{mn}(f_1;x,y) \to \frac{x}{1+x} = f_1(x,y),$$

$$B_{mn}(f_2;x,y) \to \frac{y}{1+y} = f_2(x,y),$$

and

$$B_{mn}(f_3;x,y) \to \left(\frac{x}{1+x}\right)^2 + \left(\frac{y}{1+y}\right)^2 = f_3(x,y).$$

Let the operator $L_{mn} : H_\omega(K) \to C_B(K)$ be defined by

$$L_{mn}(f;x,y) = (1+z_{mn})B_{mn}(f;x,y).$$

It is easy to see that the sequence $(L_{mn})$ satisfies conditions (3.2)-(3.5). Hence by Theorem 3.1, we have

$$\sigma\text{-}\lim\left\|L_{mn}(f;x,y) - f(x,y)\right\|_{C_B(K)} = 0.$$

On the other hand, the sequence $(L_{mn})$ does not satisfy the conditions of Theorem B, since $(L_{mn})$ is $\sigma$-convergent to 0 but not $P$-convergent. That is, Theorem B does not work for our operators $L_{mn}$. Hence our Theorem 3.1 is stronger than Theorem B.

## References


[1] G. A. Anastassiou, M. Mursaleen and S. A. Mohiuddine, *Some approximation theorems for functions of two variables through almost convergence of double sequences*, J. Comput. Analy. Appl. **13** (2011), 37–46.

[2] C. Belen and S. A. Mohiuddine, *Generalized weighted statistical convergence and application*, Applied Math. Comput. **219** (2013), 9821–9826.

[3] G. Bleimann, P. L. Butzer and L. Hahn, *A Bernstein type operator approximating continuous functions on semiaxis*, Indag. Math. **42** (1980), 255–262.

[4] N. L. Braha, H. M. Srivastava and S. A. Mohiuddine, *A Korovkin's type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallée Poussin mean*, Applied Math. Comput. **228** (2014), 162–169.

[5] C. Çakan, B. Altay and M. Mursaleen, *The σ-convergence and σ-core of double sequences*, Applied Math. Letters **19** (2006), 1122–1128.

[6] Ö. Çakar and A. D. Gadjiev, *On uniform approximation by Bleimann, Butzer and Hahn on all positive semiaxis*, Tras. Acad. Sci. Azerb. Ser. Phys. Tech. Math. Sci. **19** (1999), 21–26.

[7] O. H. H. Edely, S. A. Mohiuddine and A. K. Noman, *Korovkin type approximation theorems obtained through generalized statistical convergence*, Applied Math. Letters **23** (2010), 1382–1387.

[8] P. P. Korovkin, *Linear operators and approximation theory*, Hindustan Publ. Co., Delhi, 1960.

[9] G. G. Lorentz, *A contribution to theory of divergent sequences*, Acta Math. **80** (1948), 167–190.

[10] W. Meyer-König and K. Zeller, *Bersteinsche Potenzreihen*, Studia Math. **19** (1960), 89–94.

[11] S. A. Mohiuddine, *An application of almost convergence in approximation theorems*, Applied Math. Letters **24** (2011), 1856–1860.

[12] S. A. Mohiuddine and A. Alotaibi, *Statistical convergence and approximation theorems for functions of two variables*, J. Comput. Anal. Appl. **15** (2013), 218–223.

[13] S. A. Mohiuddine and A. Alotaibi, *Korovkin second theorem via statistical summability* $(C, 1)$, J. Inequal. Appl. **2013**, Article ID 149 (2013).

[14] S. A. Mohiuddine, A. Alotaibi and M. Mursaleen, *Statistical summability* $(C, 1)$ *and a Korovkin type approximation theorem*, J. Inequal. Appl. **2012**, Article ID 172 (2012).

[15] F. Moricz and B. E. Rhoades, *Almost convergence of double sequences and strong regularity of summability matrices*, Math. Proc. Camb. Phil. Soc. **104** (1988), 283–294.

[16] M. Mursaleen, *On some new invariant matrix methods of summability*, Quart. J. Math. Oxford **34** (1983), 77–86.

[17] M. Mursaleen and S. A. Mohiuddine, *Double $\sigma$-multiplicative matrices*, J. Math. Anal. Appl. **327** (2007), 991–996.

[18] M. Mursaleen and S. A. Mohiuddine, *Regularly $\sigma$-conservative and $\sigma$-coercive four dimensional matrices*, Comp. Math. Appl. **56** (2008), 1580–1586.

[19] M. Mursaleen and S. A. Mohiuddine, *On $\sigma$-conservative and boundedly $\sigma$-conservative four dimensional matrices*, Comput. Math. Appl. **59** (2010), 880–885.

[20] A. Pringsheim, *Zur theorie der zweifach unendlichen Zahlenfolgen*, Math. Z. **53** (1900), 289–321.

[21] F. Taşdelen and A. Erençin, *The generalization of bivariate MKZ operators by multiple generating functions*, J. Math. Anal. Appl. **331** (2007), 727–735.


*Manuscript received October 29, 2013*
*revised March 12, 2014*

A. Alotaibi
Department of Mathematics, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia
*E-mail address*: mathker11@hotmail.com

M. Mursaleen
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
*E-mail address*: mursaleenm@gmail.com

S. A. Mohiuddine
Department of Mathematics, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia
*E-mail address*: mohiuddine@gmail.com