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SOME ALGORITHMS FOR SOLVING OPTIMIZATION PROBLEMS IN HILBERT SPACES

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Abstract. Let C be a nonempty closed subspace of a real Hilbert space H, let $T : C \to H$ be a k-strictly pseudocontractive mapping T with the fixed point set $F(T) \neq \emptyset$ for some $0 \leq k < 1$, and let $f : C \to C$ be a contractive mapping with a constant $\alpha \in (0, 1)$. We devise two new iterative algorithms (one implicit and one explicit) for T, which are used to find the solution the optimization problem

$$\min_{x \in F(T)} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x),$$

where A is a strongly positive bounded linear self-adjoint operator on C, $\mu \geq 0$ is some constant, u is a given point in C, and h is a potential function for γf with $\gamma > 0$ (i.e., $h' = \gamma f$). As a direct consequence, we obtain the unique minimum-norm fixed point of T.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H, and let $T : C \to C$ be a self-mapping on C. We denote by $F(T)$ the set of fixed points of T, that is, $F(T) := \{x \in C : Tx = x\}$.

The class of pseudocontractive mappings is one of the most important classes of mappings among nonlinear mappings. Recently, many authors have been devoting the studies on the problems of finding fixed points for pseudocontractive mappings, see, for example, [1, 5, 11, 16] and the references therein. We recall that a mapping $T : C \to H$ is said to be *k-strictly pseudocontractive* if there exists a constant $k \in [0, 1)$such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C.$$

Note that the class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, T is nonexpansive (*i.e.*, $\|Tx - Ty\| \leq \|x - y\|$, $\forall x,\ y \in C$) if and only if T is 0-strictly pseudocontractive. The mapping T is also said to be pseudocontractive if $k = 1$, and T is said to be strongly pseudocontractive if there exists a constant $\lambda \in (0, 1)$ such that $T - \lambda I$ is pseudocontractive. Clearly, the class of k-strictly pseudocontractive mappings

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falls into the one between classes of nonexpansive mappings and pseudocontractive mappings. Also, we remark that the class of strongly pseudocontractive mappings is independent of the class of k-strictly pseudo-contractive mappings (see [3, 4]).

Let A be a strongly positive bounded linear self-adjoint operator on H with a constant $\overline{\gamma} > 0$, that is, there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H.$$

Let $f : C \to C$ be a contractive mapping with constant $\alpha \in (0, 1)$, that is, there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x,\ y \in C$.

The following optimization problem has been studied extensively by many authors:

$$\min_{x\in\Omega} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x),$$

where $\Omega = \bigcap_{i=1}^{\infty} C_i$, $C_1, C_2, \cdots$, are infinitely many closed convex subsets of H such that $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$, $u \in H$, $\mu \geq 0$ is a real number, A is a strongly positive bounded linear self-adjoint operator on H and h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$ for all $x \in H$). For this kind of minimization problems, see, for example, Bauschke and Borwein [2], Combettes [7], Deutsch and Yamada [8], Jung [10] and Xu [18] when $h(x) = \langle x, b\rangle$ for b is a given point in H.

Iterative algorithms for nonexpansive mappings and strictly pseudocontractive mappings have recently been applied to solve the optimization problem, where the constraint set is the set of fixed points of the mapping, see, e.q., [5, 8, 11, 15, 19, 20] and the references therein. Some iterative algorithms for equilibrium problems, variational inequality problems and fixed point problems to solve optimization problem, where the constraint set is the common set of the set of solutions of the problems and the set of fixed points of the mappings, were also investigated by many authors recently, see, e.q., [12, 21, 22] and the references therein.

Inspired and motivated by the recent works in this direction, in this paper, we consider the following optimization problem

$$\min_{x\in F(T)} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{1.1}$$

where $F(T)$ is the set of fixed points of a k-strictly pseudocontractive mapping T. We introduce new implicit and explicit iterative algorithms for a k-strictly pseudocontractive mapping T related to the optimization problem (1.1), and then prove that the sequences generated by the proposed iterative algorithms converge strongly to a fixed point of the mapping T, which solves the optimization problem (1.1). In particular, in order to establish strong convergence of explicit iterative algorithm, we utilize weak and different control conditions in comparison with previous ones. As a direct consequence, we obtain the unique minimum-norm point in the set $F(T)$.

2. Preliminaries and lemmas

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. In the following, when $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$) will denote strong (resp., weak) convergence of the sequence $\{x_n\}$ to x.

We need some facts and tools in a real Hilbert space which are listed as lemmas below. We will use them in the proofs for the main results in next section.

Recall that for every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|$$

for all $y \in C$. P_C is called the *metric projection* of H onto C. It is well known that P_C is nonexpansive.

Lemma 2.1 ([9]). *Let H a real Hilbert space, let C be a nonempty closed convex subset of H, and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$.*

The following Lemmas 2.2 and 2.3 are not hard to prove (see also Lemmas 2.3 and 2.5 in [15]).

Lemma 2.2. *Let $\mu > 0$, and let $A : H \to H$ be a strongly positive linear bounded operator on a Hilbert space H with a constant $\overline{\gamma} \in (0,1)$ such that $(1+\mu)\overline{\gamma} < 1$. Let $0 < \rho \leq (1 + \mu\|A\|)^{-1}$. Then $\|I - \rho(I + \mu A)\| < 1 - \rho(1+\mu)\overline{\gamma}$*

Lemma 2.3. *Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $f : C \to C$ be a contractive mapping with constant $\alpha \in (0,1)$, and let $A : C \to C$ be a strongly positive bounded linear operator with a constant $\overline{\gamma} \in (0,1)$. Let $\mu > 0$ and $0 < \gamma < (1+\mu)\overline{\gamma}/\alpha$ with $(1+\mu)\overline{\gamma} < 1$. Then for all $x,\ y \in C$,*

$$\langle x - y, ((I+\mu A) - \gamma f)x - ((I+\mu A) - \gamma f)y\rangle \geq ((1+\mu)\overline{\gamma} - \gamma\alpha)\|x - y\|^2.$$

That is, $(I+\mu A) - \gamma f$ is strongly monotone with a constant $(1+\mu)\overline{\gamma} - \gamma\alpha$.

Lemma 2.4 ([23]). *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let $T : C \to H$ be a k-strictly pseudo-contractive mapping. Then the following hold:*

(i) *The fixed point set $F(T)$ is closed convex so that the projection $P_{F(T)}$ is well defined,*
(ii) *$F(P_C T) = F(T)$,*
(iii) *If we define a mapping $S : C \to H$ by $Sx = \lambda x + (1-\lambda)Tx$ for all $x \in C$. Then, as $\lambda \in [k,1)$, S is a nonexpansive mapping such that $F(S) = F(T)$.*

Lemma 2.5 ([14, 18]). *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n\delta_n + r_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) *$\{\lambda_n\} \subset [0,1]$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$,*
(ii) *$\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=0}^{\infty} \lambda_n|\delta_n| < \infty$,*
(iii) *$r_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} r_n < \infty$.*

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.6. *In a Hilbert space H, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,\ y \in H.$$

Let LIM be a Banach limit. According to time and circumstances, we use $LIM_n(a_n)$ instead of $LIM(a)$ Then the following are well-known:

(i) for all $n \geq 1, a_n \leq c_n$ implies $LIM_n(a_n) \leq LIM_n(c_n)$,
(ii) $LIM_n(a_{n+N}) = LIM_n(a_n)$ for any fixed positive integer N,
(iii) $\liminf_{n\to\infty} a_n \leq LIM_n(a_n) \leq \limsup_{n\to\infty} a_n$ for all $\{a_n\} \in l^\infty$

The following lemma was given in Proposition 2 in [17].

Lemma 2.7. *Let $a \in \mathbb{R}$ be a real number, and let a sequence $\{a_n\} \in \ell^\infty$ satisfy the condition $LIM_n(a_n) \leq a$ for all Banach limit LIM. If $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$, then $\limsup_{n\to\infty} a_n \leq a$.*

The following lemma can be found in [21](see also Lemma 2.1 in [10]).

Lemma 2.8. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $g : C \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontiunous differentiable convex function. If x^* is a solution to the minimization problem*

$$g(x^*) = \inf_{x\in C} g(x),$$

then

$$\langle g'(x^*), x - x^* \rangle \geq 0, \quad x \in C.$$

In particular, if x^* solves the optimization problem

$$\min_{x\in C} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x),$$

then

$$\langle u + (\gamma f - (I + \mu A))x^*, x - x^* \rangle \leq 0, \quad x \in C,$$

where h is a potential function for γf.

Finally, we recall that the sequence $\{x_n\}$ in H is said to be *weakly asymptotically regular* if

$$w - \lim_{n\to\infty}(x_{n+1} - x_n) = 0, \quad \text{that is,} \quad x_{n+1} - x_n \rightharpoonup 0$$

and *asymptotically regular* if

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$$

respectively.

3. Main results

Throughout the rest of this paper, we always assume the following:

- H is a real Hilbert space;
- C is a nonempty closed subspace of H;
- $T : C \to H$ is a k-strictly pseudocontractive mapping with $F(T) \neq \emptyset$ for some $0 \leq k < 1$;
- $S : C \to H$ is a mapping defined by $Sx = kx + (1 - k)Tx$;
- $A : C \to C$ is a strongly positive bounded linear self-adjoint operator with a constant $\overline{\gamma} \in (0, 1)$;
- $f : C \to C$ is a contractive mapping with a constant $\alpha \in (0, 1)$;
- $\mu > 0$ and $0 < \gamma < (1 + \mu)\overline{\gamma}/\alpha$ with $(1 + \mu)\overline{\gamma} < 1$;

- $u \in C$ is a fixed element;
- P_C is a metric projection of H onto C.

First, in order to find a solution of the optimization problem (1.1), we construct the following iterative algorithm which generates a net $\{x_t\}$ in an implicit way:

$$x_t = t(u + \gamma f(x_t)) + (I - t(I + \mu A))P_C S x_t, \quad \forall t \in \left(0, \frac{1}{1 + \mu\|A\|}\right). \tag{3.1}$$

To this end, for $t \in (0, 1)$ such that $t < (1 + \mu\|A\|)^{-1}$, consider a mapping $Q_t : C \to C$ by

$$Q_t x = t(u + \gamma f(x)) + (I - t(I + \mu A))P_C S x, \quad \forall x \in C.$$

It is easy to see that Q_t is a contraction with constant $1 - t((1+\mu)\overline{\gamma} - \gamma\alpha)$. Indeed, by Lemma 2.2, we have

$$\begin{aligned}
\|Q_t x - Q_t y\| &\le t\gamma\|f(x) - f(y)\| + \|(I - t(I + \mu A))(P_C S x - P_C S y)\| \\
&\le t\gamma\alpha\|x - y\| + (1 - t(1 + \mu)\overline{\gamma})\|x - y\| \\
&= (1 - t((1 + \mu)\overline{\gamma} - \gamma\alpha))\|x - y\|.
\end{aligned}$$

Hence Q_t has a unique fixed point, denoted x_t, which uniquely solve the fixed point equation

$$x_t = t(u + \gamma f(x_t)) + (I - t(I + \mu A))P_C S x_t.$$

If we take $\mu = 0$, $u = 0$ and $f = 0$ in (3.1), then we have

$$x_t = (1 - t)P_C S x_t, \quad \forall t \in (0, 1). \tag{3.2}$$

We summary the basic properties of the net $\{x_t\}$, which can be proved by the same method in [15]. We includes its proof for the sake of completeness.

Proposition 3.1. *Let $\{x_t\}$ be defined by the implicit algorithm* (3.1). *Then*

(i) *$\{x_t\}$ is bounded for $t \in (0, (1 + \mu\|A\|)^{-1})$;*
(ii) *$\lim_{t\to 0} \|x_t - P_C S x_t\| = 0$;*
(iii) *x_t defines a continuous path from $(0, (1 + \mu\|A\|)^{-1})$ in C.*

Proof. (1) Set $\overline{A} = (I + \mu A)$ and Pick $p \in F(T)$. Observing $F(T) = F(S)$ by Lemma 2.4 (iii), from Lemma 2.2, we have

$$\begin{aligned}
\|x_t - p\| &= \|tu + t(\gamma f(x_t) - \overline{A}p) + (I - t\overline{A})(P_C S x_t - p)\| \\
&\le \|(I - t\overline{A})(P_C S x_t - p)\| + t\|u\| + t\gamma\|f(x_t) - f(p)\| + t\|\gamma f(p) - \overline{A}p\| \\
&\le (1 - t((1 + \mu)\overline{\gamma} - \gamma\alpha))\|x_t - p\| + t(\|u\| + \|\gamma f(p) - \overline{A}p\|).
\end{aligned}$$

So, it follows that

$$\|x_t - p\| \le \frac{\|u\| + \|\gamma f(p) - \overline{A}p\|}{(1 + \mu)\overline{\gamma} - \gamma\alpha}.$$

Hence $\{x_t\}$ is bounded and so are $\{f(x_t)\}$, $\{P_C S x_t\}$ and $\{\overline{A}P_C S x_t\}$.

(ii) We have $\|x_t - P_C S x_t\| = t\|u + \gamma f(x_t) - \overline{A}P_C S x_t\| \to 0$ as $t \to 0$ by the boundedness of $\{f(x_t)\}$ and $\{\overline{A}P_C S x_t\}$ in (i).

(iii) Let $t,\ t_0 \in (0, (1+\mu\|A\|)^{-1})$ and calculate

$$\begin{aligned}\|x_t - x_{t_0}\| &= \|(t-t_0)u + (t-t_0)\gamma f(x_t) + t_0\gamma(f(x_t) - f(x_{t_0}))\\ &\qquad - (t-t_0)\overline{A}P_C S x_t + (I - t_0\overline{A})(P_C S x_t - P_C S x_{t_0})\|\\ &\le |t-t_0|\|u\| + |t-t_0|\gamma\|f(x_t)\| + t_0\gamma\alpha\|x_t - x_{t_0}\|\\ &\qquad |t-t_0|\|\overline{A}P_C S x_t\| + (1 - t_0(1+\mu)\overline{\gamma})\|x_t - x_{t_0}\|.\end{aligned}$$

It follows that

$$\|x_t - x_{t_0}\| \le \frac{\|u\| + \gamma\|f(x_t)\| + \|\overline{A}P_C S x_t\|}{t_0((1+\mu)\overline{\gamma} - \gamma\alpha)}|t - t_0|.$$

This show that x_t is locally Lipschizian and hence continuous. $\square$

We provide the following result for the existence of solutions of the optimization problem (1.1).

Theorem 3.2. *The net* $\{x_t\}$ *defined by the implicit algorithm* (3.1) *converges strongly to a fixed point* $\widetilde{x}$ *of* T *as* $t \to 0$*, which solves the following variational inequality:*

$$\langle u + (\gamma f - (I+\mu A))\widetilde{x}, p - \widetilde{x}\rangle \le 0, \quad p \in F(T). \tag{3.3}$$

This $\widetilde{x}$ *is a solution of the optimization problem* (1.1).

Proof. We first show that the uniqueness of a solution of the variational inequality (3.3), which is indeed a consequence of the strong monotonicity of $(I+\mu A) - \gamma f$. Suppose that $\widetilde{x} \in F(T)$ and $\widehat{x} \in F(T)$ both are solutions to (3.3). Then we have

$$\langle u + (\gamma f - (I+\mu A))\widetilde{x}, \widehat{x} - \widetilde{x}\rangle \le 0 \tag{3.4}$$

and

$$\langle u + (\gamma f - (I+\mu A))\widehat{x}, \widetilde{x} - \widehat{x}\rangle \le 0. \tag{3.5}$$

Adding up (3.4) and (3.5) yields

$$\langle((I+\mu A) - \gamma f)\widetilde{x} - ((I+\mu A) - \gamma f)\widehat{x}, \widetilde{x} - \widehat{x}\rangle \le 0.$$

The strong monotonicity of $(I+\mu A) - \gamma f$ (Lemma 2.3) implies that $\widetilde{x} = \widehat{x}$ and the uniqueness is proved.

Next, we prove that $x_t \to \widetilde{x}$ as $t \to 0$. Observing $F(T) = F(S)$ by Lemma 2.4 (iii), from (3.1), we write, for given $p \in F(T)$,

$$x_t - p = t(u + \gamma f(x_t) - (I+\mu A)p) + (I - t(I+\mu A))(P_C S x_t - p)$$

to derive that

$$\begin{aligned}\|x_t - p\|^2 &= t\langle u + \gamma f(x_t) - (I+\mu A)p, x_t - p\rangle\\ &\qquad + \langle(I - t(I+\mu A))(P_C S x_t - p), x_t - p\rangle\\ &\le (1 - t(1+u)\overline{\gamma})\|x_t - p\|^2 + t\langle u + \gamma f(x_t) - (I+\mu A)p, x_t - p\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_t - p\|^2 &\leq \frac{1}{(1+\mu)\overline{\gamma}}\langle u + \gamma f(x_t) - (I+\mu A)p, x_t - p\rangle \\ &\leq \frac{1}{(1+\mu)\overline{\gamma}}\{\gamma\alpha\|x_t - p\|^2 + \langle u + \gamma f(p) - (I+\mu A)p, x_t - p\rangle\}.\end{aligned}$$

Therefore

$$\|x_t - p\|^2 \leq \frac{1}{(1+\mu)\overline{\gamma} - \gamma\alpha}\langle u + \gamma f(p) - (I+\mu A)p, x_t - p\rangle. \tag{3.6}$$

Since $\{x_t\}$ is bounded as $t \to 0$ (by Proposition 3.1 (i)), we see that if $\{t_n\}$ is a subsequence in (0,1) such that $t_n \to 0$ and $x_{t_n} \rightharpoonup \widetilde{x}$, then by Proposition 3.1 (ii), $\lim_{n\to\infty}(I - P_C S)x_{t_n} = 0$. By Lemma 2.1 and Lemma 2.4 (ii) and (iii), $\widetilde{x} \in F(T)$. Thus from (3.6), we see $x_{t_n} \to \widetilde{x}$.

Now, we prove that $\widetilde{x}$ is a solution of the variational inequality (3.3). Since

$$x_t = t(u + \gamma f(x_t)) + (I - t(I+\mu A))P_C S x_t,$$

we have

$$(I+\mu A)x_t - (u + \gamma f(x_t)) = -\frac{1}{t}(I - t(I+\mu A))(I - P_C S)x_t.$$

It follows that, for $p \in F(T)$,

$$\begin{aligned}\langle (I+\mu A)x_t - (u+\gamma f(x_t)), x_t - p\rangle &= -\frac{1}{t}\langle (I - t(I+\mu A))(I - P_C S)x_t, x_t - p\rangle \\ &= -\frac{1}{t}\langle (I - P_C S)x_t - (I - P_C S)p, x_t - p\rangle \\ &\quad + \langle (I+\mu A)(I - P_C S)x_t, x_t - p\rangle \\ &\leq \langle (I+\mu A)(I - P_C S)x_t, x_t - p\rangle \end{aligned} \tag{3.7}$$

since $I - P_C S$ is monotone (i.e., $\langle x - y, (I - P_C S)x - (I - P_C S)y\rangle \geq 0,\ x, y \in C$, which is due to the nonexpansivity of $P_C S$). Now, replacing t in (3.7) with t_n and letting $n \to \infty$, and noticing that $(I - P_C S)x_{t_n} \to (I - P_C S)\widetilde{x} = 0$ for $\widetilde{x} \in F(T) = F(S)$, we obtain

$$\langle u + (\gamma f - (I+\mu A))\widetilde{x}, p - \widetilde{x}\rangle = \langle (I+\mu A)\widetilde{x} - (u + \gamma f(\widetilde{x})), \widetilde{x} - p\rangle \leq 0.$$

That is, $\widetilde{x} \in F(T)$ is a solution of the variational inequality (3.3).

Moreover, if $\{t_j\}$ is another subsequence in $(0,1)$ such that $t_j \to 0$ and $x_{t_j} \rightharpoonup \widehat{x}$. By the same argument, we can show that $\widehat{x} \in F(T)$ and $\widehat{x}$ solves the variational inequality (3.3); hence $\widehat{x} = \widetilde{x}$ by uniqueness. In a summary, we have shown that each cluster point of $\{x_t\}$ (at $t \to 0$) equals $\widetilde{x}$. Therefore $x_t \to \widetilde{x}$ as $t \to 0$. By (3.3) and Lemma 2.8, we deduce immediately the desired result. This completes the proof. □

From Theorem 3.2, we can deduce the following result.

Corollary 3.3. *The net $\{x_t\}$ defined by the implicit algorithm* (3.2) *converges strongly to a fixed point $\widetilde{x}$ of T as $t \to 0$, which solves the following minimization problem: find $x^* \in F(T)$ such that*

$$\|x^*\| = \min_{x\in F(T)} \|x\|.$$

Now, we propose the following iterative algorithm which generates a sequence $\{x_n\}$ in an explicit way:

$$x_{n+1} = \alpha_n(u + \gamma f(x_n)) + (I - \alpha_n(I + \mu A))P_C S x_n, \quad n \geq 0, \tag{3.8}$$

where $\{\alpha_n\}$ is a sequence in (0,1) and $x_0 \in C$ is selected arbitrarily.

First, we prove the following main result.

Theorem 3.4. *Let $\{x_n\}$ be a sequence in C generated by the iterative algorithm* (3.8)*, and let $\{\alpha_n\}$ be a sequence in $(0, 1)$ which satisfies condition:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$.

Let LIM be a Banach limit. Then

$$LIM_n(\langle u + \gamma f(q) - (I + \mu A)q, x_n - q\rangle) \leq 0,$$

where $q = \lim_{t\to 0^+} x_t$ with x_t being defined by the implicit algorithm (3.1)*.*

Proof. Let $\{x_t\}$ be a net defined by (3.1) for $0 < t < 1$ and $t < (1+\mu\|A\|)^{-1}$. Then, by Theorem 3.2, there exists $\lim_{t\to 0} x_t := q \in F(T)$. Moreover q is a solution of the variational inequality

$$\langle u + (\gamma f - (I + \mu A))q, p - q\rangle \leq 0, \quad p \in F(T).$$

It follows from (3.1) that

$$\|x_t - x_{n+1}\| = \|(I - t(I + \mu A))(P_C S x_t - x_{n+1}) + t(u + \gamma f(x_t) - (I + \mu A)x_{n+1})\|.$$

Applying Lemma 2.2 and Lemma 2.6, we have

$$\begin{aligned} \|x_t - x_{n+1}\|^2 \leq\ & (1 - t(1 + \mu)\overline{\gamma})^2\|P_C S x_t - x_{n+1}\|^2 \\ & + 2t\langle u + \gamma f(x_t) - (I + \mu A)x_{n+1}, x_t - x_{n+1}\rangle. \end{aligned} \tag{3.9}$$

First we note from Proposition 3.1 (i) that $\{x_t\}$, $\{f(x_t)\}$, $\{P_C S x_t\}$ and $\{(I + \mu A)P_C S x_t\}$ are bounded.

Next we show that $\{x_n\}$ is bounded. To this end, let $p \in F(T)$ and set $\overline{A} = I + \mu A$. From condition (C1), we may assume, without loss of generality, that $\alpha_n < (1 + \mu\|A\|)^{-1}$ for all $n \geq 0$. Then, from Lemma 2.2, we derive

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(u + \gamma f(x_n) - \overline{A}p) + (I - \alpha_n\overline{A})(x_n - p)\| \\ &\leq (1 - \alpha_n(1 + \mu)\alpha_n\overline{\gamma})\|x_n - p\| + \alpha_n(\|u\| + \|\gamma f(x_n) - \overline{A}p\|) \\ &\leq (1 - \alpha_n(1 + \mu)\overline{\gamma})\|x_n - p\| \\ &\qquad + \alpha_n(\|u\| + \gamma\|x_n - p\| + \|\gamma f(p) - \overline{A}p\|) \\ &\leq [1 - ((1 + \mu)\overline{\gamma} - \gamma\alpha)\alpha_n]\|x_n - p\| \\ &\qquad + ((1 + \mu)\overline{\gamma} - \gamma\alpha)\alpha_n \frac{\|u\| + \|\gamma f(p) - \overline{A}p\|}{(1 + \mu)\overline{\gamma} - \gamma\alpha}. \end{aligned}$$

It follows that

$$\|x_{n+1} - p\| \leq \max\left\{\|x_n - p\|, \frac{\|u\| + \|\gamma f(p) - \overline{A}p\|}{(1+\mu)\overline{\gamma} - \gamma\alpha}\right\}.$$

Using an induction, we have

$$\|x_{n+1} - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|u\| + \|\gamma f(p) - \overline{A}p\|}{(1+\mu)\overline{\gamma} - \gamma\alpha}\right\}.$$

Hence $\{x_n\}$ is bounded, and so are $\{f(x_n)\}$, $\{P_C S x_n\}$, and $\{\overline{A} P_C S x_n\}$.

Now, as a consequence with condition (C1), we get

$$\|x_{n+1} - P_C S x_n\| = \alpha_n \|\gamma f(x_n) - \overline{A} P_C S x_n\| \to 0 \quad (n \to \infty),$$

and

$$\|P_C S x_t - x_{n+1}\| \leq \|x_t - x_n\| + e_n, \tag{3.10}$$

where $e_n = \|x_{n+1} - P_C S x_n\| \to 0$ as $n \to \infty$ and, noticing that A is strongly positive bounded linear,

$$\begin{aligned}\langle \overline{A}x_t - \overline{A}x_n, x_t - x_n\rangle = \langle \overline{A}(x_t - x_n), x_t - x_n\rangle &\geq (1+\mu\overline{\gamma})\|x_t - x_n\|^2 \\ &> (1+\mu)\overline{\gamma}\|x_t - x_n\|^2. \end{aligned} \tag{3.11}$$

From (3.9), (3.10) and (3.11), we obtain

$$\begin{aligned}
\|x_t - x_{n+1}\|^2 &\leq (1 - (1+\mu)\overline{\gamma})^2 (\|x_t - x_n\| + e_n)^2 \\
&\quad + 2t\langle u + \gamma f(x_t) - \overline{A}x_t, x_t - x_{n+1}\rangle \\
&\quad + 2t\langle \overline{A}x_t - \overline{A}x_{n+1}, x_t - x_{n+1}\rangle \\
&\leq (((1+\mu)\overline{\gamma})^2 t^2 - 2(1+\mu)\overline{\gamma}t)\|x_t - x_n\|^2 + \|x_t - x_n\|^2 \\
&\quad + (1 - t(1+\mu)\overline{\gamma})^2 (2\|x_t - x_n\| e_n + e_n^2) \\
&\quad + 2t\langle u + f(x_t) - \overline{A}x_t, x_t - x_{n+1})\rangle \\
&\quad + 2t\langle \overline{A}x_t - \overline{A}x_{n+1}, x_t - x_{n+1}\rangle \\
&\leq ((1+\mu)\overline{\gamma}t^2 - 2t)\langle \overline{A}x_t - \overline{A}x_n, x_t - x_n\rangle + \|x_t - x_n\|^2 \\
&\quad + (1 - t(1+\mu)\overline{\gamma})^2 (2\|x_t - x_n\| e_n + e_n^2) \\
&\quad + 2t\langle u + \gamma f(x_t) - \overline{A}x_t, x_t - x_{n+1})\rangle \\
&\quad + 2t\langle \overline{A}x_t - \overline{A}x_{n+1}, x_t - x_{n+1}\rangle \\
&= (1+\mu)\overline{\gamma}t^2 \langle \overline{A}x_t - \overline{A}x_n, x_t - x_n\rangle + \|x_t - x_n\|^2 \\
&\quad + (1 - t((1+\mu)\overline{\gamma})^2 (2\|x_t - x_n\| e_n + e_n^2) \\
&\quad + 2t\langle u + \gamma f(x_t) - \overline{A}x_t, x_t - x_{n+1}\rangle \\
&\quad + 2t(\langle \overline{A}x_t - \overline{A}x_{n+1}, x_t - x_{n+1}\rangle - \langle \overline{A}x_t - \overline{A}x_n, x_t - x_n\rangle).
\end{aligned} \tag{3.12}$$

Applying the Banach limit LIM to (3.12) together with $\lim_{n\to\infty} e_n = 0$, we have

$$\begin{aligned}
LIM_n(\|x_t - x_{n+1}\|^2) &\leq (1+\mu)\overline{\gamma}t^2 LIM_n(\langle \overline{A}x_t - \overline{A}x_n, x_t - x_n\rangle) \\
&\quad + LIM_n(\|x_t - x_n\|^2)
\end{aligned}$$

$$
\begin{aligned}
&+ 2tLIM_n(\langle u+\gamma f(x_t)-\overline{A}x_t, x_t-x_{n+1}\rangle)\\
&+ 2t(LIM_n(\langle \overline{A}x_t-\overline{A}x_{n+1}, x_t-x_{n+1}\rangle)\\
&- LIM_n(\langle \overline{A}x_t-\overline{A}x_n, x_t-x_n\rangle)). \qquad (3.13)
\end{aligned}
$$

Using the property $LIM_n(a_n) = LIM_n(a_{n+1})$ of Banach limit in (3.13), we obtain

$$
\begin{aligned}
LIM_n(\langle \overline{A}x_t-(u+\gamma f(x_t)), x_t-x_n\rangle) &= LIM_n(\langle \overline{A}x_t-(u+\gamma f(x_t)), x_t-x_{n+1}\rangle)\\
&\le \frac{(1+\mu)\overline{\gamma}t}{2}LIM_n(\langle \overline{A}x_t-\overline{A}x_n, x_t-x_n\rangle). \qquad (3.14)
\end{aligned}
$$

Since

$$
\begin{aligned}
t\langle \overline{A}x_t-\overline{A}x_n, x_t-x_n\rangle &\le t\|\overline{A}\|\|x_t-x_n\|^2\\
&\le t\|\overline{A}\|(\|x_t-p\|+\|p-x_n\|)^2\\
&\le t\|\overline{A}\|\left(\frac{2}{(1+\mu)\overline{\gamma}-\gamma\alpha}(\|u\|+\|\gamma f(p)-Ap\|)+\|x_0-p\|\right)^2\\
&\to 0 \quad (\text{as } t\to 0), \qquad (3.15)
\end{aligned}
$$

we conclude from (3.14) and (3.15) that

$$
\begin{aligned}
&LIM_n(\langle u+\gamma f(q)-\overline{A}q, x_n-q\rangle)\\
&\quad = LIM_n(\langle \overline{A}q-(u+\gamma f(q)), q-x_n\rangle)\\
&\quad \le \limsup_{t\to 0} LIM_n(\langle \overline{A}x_t-(u+\gamma f(x_t)), x_t-x_n\rangle)\\
&\quad \le \limsup_{t\to 0}\frac{(1+\mu)\overline{\gamma}t}{2}LIM_n(\langle \overline{A}x_t-\overline{A}x_n, x_t-x_n\rangle)\le 0,
\end{aligned}
$$

where $q = \lim_{t\to 0} x_t$. This completes the proof. □

Now, using Theorem 3.4, we establish the strong convergence of the explicit algorithm (3.8) for finding a solution of the optimization problem (1.1).

Theorem 3.5. *Let $\{x_n\}$ be a sequence in C generated by the iterative algorithm* (3.8)*, and let $\{\alpha_n\}$ be a sequence in $(0,1)$ which satisfies conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$*;*
(C2) $\sum_{n=0}^{\infty}\alpha_n = \infty$.

If $\{x_n\}$ is weakly asymptotically regular, then $\{x_n\}$ converges strongly to $q\in F(T)$, which solves the optimization problem (1.1).

Proof. First we note that from condition (C1), without loss of generality, we assume that $\alpha_n \le (1+\mu\|A\|)^{-1}$ and $\frac{2((1+\mu)\overline{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}\alpha_n < 1$ for $n\ge 0$. Let $q=\lim_{t\to 0}x_t$ with x_t being defined by (3.1). Then we know from Theorem 3.2 that $q\in F(T)$, and q is unique solution of the optimization problem (1.1).

We divide the proof into three steps:

Step 1. We show that $\{x_n\}$ is bounded. Indeed, we know that $\|x_n-p\| \le \max\left\{\|x_0-p\|, \frac{\|u\|+\|\gamma f(p)-(I+\mu A)p\|}{(1+\mu)\overline{\gamma}-\gamma\alpha}\right\}$ for all $n\ge 0$ and all $p\in F(T)$ in the proof of Theorem 3.4. Hence $\{x_n\}$ is bounded and so are $\{f(x_n)\}$, $\{P_C Sx_n\}$ and $\{(I+\mu A)P_C Sx_n\}$.

Step 2. We show that $\limsup_{n\to\infty}\langle u+\gamma f(q)-(I+\mu A)q, x_n-q\rangle \le 0$, where $q=\lim_{t\to 0} x_t$ with x_t being defined by (3.1). To this end, put

$$a_n := \langle u+\gamma f(q)-(I+\mu A)q, x_n-q\rangle, \quad n\ge 1.$$

Then Theorem 3.4 implies that $LIM_n(a_n)\le 0$ for any Banach limit LIM. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}(a_{n+1}-a_n)=\lim_{j\to\infty}(a_{n_j+1}-a_{n_j})$$

and $x_{n_j}\rightharpoonup v\in H$. This implies that $x_{n_j+1}\rightharpoonup v$ since $\{x_n\}$ is weakly asymptotically regular. Therefore, we have

$$w-\lim_{j\to\infty}(q-x_{n_j+1})=w-\lim_{j\to\infty}(q-x_{n_j})=(q-v),$$

and so

$$\limsup_{n\to\infty}(a_{n+1}-a_n)=\lim_{j\to\infty}\langle u+\gamma f(q)-(I+\mu A)q,(q-x_{n_j+1})-(q-x_{n_j})\rangle=0.$$

Then Lemma 2.7 implies that $\limsup_{n\to\infty} a_n\le 0$, that is,

$$\limsup_{n\to\infty}\langle u+\gamma f(q)-(I+\mu A)q, x_n-q\rangle\le 0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_n-q\|=0$. To do this, set $\overline{A}=I+\mu A$. Indeed, from Lemma 2.2 and Lemma 2.6, we derive

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\alpha_n(u+\gamma f(x_n)-\overline{A}q)+(I-\alpha_n\overline{A})(P_CSx_n-q)\| \\
&\le \|(I-\alpha_n\overline{A})(P_CSx_n-q)\|^2 \\
&\quad +2\alpha_n\langle u+\gamma f(x_n)-\overline{A}q, x_{n+1}-q\rangle \\
&\le (I-\alpha_n(1+\mu)\overline{\gamma})^2\|x_n-q\|^2 \\
&\quad +2\alpha_n\gamma\langle f(x_n)-f(q), x_{n+1}-q\rangle \\
&\quad +2\alpha_n\langle u+\gamma f(q)-\overline{A}q, x_{n+1}-q\rangle \\
&\le (1-(1+\mu)\overline{\gamma}\alpha_n)^2\|x_n-q\|^2 \\
&\quad +2\alpha_n\gamma\alpha\|x_n-q\|\|x_{n+1}-q\| \\
&\quad +2\alpha_n\langle u+\gamma f(q)-\overline{A}q, x_{n+1}-q\rangle \\
&\le (1-(1+\mu)\overline{\gamma})\alpha_n)^2\|x_n-q\|^2 \\
&\quad +\alpha_n\gamma\alpha[\|x_n-q\|^2+\|x_{n+1}-q\|^2] \\
&\quad +2\alpha_n\langle u+\gamma f(q)-\overline{A}q, x_{n+1}-q\rangle,
\end{aligned}$$

that is,

$$\begin{aligned}
&\|x_{n+1}-q\|^2 \\
\le\ &\frac{1-2(1+\mu)\overline{\gamma}\alpha_n+((1+\mu)\overline{\gamma})^2\alpha_n^2+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle u+\gamma f(q)-\overline{A}q, x_{n+1}-q\rangle \\
&= \left(1-\frac{2((1+\mu)\overline{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\right)\|x_n-q\|^2 + \frac{((1+\mu)\overline{\gamma})^2\alpha_n^2}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle u+\gamma f(q)-\overline{A}q, x_{n+1}-q\rangle \\
&\le \left(1-\frac{2((1+\mu)\overline{\gamma}-\gamma\alpha)}{1-\alpha_n\gamma\alpha}\alpha_n\right)\|x_n-q\|^2 + \frac{2((1+\mu)\overline{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\times \\
&\quad \left(\frac{((1+\mu)\overline{\gamma})^2\alpha_n}{2((1+\mu)\overline{\gamma}-\gamma\alpha)}M + \frac{1}{(1+\mu)\overline{\gamma}-\gamma\alpha}\langle u+\gamma f(q)-\overline{A}q, x_{n+1}-q\rangle\right) \\
&= (1-\lambda_n)\|x_n-q\|^2 + \lambda_n\delta_n,
\end{aligned}
$$

where $M = \sup\{\|x_n-q\|^2 : n \ge 0\}$, $\lambda_n = \frac{2((1+\mu)\overline{\gamma}-\gamma\alpha)}{1-\alpha_n\gamma\alpha}\alpha_n$ and

$$
\delta_n = \frac{((1+\mu)\overline{\gamma})^2\alpha_n}{2((1+\mu)\overline{\gamma}-\gamma\alpha)}M + \frac{1}{(1+\mu)\overline{\gamma}-\gamma\alpha}\langle u+\gamma f(q)-\overline{A}q, x_{n+1}-q\rangle.
$$

From conditions (C1) and (C2) and Step 2, it is easy to see that $\lambda_n \to 0$, $\sum_{n=0}^{\infty}\lambda_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Hence, by Lemma 2.5, we conclude $x_n \to q$ as $n \to \infty$. This completes the proof. □

Corollary 3.6. *Let $\{x_n\}$ be a sequence in C generated by the iterative algorithm* (3.8)*, and let $\{\alpha_n\}$ be a sequence in* $(0,1)$ *which satisfies conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$*;*
(C2) $\sum_{n=0}^{\infty}\alpha_n = \infty$*.*

If $\{x_n\}$ is asymptotically regular, then $\{x_n\}$ converges strongly to $q \in F(T)$, which solves the optimization problem (1.1)*.*

Remark 3.7. If $\{\alpha_n\}$ in Corollary 3.6 satisfies conditions (C1), (C2) and

(C3) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$; or $\lim_{n\to\infty}\frac{\alpha_n}{\alpha_{n+1}} = 1$; or
(C4) $|\alpha_{n+1}-\alpha_n| \le o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty}\sigma_n < \infty$ (the perturbed control condition),

then the sequence $\{x_n\}$ generated by the iterative algorithm (3.8) is asymptotically regular. Now, we give only the proof in case when $\{\alpha_n\}$ satisfies conditions (C1), (C2) and (C4). By Step 1 in the proof of Theorem 3.5, there exists a constant $L > 0$ such that for all $n \ge 0$,

$$
\|\overline{A}P_CSx_n\| + \gamma\|f(x_n)\| \le L.
$$

So,we obtain, for all $n \ge 0$,

$$
\begin{aligned}
\|x_{n+1}-x_n\| &= \|(I-\alpha_n\overline{A})(P_CSx_n - P_CSx_{n-1}) + (\alpha_n-\alpha_{n-1})\overline{A}P_CSx_{n-1} \\
&\qquad + \gamma[\alpha_n(f(x_n)-f(x_{n-1})) + f(x_{n-1})(\alpha_n-\alpha_{n-1})]\| \\
&\le (1-\alpha_n(1+\mu)\overline{\gamma})\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|\|\overline{A}P_CSx_{n-1}\| \\
&\quad + \gamma[\alpha_n\alpha\|x_n-x_{n-1}\| + \|f(x_{n-1})\||\alpha_n-\alpha_{n-1}|] \\
&\le (1-\alpha_n((1+\mu)\overline{\gamma}-\gamma\alpha))\|x_n-x_{n-1}\| + L|\alpha_n-\alpha_{n-1}|
\end{aligned}
$$

$$\leq (1-\alpha_n((1+\mu)\overline{\gamma}-\gamma\alpha))\|x_n-x_{n-1}\|+(o(\alpha_n)+\sigma_{n-1})L. \tag{3.16}$$

By taking $s_{n+1} = \|x_{n+1}-x_n\|$, $\lambda_n = \alpha_n((1+\mu)\overline{\gamma}-\gamma\alpha)$, $\lambda_n\delta_n = o(\alpha_n)L$ and $r_n = \sigma_{n-1}L$, from (3.16) we have

$$s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n\delta_n + r_n.$$

Hence, by (C1), (C2), (C4) and Lemma 2.5, we obtain

$$\lim_{n\to\infty} \|x_{n+1}-x_n\| = 0.$$

In view of this observation, we have the following:

Corollary 3.8. *Let $\{x_n\}$ be a sequence in C generated by the iterative algorithm* (3.8)*, and let $\{\alpha_n\}$ be a sequence in $(0,1)$ which satisfies conditions* (C1), (C2) *and* (C4) *(or conditions* (C1), (C2) *and* (C3)*). Then $\{x_n\}$ converges strongly to $q \in F(T)$, which solves the optimization problem* (1.1).

From Theorem 3.5, we can also deduce the following result.

Corollary 3.9. *Let $\{x_n\}$ be a sequence in C generated by*

$$x_{n+1} = (1-\alpha_n)P_C S x_n, \quad \forall n \geq 0,$$

and let $\{\alpha_n\}\subset (0,1)$ be a sequence satisfying conditions (C1) *and* (C2)*. If $\{x_n\}$ is weakly asymptotically regular, then $\{x_n\}$ converges strongly to a fixed point q of T as $n \to \infty$, which solves the following minimization problem: find $x^* \in F(T)$ such that*

$$\|x^*\| = \min_{x\in F(T)} \|x\|.$$

Remark 3.10. (1) From Proposition 2.6 of Acedo and Xu [1], we know that if, for any $N \geq 1$ and for each $1 \leq i \leq N$, $T_i : C \to H$ is a k_i-strictly pseudocontractive mapping for some $0 \leq k_i < 1$, and $\{\eta_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \eta_i = 1$, then $\sum_{i=1}^N \eta_i T_i$ is a k-strictly pseudocontractive mapping with $k = \max\{k_i : 1 \leq i \leq N\}$ and $F(\sum_{i=1}^N \eta_i T_i) = \bigcap_{i=1}^N F(T_i)$. So, by putting $Sx = kx+(1-k)\sum_{i=1}^N \eta_i T_i x$ in Theorem 3.2, Theorem 3.4 and Theorem 3.5, we obtain the corresponding results for a finite family of k_i-strictly pseudocontractive mappings for some $0 \leq k_i < 1$ $(1 \leq i \leq N)$, which can be utilized to solve the following optimization problem

$$\min_{x\in\bigcap_{i=1}^N F(T_i)} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x-u\|^2 - h(x).$$

(2) In Remark 3.7, condition (C4) on $\{\alpha_n\}$ is independent of condition (C3), which was imposed by Cho *et al.* [5], Marino and Xu [15] and others. For this fact, see [6, 13].

(3) We point out the our iterative algorithms (3.1) and (3.8) are different from those in the recent works in this direction.

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