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η-α-PSEUDOMONOTONICITY AND EQUILIBRIUM PROBLEM IN TOPOLOGICAL VECTOR SPACE

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Abstract. In this paper, two classes of generalized variational-like inequality problems for multivalued mappings are introduced and then by using KKM technique and Kakutani-Fan-Glicksberg fixed point theorem the solvability of them are investigated when the mappings are relaxed η-α-monotone. The results of this note extend and improve the corresponding results in the literature and it can be considered as a topological vector space version of reference of the paper [N.K. Mahato, C. Nahak, *Mixed equilibrium problems with relaxed α-monotone mapping in Banach spaces*, Rend. Circ. Mat. Palermo. DOI 10.1007/s12215-013-0103-0].

1. Introduction

The existence of a solution for variational inequality problems, complementarity problems, equilibrium problems and others is mainly dependent on the monotonicity of a map (see, for examples, [1, 2, 4, 5, 7, 9, 11, 15, 22]). Recently, many authors, see [8–12] considered the quasimonotonicity in dealing with variational inequality problems. Verma [20, 21] studied and established some existence theorems of a solution for a class of nonlinear variational inequality problems with p-monotone and p-Lipschitz mappings in the setting of reflexive Banach spaces.

Inspired and motivated by the references [1, 4, 5, 8, 10, 13, 14, 18, 23], we introduce two new concepts of the relaxed η-α-semimonotonicity and two classes of variational-like inequality problems with relaxed η-α-monotone mappings and relaxed η-α-semimonotone mappings. Using the KKM-technique, we obtain the existence of a solution for variational-like inequality problems with relaxed η-α-monotone mappings in the setting of reflexive Banach spaces. We also present the solvability of variational-like inequalities problems with η-α-semimonotone mappings for an arbitrary Banach space by applying of Kakutani-Fan fixed point theorem [6, 23].

Let K be a nonempty subset of a real reflexive Banach space X. Let $\varphi : K \to \Re$ (the real line) be a real valued function and $f : K \times K \to \Re$ be an equilibrium bi-function, i.e., $f(x, x) = 0$, for all $x \in K$. Then the mixed equilibrium problem [for short, MEP] is to find $\overline{x} \in k$

$$f(\overline{x}, y) + \varphi(y) - \varphi(\overline{x}) \geq 0, \ \forall y \in K. \tag{1.1}$$

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In particular, if $\varphi = 0$, the MEP problem reduces to the classical equilibrium problem [for short, EP], which is to find $\overline{x} \in k$ such that

$$f(\overline{x}, y) \geq 0, \ \forall y \in K. \tag{1.2}$$

Equilibrium problems have numerous applications, including but not limited to problems in nonlinear analysis, Economics, game theory, finance and traffic analysis. The equilibrium problem (1.2) includes many mathematical problems as particular cases for examples, mathematical programming problems, complementary problems, variational inequality problems, Nash equilibrium problems in noncooperative games, minimax inequality problems, and fixed point problems [1, 3].

The generalized monotonicity plays an important role in the literature of equilibrium problems and variational inequality problems. There is a substantial number of papers on existence results for solving equilibrium problems and variational inequality problems based on different relaxed monotonicity notions such as monotonicity, pseudomonotonicity, quasimono-tonicity, relaxed monotonicity, C_x-pseudomonotonicity (for more details, see [2–6, 8, 9, 12–15]).

In 2003, Fang and Huang [8] considered two types of the variational-like inequality problems with the relaxed η-α-monotone and relaxed η-α-semimonotone mappings. They obtained the existence solutions of these variational like inequalities with relaxed η-α-monotone and relaxed η-α-semimonotone mappings in the reflexive Banach spaces using the KKM technique.

In 2006, Bai et al. [1] introduced a new concept of the relaxed η-α-pseudomonotone for a mapping and by using it they established an existence result of a solution for a variational-like inequality problem. After the work reported in [1] very recently, Mahato and Nahak [16] defined weakly relaxed η-α-pseudomonotone bi-function in order to study EP.

Bianchi and Schaible [2] introduced various kinds of generalized monotone mappings. It is well-known that, under some suitable modifications, many results of variational inequality problems can be adapted to equilibrium problems , which is a very general phenomenon in this field.

In this paper, inspired and motivated by the recent researches [16, 17], we introduce a new concept of the relaxed η-α-monotonicity for bi-functions and by using it and the KKM technique we present some existence results of a solution for the MEP. Our results can be viewed as a generalization of the main results given in [1, 8, 11, 15–17].

2. Variational-like inequalities with relaxed η-α-monotone mappings

Throughout this paper, unless otherwise specified, we always let E be a Hausdorff topological vector space, θ denotes the zero vector of E and $E^\star$ is its topological dual space of E, K a nonempty closed convex subset of E, $T : K \to 2^{E^\star}\setminus\{\emptyset\}$, a multivalued mapping from K to $E^\star$, and $\eta : K \times K \to K$, $\alpha : E \to \Re$ are two mappings. Furthermore, we assume that $\alpha(\theta) = 0$ and $\lim_{t\to 0^+} \frac{\alpha(tz)}{t} = 0$, for all $z \in K$. This means that the directional derivative of α at θ at every direction $z \in K$ exists and equals to zero. For examples of these mappings, one can consider all α which has the property $\alpha(tz) = t^p\alpha(z)$ for all $t \geq 0$, $p > 1$ and $z \in E$. We note that if we take $E = \Re$ then it is easy to see that the directional derivative of the mapping

$\alpha(x) = |x|$ at θ in each direction $z \in E$ is zero but it dose not satisfy $\alpha(tz) = t^p\alpha(z)$ for all $t \geq 0$, $p > 1$ and $z \in E$.

Definition 2.1. A mapping $f : K \times K \to \Re$ is called relaxed η-α-monotone if there exist mappings $\eta : E \times E \to E$ and $\alpha : E \to \Re$ with $\lim_{t\to 0^+} \frac{\alpha(\eta(\theta,tz))}{t} = 0$, for all $z \in K$, such that the following inequality holds,

$$f(x,y) + f(y,x) \ \leq \ \alpha(\eta(x,y)), \text{ for all } x, y \in K. \tag{2.1}$$

Definition 2.2. The mapping $f : K \times K \to \Re$ is called relaxed η-α-pseudomonotone if there exist mappings $\eta : E \times E \to E$ and $\alpha : E \to \Re$ with $\lim_{t\to 0^+} \frac{\alpha(\eta(\theta,tz))}{t} = 0$, for all $z \in K$, such that the following inequality, for every pair of points $x, y \in K$, holds

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq \ \alpha(\eta(x,y)). \tag{2.2}$$

Note that η-α-monotone mapping is a η-α-pseudomonotone.

Remark 2.3. (1) If we define $\eta(x,y) = y - x$, for all $x, y \in K$ and $\alpha(tx) = t^p\alpha(x)$, for all $t > 0$ and $x \in K$, then the problem given by (2.1) collapses to the Definition 2.1 of [16] and

$$\lim_{t\to 0^+} \frac{\alpha(\eta(\theta,tz))}{t} = 0, \forall z \in K.$$

While if we take $E = K = \Re, \eta(x,y) = y - x$ and $\alpha(x) = |x|$, for all $x, y \in \Re$, then

$$\lim_{t\to 0^+} \frac{\alpha(\eta(\theta,tz))}{t} = 0, \forall z \in K$$

but α dose not satisfy the following equality

$$\alpha(tz) = t^p\alpha(z), \forall(t > 0, p > 1, z \in E).$$

Hence Definition 2.2 extends Definition 2.1 of [16].

(2) In [17] the authors considered the mapping $\eta(x,y) = y - x$ and α with the property

$$\alpha(tx) = k(t)\alpha(x),\ \forall(t > 0, x \in K),$$

where $K : (0,\infty) \to (0,\infty)$ with

$$\lim_{t\to 0^+} \frac{K(t)}{t} = 0.$$

It is easy to check that

$$\lim_{t\to 0^+} \frac{\alpha(tz)}{t} = 0, \forall z \in K.$$

If we take $E = K = \Re$ and define

$$\alpha(x) = \begin{cases} x^2, & x \in Q \text{ (rational numbers);} \\ 0, & x \in Q^c\text{(irrational numbers),} \end{cases}$$

then it is easy to check that there is no $K : (0,\infty) \to (0,\infty)$ such that

$$\alpha(tx) = k(t)\alpha(x),\ \forall(t > 0, x \in K),$$

while
$$\lim_{t\to 0^+}\frac{\alpha(\eta(\theta,tz))}{t}=0, \forall z\in K.$$
Hence the example shows that Definition 2.1 and Definition 2.2 include the corresponding definitions given in [17] and are extended definitions of them.

Definition 2.4. Let X and Y be two topological spaces. A set-valued mapping $G: X\to 2^Y$ is called:

(i) **upper semi-continuous** (u.s.c.) at $x\in X$ if for each open set V containing $G(x)$, there is an open set U containing x such that for each $t\in U,\ G(t)\subseteq V$; G is said to be u.s.c. on X if it is u.s.c. at all $x\in X$.

(ii) **lower semi-continuous** (l.s.c.) at $x\in X$ if for each open set V with $G(x)\cap V\neq\emptyset$, there is an open set U containing x such that for each $t\in U,\ G(t)\cap V\neq\emptyset$; G is said to be l.s.c. on X if it is l.s.c. at all $x\in X$.

(iii) **continuous** if G is both lower semi-continuous and upper semi-continuous.

Proposition 2.5 ([19]). *Let X and Y be two topological spaces. A set-valued mapping $T: X\to 2^Y$ is l.s.c. at $x\in X$ if and only if for any $y\in T(x)$ and any net $\{x_\alpha\}$ which converges to x there is a net $\{y_\alpha\}$ such that $y_\alpha\in T(x_\alpha)$ and $y_\alpha\to y$.*

Definition 2.6. A real valued mapping $T: K\to E$ is called lower hemi-continuous if, for all $x,y\in K$, the mapping $F:[0,1]\to X$ defined by $F(t)=f(tx+(1-t)y)$ is lower semi-continuous at 0 from the right.

Remark that Definition 2.6 is weaker than Definition 2.2 of [16].

Definition 2.7 ([7]). A mapping $F: K\to 2^E$ is said to be a KKM-mapping, if for any $\{x_1,x_2,\dots,x_n\}\subset K$, co$\{x_1,x_2,\dots,x_n\}\subset\bigcup_{i=1}^{n}F(x_i)$, where $2^E\backslash\{\emptyset\}$ denotes the family of all nonempty subsets of E.

Lemma 2.8 ([7]). *Let K be a nonempty subset of a topological vector space X and $F: K\to 2^X$ a KKM mapping with closed values in K. Assume that there exists a nonempty compact convex subset B of K such that $\bigcap_{x\in B}F(x)$ is compact. Then*
$$\bigcap_{x\in K}F(x)\neq\emptyset.$$

Now, we are ready to present the first result of the paper.

Theorem 2.9. *Let $f: K\times K\to\Re\cup\{+\infty\}$ be a proper function (that is $f\neq+\infty$) be relaxed η-α-pseudomonotone mapping and $\eta: K\times K\to E$ be a mapping. Let $T: K\to 2^{E^*}\backslash\{\emptyset\}$ be lower η-hemicontinuous (that is the restriction of the multivalued mapping $x\to\langle T(x),\eta(y,x)\rangle$ to line segments is l.s.c. for each $y\in K$) multivalued mapping. Assume that*

(i) *$\eta(x,x)=0$, for all $x\in K$,*
(ii) *for any fixed $x\in K$ and $u\in Ty$, the mapping $y\to\langle u,\eta(y,x)\rangle$ is convex,*
(iii) *for any fixed $x\in K$, the mapping $y\to f(y,x)$ is convex.*

Then the MEP and the following problem are equivalent (that is, their solution sets are equal):

Find $\overline{x} \in K$ such that

$$f(y, \overline{x}) + \varphi(\overline{x}) - \phi(y) \leq \alpha(\eta(\overline{x}, y)), \ \forall y \in K. \tag{2.3}$$

Proof. Let $x \in K$ be a solution of MEP. Then

$$f(\overline{x}, y) + \varphi(y) - \varphi(\overline{x}) \leq 0, \ \forall y \in K.$$

Since f is relaxed η-α-monotone, we have

$$f(y, \overline{x}) + \varphi(\overline{x}) - \varphi(x) \ \leq \ \alpha(\eta(\overline{x}, y)) - f(\overline{x}, y) + \varphi(\overline{x}) - \varphi(y))$$

for all $y \in K$, $v \in T(y)$. Then $x \in K$ is a solution of (2.3).

To see the converse, let $x \in K$ be a solution of (2.3). Assume that y is an arbitrary element of K and $u \in T(x)$. Since x is a solution of (2.3) then $f(x, x) < \infty$. Letting

$$y_t \ = \ (1 - t)x + ty, \ t \in [0, 1],$$

(note K is a convex set) then $y_t \in K$. Moreover y_t approaches to x when t converges to zero and so by Proposition 2.5 (note $u \in T(x)$ and T is lower η-hemicontinuous) there is $v_t \in T(y_t)$, such that

$$\langle v_t, \eta(y, x)\rangle \to \langle u, \ \eta(y, x)\rangle \text{ if } t \to 0 \tag{2.4}$$

and hence (note that x is a solution of (2.3))

$$\langle v_t, \eta(y_t, x)\rangle + f(y_t, x) - f(x, x) \ \geq \ \alpha(y_t - x) \ = \ \alpha(t(y - x)). \tag{2.5}$$

By condition (iii), we get

$$f(y_t, x) - f(x, x) \ = \ f((1 - t)x + ty, x) - f(x, x) \ \leq \ t(f(y, x) - f(x, x)) \tag{2.6}$$

and also conditions (ii) and (i) imply that

$$\begin{aligned} \langle v_t, \eta(y_t, x)\rangle &= \langle v_t, \eta((1 - t)x + ty, x)\rangle \\ &\leq (1 - t)\langle v_t, \eta(x, x)\rangle + t\langle v_t, \eta(y, x)\rangle \\ &= t\langle v_t, \eta(y, x)\rangle \ . \end{aligned} \tag{2.7}$$

It follows from (2.5)-(2.7), for $t \in]0, 1]$, that,

$$\langle v_t, \eta(y, x)\rangle + f(y, x) - f(x, x) \ \geq \ \frac{\alpha(t(y - x))}{t} = \frac{\alpha(t(y - x)) - \alpha(\theta)}{t}, \tag{2.8}$$

for all $y \in K$ and $v_t \in T(y_t)$. Now the result follows by letting $t \to 0$ in (2.8), using (2.4) and the fact that α has nonnegative directional derivative at zero in each direction. That is

$$\langle u, \eta(y, x)\rangle + f(y, x) - f(x, x) \ \geq \ 0, \text{ for all } y \in K \text{ and } u \in T(x).$$

Hence $x \in K$ is a solution of (2.2). This completes the proof. □

We need the next theorem in the sequel.

Theorem 2.10. *Let K be a nonempty closed convex subset of a topological vector space E. Let $T : K \to 2^{E^\star} \backslash \{\emptyset\}$, $f : K \times K \to R \cup \{+\infty\}$ and $\eta : K \times K \to E$ be three mappings such that,*

(i) $\eta(x, y) + \eta(y, x) = 0$, *for all* $x \in K$,
(ii) *for any fixed* $y \in K$, *the mapping* $x \to \langle Tx, \eta(y, x)\rangle + f(y, x) - f(x, x)$ *is lower semi-continuous,*
(iii) *for any fixed* $y \in K$, *the mappings* $x \to \eta(x, y)$ *and* $x \to f(x, y)$ *are concave and convex, respectively,*
(iv) $\langle u_i - u_j, \eta(a_i, a_j)\rangle \geq 0$, *for each finite subset* $A = \{a_1, a_2, \ldots, a_n\}$ *of* K, $y \in coA$ *and* $u_i \in T(y)$,
(v) *there exist a compact convex subset* D *of* K *and a compact subset* B *of* K *such that*

$$\forall x \in K\backslash B\ \exists z \in D:\ \langle u, \eta(z, x)\rangle + f(z, x) - f(x, x) \ < \ 0, \ \textit{for some } u \in T(z).$$

Then the solution set of problem (2.2) is nonempty and compact.

Proof. Define set-valued mapping, $F : K \to 2^E$ as follows:

$$F(y) \ = \ \{x \in K \ : \forall \ u \in T(x), \ \langle u, \eta(y, x)\rangle + f(y, x) - f(x, x) \ \geq \ 0\}.$$

We claim that F is a KKM mapping. If F is not a KKM-mapping, then there exist subset $\{y_1, y_2, \ldots, y_n\} \subset K$ and $t_i > 0$, $i = 1, 2, \ldots, n$, such that $\sum_{i=1}^{n} t_i = 1$,

$$z \ = \ \sum_{i=1}^{n} t_i \ y_i \ \notin \ \bigcup_{i=1}^{n} F(y_i),$$

and hence there exist $u_i \in T(y)$, for $i = 1, 2, \ldots, n$ such that

$$\langle u_i, \ \eta(y_i, z)\rangle + f(y_i, z) - f(z, z) < 0, \ \text{for } i = 1, 2, \ldots, n,$$

and so

$$\sum_{i=1}^{n} t_i \langle u_i, \ \eta(y_i, z)\rangle + \sum_{i=1}^{n} t_i f(y_i, z) - f(z, z) < 0,$$

and by (iii) (f is convex in the first variable) we have

$$\sum_{i=1}^{n} t_i \langle u_i, \ \eta(y_i, z) < 0,$$

and by (i) (note $\eta(y_i, z) = -\eta(z, y_i)$ and $z = \sum_{j=1}^{n} t_j y_j$) we get

$$-\sum_{i=1}^{n} t_i \langle u_i, \ \eta(z, y_i) < 0,$$

and it follows from (iii) and (i) that

$$-\sum_{j=1}^{n} \sum_{i=1}^{n} t_i t_j \langle u_i, \ \eta(y_j, y_i) < 0,$$

and so by (i) (note $\eta(y_i, y_i) = 0, \eta(y_i, y_j) = -\eta(y_j, y_i)$) we get

$$\sum_{i<j} t_i t_j \langle u_i - u_j, \ \eta(y_i, y_j)\rangle \ < 0,$$

and so $\langle u_i - u_j, \ \eta(y_i, y_j)\rangle \ < 0$, for some $i < j$,which is contradicted (by (iv)). This implies that F is a KKM-mapping. We claim that $F(y)$ is closed for all $y \in K$.

Indeed, let $\{x_\alpha\}$ be a net in $F(y)$ which converges to $x \in K$. We have to show that $x \in F(y)$. To see this let $v \in T(x)$ be an arbitrary element. By (ii) through Proposition 2.5 there is net $\{v_\alpha\}$ in E^* with $v_\alpha \in T(x_\alpha)$ such that

$$\langle v_\alpha, \eta(y, x_\alpha)\rangle + f(y, x_\alpha) - f(x_\alpha, x_\alpha) \to \langle v, \eta(y, x)\rangle + f(y, x) - f(x, x) \tag{2.9}$$

and since $x_\alpha \in F(y)$ we deduce from (2.9) that

$$\langle v, \eta(y, x)\rangle + f(y, x) - f(x, x) \geq 0,$$

and hence $x \in F(y)$. Also it follows from (v) that $\cap_{z\in D} F(z) \subseteq B$, and so F satisfies all the assumptions of Lemma 2.8 and then there exists $\overline{x} \in \bigcap_{y\in K} F(y)$. This means that $\overline{x}$ is a solution of problem (2.2). Furthermore the solution set of problem (2.2) equals to the intersection $\bigcap_{y\in K} F(y)$ which by using (v) is a subset of the compact set B and, note $\bigcap_{y\in K} F(y)$ is closed, so it is compact. This completes the proof of theorem. □

Remark 2.11. (i) It is clear that one can omit condition (v) in Theorem 2.10 when the set K is compact.

(ii) In [8], the authors, instead of condition (v) in Theorem 2.10, considered the following condition for a reflexive Banach space, which consists of finding $x_0 \in K$ such that,

$$\frac{\langle u - u_0,\ \eta(x, x_0)\rangle - f(x_0, x) + f(x, x)}{\|\eta(x, x_0)\|} \to +\infty \tag{2.10}$$

whenever $\|x\| \to \infty$, for all $u \in T(x)$, $u_0 \in T(x_0)$.
They (2.10), called η-coercive. It is clear that (2.10) is a special case of condition (v) in Theorem 2.10. Because for each positive real number M there is another positive number N such that

$$\|x\| > N \Rightarrow \frac{\langle u - u_0,\ \eta(x, x_0)\rangle - f(x_0, x) + f(x, x)}{\|\eta(x, x_0)\|} > M. \tag{2.11}$$

Now we can take $B = \{x : \|x\| \leq N\}$ and $D = \{x_0\}$ which are weakly compact (note E is a reflexive Banach space) and convex. Moreover, by condition (i) of Theorem 2.10, $\eta(x, x_0) = -\eta(x_0, x)$ and by multiplying the relation (2.11) by -1 we get condition (v) in Theorem 2.10.

An special case of (2.10) has been given in [22] as follows,

$$\frac{\langle u - u_0,\ \eta(x, x_0)\rangle + f(x) - f(x_0)}{\|\eta(x, x_0)\|} \to +\infty,$$

whenever $\|x\| \to \infty$, for all $u \in T(x)$, $u_0 \in T(x_0)$.

By combining Theorems 2.9 and 2.10 one can deduce the next result.

Theorem 2.12. *Let K be a nonempty closed convex subset of a topological vector space E and $E^\star$ the dual space of E. Let $T : K \to 2^{E^\star}\backslash\{\emptyset\}$ be lower η-hemicontinuous and relaxed η-α-monotone and the conditions (i)-(v) of Theorem 2.10 and condition (ii) of Theorem 2.9 hold. Then the solution sets of problems (2.2) and (2.3) are equal and a nonempty compact subset of K.*

We note that if T is a single valued mapping and f is a zero map, then the Theorems 2.9 and 2.10 are equivalent to the problems considered and studied by Bai et al [1].

3. Variational-like inequalities with relaxed η-α-semimonotone mappings

Throughout this section, let E be an arbitrary locally convex topological vector space (briefly, locally convex space) with its topological dual E^* and K a nonempty closed convex subset of E.

Definition 3.1. Let $A : K \times K \to 2^{E^*}$, $\eta : K \times K \to E$ and $\alpha : E \to \Re$ be three mappings. The mapping A is called relaxed η-α-semimonotone if the mapping $y \to A(w, y)$ is relaxed η-α-monotone, for each $w \in K$. In this section we consider the following problem of finding $x \in K$ such that

$$\langle u, \eta(y, x)\rangle + f(y, x) - f(x, x) \geq 0, \text{ for all } y \in K \text{ and } u \in A(x, y). \tag{3.1}$$

where $f : K \times K \to \Re$.

In order to prove our existence theorem we need the following result.

Theorem 3.2 ([6, Kakutani-Fan-Glicksberg])**.** *Let X be a locally convex Hausdorff space, $D \subseteq X$ a nonempty, convex compact subset. Let $T : D \to 2^D$ be upper semicontinuous with nonempty, closed convex values $T(x)$, for all $x \in D$. Then T has a fixed point in D.*

Theorem 3.3. *Let E be a locally convex Hausdorff space, $K \subseteq E$ a nonempty closed convex set, $A : K \times K \to 2^E$ a relaxed η-α-semimonotone mapping, $f : K \times K \to \Re \bigcup \{+\infty\}$ a proper convex and weakly lower semicontinuous function, and $\eta : K \times K \to E$ a mapping. If for all $w \in K$, the mapping $y \in A(w, y)$ satisfies all the assumptions of Theorem 2.10 and the mapping, for all $w \in K$, $x \to \langle u, \eta(y, x)\rangle + f(y, x) - f(x, x) \geq 0$, for all $y \in K$ and $u \in A(w, y)$, is convex and upper semicontinuous, then problem (3.1) has a solution. Moreover the solution set of problem (3.1) is compact and convex.*

Proof. By Theorem 2.10, for each $w \in coB$, the set

$$G(w) = \{x \in coB : \langle u, \eta(y, x)\rangle + f(y, x) - f(x, x) \geq 0, \forall y \in K \text{ and } u \in A(w, y)\}$$

is a nonempty convex and compact subset of $B \subset K$. Now the mapping $G : coB \to 2^{coB}$ defined by $w \to G(w)$ fulfils all the conditions of Theorem 3.2. Hence there is $x \in coB \subset K$ such that $x \in G(x)$ and so x is a solution of problem (3.1) and so the solution set of the problem (3.1) is nonempty. It is clear that the solution set of problem (3.1) is equal to the intersection

$$\bigcap_{w \in K} G(w) \subseteq \bigcap_{x \in coB} G(w) \subset D$$

and since $G(w)$, for all $w \in K$ is closed and D is compact then the solution set problem (3.1) is compact and the convexity of the solution set is obvious from the assumptions. This completes the proof. □

Remark 3.4. If A is a single valued mapping and f is a zero map, then problem (3.1) is equivalent to the problem (3.1) considered and studied by Bai et al [1]. Note that Theorems 2.10 and 3.3 are topological vector space version of Theorems 2.1 and 2.6, respectively, in [4].

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