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VISCOSITY ITERATIVE METHOD FOR A FINITE FAMILY OF GENERALIZED ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN CONVEX METRIC SPACES

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Abstract. We introduce a general viscosity iterative method for a finite family of generalized asymptotically quasi-nonexpansive mappings in a convex metric space. The new iterative method contains several well-known iterative methods as its special case including multistep iterative method of Khan et al. [Common fixed points Noor iteration for a finite family of asymptotically quasi-nonexpansive mappings in Banach space, J. Math. Anal. Appl. 341(2008), 1-11] and viscosity iterative method of Chang et al.[Approximating solutions of variational inequalities for asymptotically nonexpansive mappings, Appl. Math. Comput., 212(2009), 51-59]. Our results are new in convex metric spaces and generalize many known results in Banach spaces and $CAT(0)$ spaces simultaneously.

1. Introduction and preliminaries

Let C be a nonempty subset of a metric space X and $T : C \to C$ be a mapping. Throughout this paper, we assume that $F(T)$, the set of fixed points of T is nonempty and $I = \{1, 2, 3, \ldots, r\}$. The mapping T is (i) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for $x, y \in C$ (ii) quasi-nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for $x \in C, y \in F(T)$ (iii) asymptotically nonexpansive [7] if there exists a sequence of real numbers $\{u_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $d(T^n x, T^n y) \leq (1 + u_n) d(x, y)$ for all $x, y \in C$ and $n \geq 1$ (iv) asymptotically quasi-nonexpansive if there exists a sequence of real numbers $\{u_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $d(T^n x, p) \leq (1 + u_n) d(x, p)$ for all $x \in C, p \in F(T)$ and $n \geq 1$ (v) generalized asymptotically quasi-nonexpansive [18] if there exist two sequences of real numbers $\{u_n\}$ and $\{c_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0 = \lim_{n\to\infty} c_n$ such that $d(T^n x, p) \leq d(x, p) + u_n d(x, p) + c_n$ for all $x \in C, p \in F(T)$ and $n \geq 1$ (vi) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \leq L d(x, y)$, for all $x, y \in C$ and $n \geq 1$ (vii) uniformly Hölder continuous if there are constants $L > 0, \gamma > 0$ such that $d(T^n x, T^n y) \leq L d(x, y)^\gamma$ for all $x, y \in C$ and $n \geq 1$ and (viii) semi-compact if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that x_{n_i} converges to a point in C.

From these definitions, it is clear that
(1) the class of generalized asymptotically quasi-nonexpansive mappings includes

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the class of asymptotically quasi-nonexpansive mappings and this inclusion is proper (see [3, 18, 23]).
(2) a uniformly L-Lipschitzian mapping is uniformly Hölder continuous but the converse is not true, in general.

Convergence theorems for the mappings, mentioned in (i)-(v), through different iterative methods have been obtained by a number of authors (e.g., [3, 16–18, 22, 23] and the references therein).

The reader interested in fixed points of uniformly Lipschitzian mappings in metric spaces and Mann iterative methods for nonexpansive mappings in geodesic metric spaces is referred to Dhompongsa et al. [4, 5]

In 2005, Suantai [19] introduced a general three-step iterative method as an extension of one-step, two-step and three-step iterative methods [6, 8, 14, 22].

Let C be a convex subset of a normed space. Khan et al. [9] introduced the following multi-step iterative method:

$$
\begin{aligned}
x_1 &\in C, \\
x_{n+1} &= (1-a_{rn})x_n + a_{rn}T_r^n\ y_{(r-1)n}, \\
y_{(r-1)n} &= (1-a_{(r-1)n})x_n + a_{(r-1)n}T_{r-1}^n\ y_{(r-2)n}, \\
y_{(r-2)n} &= (1-a_{(r-2)n})x_n + a_{(r-2)n}T_{r-2}^n\ y_{(r-3)n}, \\
&\vdots \\
y_{2n} &= (1-a_{2n})x_n + a_{2n}T_2^n\ y_{1n}, \\
y_{1n} &= (1-a_{1n})x_n + a_{1n}T_1^n\ y_{0n},
\end{aligned} \tag{1.1}
$$

where $\{T_i : i \in I\}$ is a family of selfmappings of C, $0 \leq a_{in} \leq 1, y_{0n} = x_n$ for all n.

The iterative method (1.1) extends Mann iterative method [14], Ishikawa iterative method [8], Khan and Takahashi iterative method [12] and the three-step iterative method of Xu and Noor [22], simultaneously.

Moudafi [15] proposed viscosity iterative method which amounts to selecting a particular fixed point of a given nonexpansive selfmapping. The so-called viscosity iterative method have been studied by many authors (see, for example, [13, 16, 21] and references therein). These methods are very important because of their applicability to convex optimization, linear programming, monotone inclusions and elliptic differential equations [15].

Recently, Chang et al. [2] introduced and studied the following viscosity iterative method:

$$
\begin{aligned}
x_{n+1} &= (1-\alpha_n)f(x_n) + \alpha_n T^n\ y_n \\
y_n &= (1-\beta_n)x_n + \beta_n T^n x_n, \quad n \geq 1
\end{aligned} \tag{1.2}
$$

where T is an asymptotically nonexpansive mapping and f is a fixed contraction.

As the iterative methods in (1.1)-(1.2) involve convex combinations, so we need some convex structure in a metric space to define and investigate their convergence on a nonlinear domain.

A mapping $W : X^2 \times J \to X$ is a convex structure [20] on a metric space X if

$$d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1-\alpha)d(u, y)$$

for all $x, y, u \in X$ and $\alpha \in J = [0, 1]$. The metric space X together with a convex structure W is known as a convex metric space. A nonempty subset C of a convex metric space X is convex if $W(x, y, \alpha) \in C$ for all $x, y \in C$ and $\alpha \in J$. All normed linear spaces are convex metric spaces but there are convex metric spaces which are not linear; for example, a $CAT\,(0)$ space [1, 11].

A convex metric space X is uniformly convex if for any $\varepsilon > 0$, there exists $\delta = \delta\,(\varepsilon) > 0$ such that for all $r > 0$ and $x, y, z \in X$ with $d\,(z, x) \leq r, d\,(z, y) \leq r$ and $d\,(x, y) \geq r\varepsilon$ imply that $d\left(z, W\left(x, y, \frac{1}{2}\right)\right) \leq (1 - \delta)\, r$.

Obviously, uniformly convex Banach spaces are uniformly convex metric spaces.

A convex metric space X has property (C) if $W\,(x, y, \alpha) = W\,(y, x, 1 - \alpha)$ for all $x, y \in X, \alpha \in J$ and property (H) if $d\,(W\,(x, y, \alpha)\,, W\,(z, w, \alpha)) \leq \alpha d\,(x, z) + (1 - \alpha)\, d\,(y, w)$ for all $x, y, z, w \in X, \alpha \in J$.

In general, a convex structure W is not continuous. However, if W satisfies properties (C) and (H), then W is continuous. In fact, the property (C) always holds in uniformly convex metric spaces. Therefore W is continuous on a uniformly convex metric space if and only if it satisfies the property (H).

In a convex metric space, we devise a general iterative method which extends the methods in (1.1) and (1.2), simultaneously.

We define S_n-mapping generated by a family $\{T_i : i \in I\}$ of generalized asymptotically quasi-nonexpansive mappings on C as:

$$S_n x = U_{rn} x \tag{1.3}$$

where $U_{0n} = I$ (the identity mapping), $U_{1n}x = W(T_1^n U_{0n}x, x, a_{1n})$, $U_{2n}x = W(T_2^n U_{1n}x, x, a_{2n}), \ldots, U_{rn}x = W(T_r^n U_{(r-1)n}x, x, a_{rn})$.

For $\{\alpha_n\} \subset J$, a fixed contractive mapping f on C and S_n given in (1.3), we define $\{x_n\}$ as follows:

$$x_1 \in C, x_{n+1} = W(f\,(x_n)\,, S_n x_n, \alpha_n), \tag{1.4}$$

and call it a general viscosity iterative method in a convex metric space.

The purpose of this paper is to:
(i) establish a necessary and sufficient condition for convergence of iterative method (1.4) to a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings on a convex metric space;
(ii) prove strong convergence results for the iterative method (1.4) to a common fixed point of a finite family of uniformly Hölder continuous and generalized asymptotically quasi-nonexpansive mappings on a uniformly convex metric space.

Our work is a significant generalization of the corresponding results in Banach spaces and $CAT\,(0)$ spaces.

We assume that $F = \bigcap_{i \in I} F(T_i) \neq \emptyset$.

We need the following known results for our convergence analysis.

Lemma 1.1 ([9]). *Let the sequences $\{a_n\}$ and $\{u_n\}$ of real numbers satisfy:*

$$a_{n+1} \leq (1 + u_n) a_n,\; a_n \geq 0,\; u_n \geq 0, \sum_{n=1}^{\infty} u_n < +\infty.$$

Then (i) $\lim_{n\to\infty} a_n$ *exists;* (ii) *if* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 1.2 ([10])**.** *Let X be a uniformly convex metric space satisfying property (H). Let $x \in X$ and $\{a_n\}$ be a sequence in $[b,c]$ for some $b,c \in (0,1)$. If $\{u_n\}$ and $\{v_n\}$ are sequences in X such that* $\limsup_{n\longrightarrow\infty} d(u_n,x) \leq r$, $\limsup_{n\longrightarrow\infty} d(v_n,x) \leq r$ *and* $\lim_{n\longrightarrow\infty} d(W(u_n,v_n,a_n),x) = r$ *for some $r \geq 0$, then* $\lim_{n\to\infty} d(u_n,v_n) = 0$.

2. Convergence theorems in convex metric spaces

The aim of this section is to prove some results for the viscosity iteration method (1.4) to converge to a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings in a convex metric space.

Lemma 2.1. *Let C be a nonempty, closed and convex subset of a convex metric space X and $\{T_i : i \in I\}$ be a family of generalized asymptotically quasi-nonexpansive selfmappings of C, i.e., $d\left(T_i^n x, p_i\right) \leq (1+u_{in})d\left(x,p_i\right) + c_{in}$ for all $x \in C$ and $p_i \in F(T_i)$, $i \in I$ where $\{u_{in}\}$ and $\{c_{in}\}$ are sequences in $[0,\infty)$ with $\sum_{n=1}^{\infty} u_{in} < \infty, \sum_{n=1}^{\infty} c_{in} < \infty$ for each i. Then, for the sequence $\{x_n\}$ in (1.4) with $\sum_{n=1}^{\infty} \alpha_n < \infty$, there are sequences $\{\nu_n\}$ and $\{\xi_n\}$ in $[0,\infty)$ satisfying $\sum_{n=1}^{\infty} \nu_n < \infty, \sum_{n=1}^{\infty} \xi_n < \infty$ such that*
(a) *$d\left(x_{n+1},p\right) \leq \left(1+\nu_n\right)^r d\left(x_n,p\right) + \xi_n$, for all $p \in F$ and all $n \geq 1$;*
(b) *$d\left(x_{n+m},p\right) \leq M_1\left(d\left(x_n,p\right) + \sum_{n=1}^{\infty} \xi_n\right)$, for all $p \in F$ and $n \geq 1, m \geq 1, M_1 > 0$.*

Proof. (a) Let $p \in F$ and $\nu_n = \max_{i\in I} u_{in}$ for all $n \geq 1$. Since $\sum_{n=1}^{\infty} u_{in} < \infty$ for each i, therefore $\sum_{n=1}^{\infty} \nu_n < \infty$.

Now we have

$$\begin{aligned}
d\left(U_{1n}x_n,p\right) &= d\left(W(T_1^n U_{0n}x_n, x_n, a_{1n}), p\right) \\
&\leq (1-\alpha_{1n})d\left(x_n,p\right) + \alpha_{1n}\left(T_1^n x_n, p\right) \\
&\leq (1-\alpha_{1n})d\left(x_n,p\right) + \alpha_{1n}\left[(1+u_{1n})d\left(x_n,p\right) + c_{1n}\right] \\
&\leq (1+u_{1n})d\left(x_n,p\right) + c_{1n} \\
&= (1+\nu_n)^1 d\left(x_n,p\right) + c_{1n}.
\end{aligned}$$

Assume that $d\left(U_{kn}x_n,p\right) \leq (1+\nu_n)^k d\left(x_n,p\right) + (1+\nu_n)^{k-1}\sum_{i=1}^{k} c_{in}$ holds for some $k > 1$.

Consider

$$\begin{aligned}
d\left(U_{(k+1)n}x_n,p\right) &= d\left(W(T_{k+1}^n U_{kn}x_n, x_n, a_{(k+1)n}), p\right) \\
&\leq (1-a_{(k+1)n})d\left(x_n,p\right) + a_{(k+1)n}d\left(T_{k+1}^n U_{kn}x_n, p\right) \\
&\leq (1-a_{(k+1)n})d\left(x_n,p\right) + a_{(k+1)n}(1+u_{(k+1)n})d\left(U_{kn}x_n,p\right) \\
&\quad + a_{(k+1)n}c_{k+1n} \\
&\leq (1-a_{(k+1)n})d\left(x_n,p\right) + a_{(k+1)n}c_{(k+1)n} \\
&\quad + a_{(k+1)n}(1+u_{(k+1)n})d\left(U_{kn}x_n,p\right) \\
&\leq (1-a_{(k+1)n})d\left(x_n,p\right) + a_{(k+1)n}c_{(k+1)n} \\
&\quad + a_{(k+1)n}(1+\nu_n)\left[(1+\nu_n)^k d\left(x_n,p\right) + (1+\nu_n)^{k-1}\sum_{i=1}^{k} c_{in}\right] \\
&\leq (1-a_{(k+1)n})(1+\nu_n)^{k+1} d\left(x_n,p\right) + a_{(k+1)n}(1+\nu_n)c_{(k+1)n}
\end{aligned}$$

$$
\begin{aligned}
& +a_{(k+1)n}(1+\nu_n)^{k+1} d\left(x_n, p\right)+a_{(k+1)n}(1+\nu_n)^{k-1} \sum_{i=1}^{k+1} c_{in} \\
\leq\ & (1+\nu_n)^{k+1} d\left(x_n, p\right)+(1+\nu_n)^{k} \sum_{i=1}^{k+1} c_{in}
\end{aligned}
$$

By mathematical induction, we have

$$
d\left(U_{jn} x_n, p\right) \leq (1+\nu_n)^{j} d\left(x_n, p\right)+(1+\nu_n)^{j-1} \sum_{i=1}^{j} c_{in}, \quad 1 \leq j \leq r. \tag{2.1}
$$

Hence

$$
d\left(S_n x_n, p\right)=d\left(U_{rn} x_n, p\right) \leq (1+\nu_n)^{r} d\left(x_n, p\right)+(1+\nu_n)^{r-1} \sum_{i=1}^{r} c_{in}. \tag{2.2}
$$

Now, by (1.4) and (2.2), we obtain

$$
\begin{aligned}
d\left(x_{n+1}, p\right) &= d\left(W\left(f\left(x_n\right), S_n, \alpha_n\right), p\right) \\
&\leq \alpha_n d\left(f\left(x_n\right), p\right)+\left(1-\alpha_n\right) d\left(S_n x_n, p\right) \\
&\leq \alpha_n \alpha d\left(x_n, p\right)+\alpha_n d\left(f\left(p\right), p\right) \\
&\quad +\left(1-\alpha_n\right)\left((1+\nu_n)^{r} d\left(x_n, p\right)+(1+\nu_n)^{r-1} \sum_{i=1}^{r} c_{in}\right) \\
&\leq (1+\nu_n)^{r} d\left(x_n, p\right)+\left(1-\alpha_n\right)\left(1+\nu_n\right)^{r-1} \sum_{i=1}^{r} c_{in} \\
&\quad +\alpha_n d\left(f\left(p\right), p\right) \\
&\leq (1+\nu_n)^{r} d\left(x_n, p\right)+\alpha_n d\left(f\left(p\right), p\right)+(1+\nu_n)^{r-1} \sum_{i=1}^{r} c_{in}.
\end{aligned}
$$

Setting $\max\left\{d\left(f\left(p\right), p\right), \sup(1+\nu_n)^{r-1}\right\}=M$, we get that

$$
d\left(x_{n+1}, p\right) \leq (1+\nu_n)^{r} d\left(x_n, p\right)+M\left(\alpha_n+\sum_{i=1}^{r} c_{in}\right).
$$

That is,

$$
d\left(x_{n+1}, p\right) \leq (1+\nu_n)^{r} d\left(x_n, p\right)+\xi_n,
$$

where $\xi_n=M\left(\alpha_n+\sum_{i=1}^{r} c_{in}\right)$ and $\sum_{n=1}^{\infty} \xi_n<\infty$.

(b) We know that $1+t \leq e^{t}$ for $t \geq 0$. Thus, by part (a), we have

$$
\begin{aligned}
d\left(x_{n+m}, p\right) &\leq (1+\nu_{n+m-1})^{r} d\left(x_{n+m-1}, p\right)+\xi_{n+m-1} \\
&\leq e^{r\nu_{n+m-1}} d\left(x_{n+m-1}, p\right)+\xi_{n+m-1} \\
&\leq e^{r(\nu_{n+m-1}+\nu_{n+m-2})} d\left(x_{n+m-2}, p\right)+\xi_{n+m-1}+\xi_{n+m-2} \\
&\ \vdots \\
&\leq e^{r\sum_{i=n}^{n+m-1} v_i} d\left(x_n, p\right)+\sum_{i=n+1}^{n+m-1} v_i \sum_{i=n}^{n+m-1} \xi_i
\end{aligned}
$$

$$\begin{aligned} &\leq e^{r\sum_{i=1}^{\infty} v_i}\left(d\left(x_n, p\right)+\sum_{i=1}^{\infty} \xi_i\right) \\ &= M_1\left(d\left(x_n, p\right)+\sum_{i=1}^{\infty} \xi_i\right), \text{where} M_1=e^{r\sum_{i=1}^{\infty} v_i}. \end{aligned}$$

□

The next result deals with a necessary and sufficient condition for the convergence of $\{x_n\}$ in (1.4) to a point of F; for this we follow the arguments of Khan et al. ([9], Theorem 2.2).

Theorem 2.2. *Let C be a nonempty, closed and convex subset of a complete convex metric space X and $\{T_i : i \in I\}$ a family of generalized asymptotically quasi-nonexpansive selfmappings of C, i.e., $d\left(T_i^n x, p_i\right) \leq (1+u_{in}) d\left(x, p_i\right)+c_{in}$ for all $x \in C$ and $p_i \in F(T_i)$, $i \in I$ where $\{u_{in}\}$ and $\{c_{in}\}$ are sequences in $[0,\infty)$ with $\sum_{n=1}^{\infty} u_{in} < \infty, \sum_{n=1}^{\infty} c_{in} < \infty$ for all i. Then, for the sequence $\{x_n\}$ in (1.4) with $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\{x_n\}$ converges strongly to a point in F if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p\in F}(x, p)$.*

Proof. The necessity is obvious; we only prove the sufficiency. By Lemma 2.1 (a), we have

$$d\left(x_{n+1}, p\right) \leq (1+\nu_n)^r d\left(x_n, p\right)+\xi_n \text{ for all } p \in F \text{ and } n \geq 1.$$

Therefore,

$$\begin{aligned} d(x_{n+1}, F) &\leq (1+\nu_n)^r d(x_n, F)+\xi_n, \\ &= \left(1+\sum_{k=1}^{r} \frac{r(r-1)\cdots(r-k+1)}{k!} \nu_n^k\right) d(x_n, F)+\xi_n. \end{aligned}$$

As $\sum_{n=1}^{\infty} \nu_n < +\infty$, so $\sum_{n=1}^{\infty}\sum_{k=1}^{r} \frac{r(r-1)\cdots(r-k+1)}{k!}\nu_n^k < \infty$. Now $\sum_{n=1}^{\infty} \xi_n < \infty$ in Lemma 2.1 (a), so by by Lemma 1.1 and $\liminf_{n\to\infty} d(x_n, F) = 0$, we get that $\lim_{n\to\infty} d(x_n, F) = 0$. Next, we prove that $\{x_n\}$ is a Cauchy sequence in X. Let $\varepsilon > 0$. From the proof of Lemma 2.1 (b), we have

$$d\left(x_{n+m}, x_n\right) \leq d\left(x_{n+m}, F\right)+d\left(x_n, F\right) \leq \left(1+M_1\right) d\left(x_n, F\right)+M_1 \sum_{i=n}^{\infty} \xi_i, \tag{2.3}$$

Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{i=1}^{\infty} \xi_i < \infty$, there exists a natural number n_0 such that

$$d(x_n, F) \leq \frac{\varepsilon}{2\left(1+M_1\right)} \text{ and } \sum_{i=n}^{\infty} \xi_i < \frac{\varepsilon}{2M_1} \text{ for all } n \geq n_0.$$

So for all integers $n \geq n_0, m \geq 1$, we obtain from (2.3) that

$$d\left(x_{n+m}, x_n\right) < \left(M_1+1\right) \frac{\varepsilon}{2\left(1+M_1\right)}+M_1 \frac{\varepsilon}{2M_1} = \varepsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X and so converges to $q \in X$. Finally, we show that $q \in F$. For any $\overline{\varepsilon} > 0$, there exists natural number n_1 such that

$$d(x_n, F) = \inf_{p \in F} d\left(x_n, p\right) < \frac{\overline{\varepsilon}}{3} \text{ and } d\left(x_n, q\right) < \frac{\overline{\varepsilon}}{2}, \text{for all } n \geq n_1.$$

There must exist $p^* \in F$ such that $d\left(x_n, p^*\right) < \frac{\overline{\varepsilon}}{2}$ for all $n \geq n_1$; in particular, $d\left(x_{n_1}, p^*\right) < \frac{\overline{\varepsilon}}{2}$ and $d\left(x_{n_1}, q\right) < \frac{\overline{\varepsilon}}{2}$.

Hence

$$d\left(p^*, q\right) \leq d\left(x_{n_1}, p^*\right) + d\left(x_{n_1}, q\right) < \overline{\varepsilon}.$$

Since $\overline{\varepsilon}$ is arbitrary, therefore $d\left(p^*, q\right) = 0$. That is, $q = p^* \in F$. □

A generalized asymptotically nonexpansive mapping is a generalized asymptotically quasi-nonexpansive, so we have the following important new results:

Corollary 2.3. *Let C be a nonempty, closed and convex subset of a complete convex metric space X and $\{T_i : i \in I\}$ a family of generalized asymsptotically nonexpansive selfmappings of C, i.e., $d\left(T_i^n x, T_i^n y\right) \leq (1 + u_{in}) d\left(x, y\right) + c_{in}$, for all $x, y \in C$ and $i \in I$ where $\{u_{in}\}$ and $\{c_{in}\}$ are sequences in $[0, \infty)$ with $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} c_{in} < \infty$ for all i. Then the sequence $\{x_n\}$ in (1.4), converges strongly to a point $p \in F$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Corollary 2.4. *Let C, $\{T_i : i \in I\}$, F and $\{u_{in}\}, \{c_{in}\}$ be as in Theorem 2.2. Then the sequence $\{x_n\}$ in (1.4), converges strongly to a point $p \in F$ if and only if there exists a subsequence $\{x_i\}$ of $\{x_n\}$ which converges strongly to p.*

Theorem 2.5. *Let C be a nonempty, closed and convex subset of a complete convex metric space X, and $\{T_i : i \in I\}$ a family of generalized asymptotically nonexpansive selfmappings of C, i.e., $d\left(T_i^n x, T_i^n y\right) \leq (1 + u_{in}) d\left(x, y\right) + c_{in}$, for all $x, y \in C$ and $i \in I$ where $\{u_{in}\}$ and $\{c_{in}\}$ are sequences in $[0, \infty)$ with $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} c_{in} < \infty$ for all i. If $\lim_{n\to\infty} d\left(x_n, T_i x_n\right) = 0$ for the sequence $\{x_n\}$ in (1.4), $i \in I$ and one of the mappings is semi-compact, then $\{x_n\}$ converges strongly to $p \in F$.*

Proof. Let T_ℓ be semi-compact for some $1 \leq \ell \leq r$. Then there exists a subsequence $\{x_i\}$ of $\{x_n\}$ such that $x_i \to p \in C$. Hence

$$d\left(p, T_\ell p\right) = \lim_{i\to\infty} d\left(x_i, T_\ell x_i\right) = 0.$$

Thus, $p \in F$ and so by Corollary 2.4, $\{x_n\}$ converges strongly to a common fixed point of the family of mappings. □

Theorem 2.6. *Let C, $\{T_i : i \in I\}$, $F, \{u_{in}\}$ and $\{c_{in}\}$ be as in Theorem 2.5. Suppose that there exists a mapping T_j which satisfies the following conditions:*
(i) $\lim_{n\to\infty} d\left(x_n, T_j x_n\right) = 0$;
(ii) *there exists a constant M such that $d\left(x_n, T_j x_n\right) \geq M d(x_n, F)$, for all $n \geq 1$. Then the sequence $\{x_n\}$ in (1.4), converges strongly to a point $p \in F$.*

Proof. From (i) and (ii), it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. By Theorem 2.2, $\{x_n\}$ converges strongly to a common fixed point of the family of mappings. □

3. Results in a uniformly convex metric space

In this section, we establish some convergence results for the iterative method (1.4) of generalized asymptotically quasi-nonexpansive mappings on a uniformly convex metric space.

Lemma 3.1. *Let C be a nonempty, closed and convex subset of a uniformly convex metric space X and $\{T_i : i \in I\}$ be a family of uniformly Hölder continuous and generalized asymptotically quasi-nonexpansive selfmappings of C, i.e., $d(T_i^n x, p_i) \leq (1 + u_{in})(x, p_i) + c_{in}$ for all $x \in C$ and $p_i \in F(T_i)$, where $\{u_{in}\}$ and $\{c_{in}\}$ are sequences in $[0, \infty)$ with $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} c_{in} < \infty$, respectively, for each $i \in I$. Then, for the sequence $\{x_n\}$ in (1.4) with $a_{in} \in [\delta, 1-\delta]$ for some $\delta \in \left(0, \frac{1}{2}\right)$ and $\sum_{n=1}^{\infty} \alpha_n < \infty$, we have*
(a) $\lim_{n\to\infty} d(x_n, p)$ *exists for all* $p \in F$
(b) $\lim_{n\to\infty} d(x_n, T_j x_n) = 0$, *for each* $j \in I$.

Proof. (a) Let $p \in F$ and $\nu_n = \max_{i\in I} u_{in}$, for all $n \geq 1$.By Lemma 1.1 (i) and Lemma 2.1 (a), it follows that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$. Assume that

$$\lim_{n\to\infty} d(x_n, p) = c. \tag{3.1}$$

(b) The inequality (2.1) together with (3.1) gives that

$$\limsup_{n\to\infty} d(U_{jn} x_n, p) \leq c, 1 \leq j \leq r. \tag{3.2}$$

By (1.4), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(f(x_n), S_n, \alpha_n), p) \\ &\leq \alpha_n d(f(x_n), p) + (1-\alpha_n) d(S_n x_n, p) \\ &\leq \alpha_n d(f(x_n), p) + \alpha_n d(f(p), p) + (1-\alpha_n) d(U_{rn} x_n, p), \end{aligned} \tag{3.3}$$

and hence

$$c \leq \liminf_{n\to\infty} d(U_{rn} x_n, p) \tag{3.4}$$

Combining (3.2) and (3.4), we get

$$\lim_{n\to\infty} d(U_{rn} x_n, p) = c.$$

Note that

$$\begin{aligned} d(U_{rn} x_n, p) &= d\left(W(T_r^n U_{(r-1)n} x_n, x_n, a_{rn}), p\right) \\ &\leq a_{rn} d\left(T_r^n U_{(r-1)n} x_n, p\right) + (1-a_{rn}) d(x_n, p) \\ &\leq a_{rn}\left[(1+\nu_n) d\left(U_{(r-1)n} x_n, p\right) + c_{rn}\right] + (1-a_{rn}) d(x_n, p) \\ &= a_{rn}(1+\nu_n) d\left(W(T_{r-1}^n U_{(r-2)n} x_n, x_n, a_{(r-1)n}) x_n, p\right) \\ &\quad + a_{rn} c_{rn} + (1-a_{rn}) d(x_n, p) \\ &\leq a_{rn}(1+\nu_n)\left[a_{(r-1)n} d\left(T_{r-1}^n U_{(r-2)n} x_n, p\right) + \left(1-a_{(r-1)n}\right) d(x_n, p)\right] \\ &\quad + a_{rn} c_{rn} + (1-a_{rn}) d(x_n, p) \\ &\leq a_{rn} a_{(r-1)n}(1+\nu_n)^2 d\left(U_{(r-2)n} x_n, p\right) \\ &\quad + \left(1-a_{rn} a_{(r-1)n}\right)(1+\nu_n)^2 d(x_n, p) \end{aligned}$$

$$
\begin{aligned}
& +a_{rn}a_{(r-1)n}(1+\nu_n)^2 c_{(r-1)n} + a_{rn}(1+\nu_n)^2 c_{rn} \\
& \vdots \\
\leq\ & \prod_{i=j+1}^{r} a_{in}(1+\nu_n)^{r-j} d(U_{jn}x_n, p) \\
& +\Big(1-\prod_{i=j+1}^{r} a_{in}\Big)(1+\nu_n)^{r-j} d(x_n, p) \\
& +\prod_{i=j+1}^{r} a_{in}(1+\nu_n)^{r-j} c_{j+1n} + \prod_{i=j+2}^{r} a_{in}(1+\nu_n)^{r-j} c_{jn} \\
& +\cdots + a_{rn}(1+\nu_n)^{r-j} c_{rn}.
\end{aligned}
$$

and therefore, we have

$$
\begin{aligned}
d(x_n, p) \quad \leq \quad & \frac{d(x_n, p)}{\delta^{r-j}} - \frac{d(U_{rn}x_n, p)}{\delta^{r-j}(1+\nu_n)^{r-j}} + d(U_{jn}x_n, p) \\
& + c_{j+1n} + \frac{c_{jn}}{\delta} + \cdots + \frac{c_{rn}}{\delta^{r-j+1}}.
\end{aligned}
$$

Hence

$$
c \leq \liminf_{n\to\infty} d(U_{jn}x_n, p),\ 1 \leq j \leq r. \tag{3.5}
$$

Using (3.2) and (3.5), we have

$$
\lim_{n\to\infty} d(U_{jn}x_n, p) = c.
$$

That is,

$$
\lim_{n\to\infty} d\big(W(T_j^n U_{(j-1)n}x_n, x_n, a_{jn}), p\big) = c \text{ for } 1 \leq j \leq r.
$$

This together with (3.1) and (3.2) gives that

$$
\lim_{n\to\infty} d\big(T_j^n U_{(j-1)n}x_n, x_n,\big) = 0 \text{ for } 1 \leq j \leq r. \tag{3.6}
$$

If $j = 1$,we have by (3.6),

$$
\lim_{n\to\infty} d(T_1^n x_n, x_n) = 0.
$$

In case $j \in \{2, 3, 4, \ldots, r\}$, we observe that

$$
\begin{aligned}
d\big(x_n, U_{(j-1)n}x_n\big) &= d\big(x_n, W\big(T_{j-1}^n U_{(j-2)n}x_n, x_n, a_{(j-1)n}\big)\big) \\
&\leq a_{(j-1)n} d\big(T_{j-1}^n U_{(j-2)n}x_n, x_n\big) \to 0.
\end{aligned} \tag{3.7}
$$

Since T_j is uniformly Hölder continuous, therefore the inequality

$$
\begin{aligned}
d\big(T_j^n x_n, x_n\big) \quad & \leq \quad d\big(T_j^n x_n, T_j^n U_{(j-1)n}x_n\big) + d\big(T_j^n U_{(j-1)n}x_n, x_n\big) \\
& \leq \quad L d\big(x_n, U_{(j-1)n}x_n\big)^{\gamma} + d\big(T_j^n U_{(j-1)n}x_n, x_n\big),
\end{aligned}
$$

together with (3.6) and (3.7) gives that

$$
\lim_{n\to\infty} d\big(T_j^n x_n, x_n\big) = 0.
$$

Hence,

$$
d\big(T_j^n x_n, x_n\big) \to 0 \text{ as } n \to \infty \text{ for } 1 \leq j \leq r. \tag{3.8}
$$

Since

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_n, W(f(x_n), S_n x_n, \alpha_n)) \\ &\leq \alpha_n d(x_n, f(x_n)) + (1-\alpha_n) d(x_n, S_n x_n) \\ &\leq \alpha_n [d(x_n, p) + d(p, f(p)) + d(f(p), f(x_n))] \\ &\quad + (1-\alpha_n) a_{rn} d\left(x_n, T_r^n U_{(r-1)n} x_n\right) \\ &\leq \alpha_n (1+\alpha) d(x_n, p) + \alpha_n d(p, f(p)) \\ &\quad + (1-\alpha_n) a_{rn} d\left(x_n, T_r^n U_{(r-1)n} x_n\right), \end{aligned}$$

therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.9}$$

Let us observe that:

$$\begin{aligned} d(x_n, T_j x_n) &\leq d(x_n, x_{n+1}) + d\left(x_{n+1}, T_j^{n+1} x_{n+1}\right) \\ &\quad + d\left(T_j^{n+1} x_{n+1}, T_j^{n+1} x_n\right) + d\left(T_j^{n+1} x_n, T_j x_n\right) \\ &\leq d(x_n, x_{n+1}) + d\left(x_{n+1}, T_j^{n+1} x_{n+1}\right) \\ &\quad + L d(x_{n+1}, x_n)^\gamma + L d\left(T_j^n x_n, x_n\right)^\gamma. \end{aligned}$$

By uniform Hölder continuity of T_j, (3.8) and (3.9) , we get

$$\lim_{n\to\infty} d(x_n, T_j x_n) = 0, 1 \leq j \leq r. \tag{3.10}$$

□

Theorem 3.2. *Under the hypotheses of Lemma 3.1, assume, for some $1 \leq j \leq r$, T_j^m is semi-compact for some positive integer m. If X is complete, then $\{x_n\}$ in (1.4), converges strongly to a point in F.*

Proof. Fix $j \in I$ and suppose T_j^m is semi-compact for some $m \geq 1$. By (3.10), we obtain

$$\begin{aligned} d\left(T_j^m x_n, x_n\right) &\leq d\left(T_j^m x_n, T_j^{m-1} x_n\right) + d\left(T_j^{m-1} x_n, T_j^{m-2} x_n\right) \\ &\quad + \cdots + d\left(T_j^2 x_n, T_j x_n\right) + d(T_j x_n, x_n) \\ &\leq d(T_j x_n, x_n) + (m-1) L d(T_j x_n, x_n)^\gamma \to 0. \end{aligned}$$

Since $\{x_n\}$ is bounded and T_j^m is semi-compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to q \in C$. Hence, by (3.10), we have

$$d(q, T_i q) = \lim_{n\to\infty} d\left(x_{n_j}, T_i x_{n_j}\right) = 0, i \in I.$$

Thus $q \in F$ and so by Corollary 2.4, $\{x_n\}$ converges strongly to a common fixed point q of the family $\{T_i : i \in I\}$. □

An immediate consequence of Lemma 3.1 and Theorem 2.6 is the following strong convergence result in a uniformly convex metric space.

Theorem 3.3. *Let C, $\{T_i : i \in I\}$, $F, \{u_{in}\}$ and $\{c_{in}\}$ be as in Lemma 3.1. If there exists a constant M such that $d(x_n, T_j x_n) \geq M d(x_n, F)$, for all $n \geq 1$ and X is complete, then the sequence $\{x_n\}$ in (1.4), converges strongly to a point in F.*

Remark 3.4. (i) Theorem 2.2 and Theorems 3.2–3.3, respectively, contain as special cases, Theorem 2.2 and Theorems 3.2–3.3 in [9] which themselves improve the results of Khan and Takahashi [12], Suantai [19] and Xu and Noor [22].
(ii) Theorems 2.3, 3.2–3.3 are analogues of the corresponding results in [18, 23] for a general viscosity iterative method in a uniformly convex metric space.
(iii) All the results in this paper are new in $CAT(0)$ spaces.

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