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CONVERGENCE OF THE PATH AND ITS DISCRETIZATION TO THE MINIMUM-NORM FIXED POINT OF PSEUDOCONTRACTIONS

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Abstract. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a Lipschitz pseudocontractive mapping with $Fix(T) \neq \emptyset$. In this paper, we first show that as $t \to 0+$, the path $x \to x_t, t \in (0, 1)$, in C, defined by $x_t = (1-\beta)P_C[(1-t)x_t] + \beta T x_t$ converges strongly to the minimum-norm fixed point of T. Subsequently, by discreting the path, we suggest an explicit method $x_{n+1} = (1-\beta_n)P_C[(1-\alpha_n)x_n] + \beta_n T x_n$. Under some assumptions, we prove the sequence $\{x_n\}$ also converges strongly to the minimum-norm fixed point of T.

1. Introduction

The interest of pseudocontractions lies in their connection with monotone operators; namely, T is a pseudocontraction if and only if the complement $I - T$ is a monotone operator. However, it is now well-known that Mann's algorithm fails to converge for Lipschitzian pseudocontractions (see the counterexample of Chidume and Mutangadura [1]). It is therefore an interesting question of inventing iterative algorithms which generate a sequence converging in the norm topology to a fixed point of a Lipschitzian pseudocontraction (if any). On the other hand, it is quite often to seek a particular solution of a given nonlinear problem, in particular, the minimum-norm solution.

Recently, in order to find the minimum-norm fixed point of Lipschitzian pseudocontractions, Yao, Colao, Marino and Xu [4] suggested the following implicit algorithm

$$x_n = \beta_n x_{n-1} + (1-\alpha_n-\beta_n)T x_n, n \geq 0. \tag{1.1}$$

They proved that under some mild assumptions on algorithm parameters $\{\alpha_n\}$ and $\{\beta_n\}$, the sequence $\{x_n\}$ defined by (1.1) converges strongly to the minimum-norm fixed point of T provided $0 \in C$. They pointed out that this assumption $0 \in C$ cannot be removed due to the algorithm (1.1) may not be well-defined. Afterwards, they further put forth the following interesting topic: It is of interest to adapt the algorithm (1.1) to suit for the general case (i.e., without assuming $0 \in C$) of find the minimum-norm fixed point of a Lipschitz pseudocontraction.

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The purpose of this paper is to construct an implicit algorithm which defines a net $\{x_t\}$ converging strongly to the minimum-norm fixed point of a Lipschitz pseudocontraction without assuming $0 \in C$. Subsequently, by discreting the net, we suggest an explicit method which generates a sequence $\{x_n\}$. Under some assumptions, we prove the sequence $\{x_n\}$ also converges strongly to the minimum-norm fixed point of T.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Recall the following notions for a mapping $T : C \to C$.

- T is called pseudocontractive (or a pseudocontraction) if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2, \quad x, y \in C;$$

- T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

The (nearest point or metric) projection from H onto C is defined as follows: for each point $x \in H$, $P_C x$ is the unique point in C with the property:

$$\|x - P_C x\| \leq \|x - y\|, \quad y \in C.$$

Note that P_C is characterized by the inequality:

$$P_C x \in C, \quad \langle x - P_C x, y - P_C x\rangle \leq 0, \quad y \in C.$$

We adopt the following notations:

- $Fix(T)$ stands for the set of fixed points of T;
- $x_n \rightharpoonup x$ stands for the weak convergence of $\{x_n\}$ to x;
- $x_n \to x$ stands for the strong convergence of $\{x_n\}$ to x.

We need the following lemma for proof of our main results.

Lemma 2.1 ([5]). *Let C be a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a Lipschitz pseudocontraction. Then $Fix(T)$ is a closed convex subset of C and the mapping $I - T$ is demiclosed at 0, i.e. whenever $\{x_n\} \subset C$ is such that $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$, then $(I - T)x = 0$.*

Lemma 2.2 ([3]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in R such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n \gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section, we introduce our algorithms and prove the strong convergence of these algorithms to the minimum norm fixed point of pseudocontractive mapping T.

First, we introduce an implicit path on pseudocontractive mappings.

Algorithm 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be Lipschitz a pseudocontraction. Let $\beta \in (0,1)$ be a constant. For each $t \in (0,1)$, let the net $\{x_t\}$ be defined as the unique solution of fixed point equation*

$$x_t = (1-\beta)P_C[(1-t)x_t] + \beta T x_t,\ t \in (0,1), \tag{3.1}$$

where $P_C : H \to C$ is the metric projection from H on C.

Remark 3.2. We note that the algorithm (3.1) is well-defined. Indeed, for $\beta, t \in (0,1)$ define a mapping $U_t : C \to C$ by

$$U_t x = (1-\beta)P_C[(1-t)x] + \beta T x, \quad x \in C.$$

It is clear that U_t is a self-mapping of C. For $x, y \in C$, we have

$$\begin{aligned}
\langle U_t x - U_t y, x - y\rangle &= (1-\beta)\langle P_C[(1-t)x] - P_C[(1-t)y], x - y\rangle \\
&\quad + \beta\langle Tx - Ty, x - y\rangle \\
&\le (1-\beta)\|P_C[(1-t)x] - P_C[(1-t)y]\|\|x - y\| \\
&\quad + \beta\|x - y\|^2 \\
&\le (1-\beta)(1-t)\|x - y\|^2 + \beta\|x - y\|^2 \\
&= [1 - (1-\beta)t]\|x - y\|^2.
\end{aligned}$$

This implies that U_t is strongly pseudocontractive. So, by Deimling [2], U_t has a unique fixed point $x_t \in C$ which is the unique solution of the fixed point equation (3.1).

We are now in a position to prove the strong convergence of the implicit algorithm (3.1) to the minimum-norm fixed point of the pseudocontractive mapping T.

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a Lipschitz pseudocontraction with $Fix(T) \neq \emptyset$. Then the net $\{x_t\}$ defined by (3.1) converges in norm, as $t \to 0^+$, to the minimum-norm fixed point of T.*

Proof. We first show that the net $\{x_t\}$ is bounded.

Taking $p \in Fix(T)$, we get from (3.1) that

$$\begin{aligned}
\|x_t - p\|^2 &= (1-\beta)\langle P_C[(1-t)x_t] - p, x_t - p\rangle + \beta\langle Tx_t - p, x_t - p\rangle \\
&\le (1-\beta)\|P_C[(1-t)x_t] - p\|\|x_t - p\| + \beta\|x_t - p\|^2 \\
&\le (1-\beta)\|(1-t)x_t - p\|\|x_t - p\| + \beta\|x_t - p\|^2 \\
&\le (1-\beta)[(1-t)\|x_t - p\| + t\|p\|]\|x_t - p\| + \beta\|x_t - p\|^2.
\end{aligned}$$

It turns that

$$\|x_t - p\| \le \|p\|.$$

Consequently, $\{x_t\}$ is bounded and so is $\{Tx_t\}$.

Next, we show that $\lim_{t\to 0^+} \|x_t - Tx_t\| = 0$.

From (3.1), we have

$$\begin{aligned}\|x_t - Tx_t\| &= \|(1-\beta)P_C[(1-t)x_t] + \beta Tx_t - Tx_t\| \\ &\le (1-\beta)\|P_C[(1-t)x_t] - Tx_t\| \\ &\le (1-\beta)[\|x_t - Tx_t\| + t\|x_t\|].\end{aligned}$$

Therefore,

$$\|x_t - Tx_t\| \le \frac{(1-\beta)t}{\beta}\|x_t\| \to 0. \tag{3.2}$$

Next we show that $\{x_t\}$ is relatively norm-compact as $t \to 0^+$. Let $\{t_n\} \subset (0,1)$ be a sequence such that $t_n \to 0^+$ as $n \to \infty$. Put $x_n := x_{t_n}$. From (3.2), we have

$$\|x_n - Tx_n\| \to 0. \tag{3.3}$$

Again from (3.1), we get

$$\begin{aligned}\|x_t - p\|^2 &= (1-\beta)\langle P_C[(1-t)x_t] - p, x_t - p\rangle + \beta\langle Tx_t - p, x_t - p\rangle \\ &\le (1-\beta)\|P_C[(1-t)x_t] - p\|\|x_t - p\| + \beta\|x_t - p\|^2 \\ &\le (1-\beta)\frac{1}{2}(\|P_C[(1-t)x_t] - p\|^2 + \|x_t - p\|^2) + \beta\|x_t - p\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_t - p\|^2 &\le \|P_C[(1-t)x_t] - p\|^2 \\ &\le \|x_t - p - tx_t\|^2 \\ &= \|x_t - p\|^2 - 2t\langle x_t, x_t - p\rangle + t^2\|x_t\|^2 \\ &= \|x_t - p\|^2 - 2t\langle x_t - p, x_t - p\rangle - 2t\langle p, x_t - p\rangle + t^2\|x_t\|^2 \\ &= (1-2t)\|x_t - p\|^2 - 2t\langle p, x_t - p\rangle + t^2\|x_t\|^2.\end{aligned}$$

It turns out that

$$\begin{aligned}\|x_t - p\|^2 &\le \langle p, p - x_t\rangle + \frac{t}{2}\|x_t\|^2 \\ &\le \langle p, p - x_t\rangle + tM. \end{aligned} \tag{3.4}$$

where $M > 0$ is some constant such that $\sup\{\frac{1}{2}\|x_t\|^2 : t \in (0,1)\} \le M$. In particular, we get from (3.4)

$$\|x_n - p\|^2 \le \langle p, p - x_n\rangle + t_n M, \quad p \in Fix(T). \tag{3.5}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point $x^* \in C$. Noticing (3.3) we can use Lemma 2.1 to get $x^* \in Fix(T)$. Therefore we can substitute x^* for p in (3.5) to get

$$\|x_n - x^*\|^2 \le \langle x^*, x^* - x_n\rangle + t_n M. \tag{3.6}$$

However, $x_n \rightharpoonup x^*$. This together with (3.6) guarantees that $x_n \to x^*$. The net $\{x_t\}$ is therefore relatively compact, as $t \to 0^+$, in the norm topology.

Now we return to (3.5) and take the limit as $n \to \infty$ to get

$$\|x^* - p\|^2 \le \langle p, p - x^*\rangle, \quad p \in Fix(T).$$

This is equivalent to

$$0 \le \langle x^*, p - x^*\rangle, \quad p \in Fix(T). \tag{3.7}$$

Therefore, $x^* = P_{Fix(T)}0$. This is sufficient to conclude that the entire net $\{x_t\}$ converges in norm to x^* and x^* is the minimum-norm fixed point of T. As a matter of fact, from (3.7), we have

$$\|x^*\|^2 \le \langle x^*, p\rangle \le \|x^*\|\|p\|, \quad p \in Fix(T).$$

It follows that

$$\|x^*\| \le \|p\|, \quad p \in Fix(T).$$

This completes the proof. □

Now, we introduce an explicit algorithm which is the discretization of (3.1) and prove its strong convergence to the minimum-norm fixed point of T.

Algorithm 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a pseudocontraction. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real number sequences in $(0, 1)$. For chosen $x_0 \in C$ arbitrarily, we define a sequence $\{x_n\}$ iteratively by the following manner*

$$x_{n+1} = (1 - \beta_n)P_C[(1 - \alpha_n)x_n] + \beta_n T x_n, \quad n \ge 0. \tag{3.8}$$

Theorem 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be an $L-$Lipschitzian pseudocontraction with $Fix(T) \ne \emptyset$. If the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the condition $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = \lim_{n\to\infty} \frac{\beta_n^2}{\alpha_n} = 0$, then the following hold:*

(*i*) *the sequence $\{x_n\}$ defined by (3.8) is bounded;*

(*ii*) *the sequence $\{x_n\}$ is asymptotically regular, that is,* $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Further, if $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \frac{\|x_{n+1}-x_n\|}{\beta_n} = 0$, then the sequence $\{x_n\}$ converges strongly to the minimum-norm fixed point of T.

Proof. First we prove that the sequence $\{x_n\}$ is bounded. We will show this fact by induction. According to the assumption, there exists a sufficiently large positive integer m such that

$$1 - \frac{1}{1/2 - \beta_m}(L + 1)(L + 2)\left(\alpha_m + 2\beta_m + (\beta_m^2/\alpha_m)\right) > 0, \quad n \ge m. \tag{3.9}$$

Fix a $p \in Fix(T)$ and take a constant $M_1 > 0$ such that

$$\max\{\|x_0 - p\|, \|x_1 - p\|, \cdots, \|x_{m-1} - p\|, 4\|x_m - p\| + 8\|p\|\} \le M_1. \tag{3.10}$$

Next, we show that $\|x_{m+1} - p\| \le M_1$. Set $y_m = P_C[(1 - \alpha_m)x_m]$, thus $x_{m+1} = (1 - \beta_m)y_m + \beta_m T x_m$. By the fact that $I - T$ is monotone, we have

$$\langle (I - T)x_{m+1} - (I - T)p, x_{m+1} - p\rangle \ge 0. \tag{3.11}$$

From (3.8), we obtain

$$\begin{aligned}
\|x_{m+1}-p\|^2 &= (1-\beta_m)\langle y_m-p, x_{m+1}-p\rangle + \beta_m\langle Tx_m-p, x_{m+1}-p\rangle \\
&= (1-\beta_m)\langle y_m-(1-\alpha_m)x_m, x_{m+1}-p\rangle \\
&\quad + (1-\beta_m)\langle (1-\alpha_m)x_m-p, x_{m+1}-p\rangle \\
&\quad + \beta_m\langle Tx_m-p, x_{m+1}-p\rangle \\
&= (1-\beta_m)\langle y_m-(1-\alpha_m)x_m, x_{m+1}-p\rangle \\
&\quad + (1-\beta_m)\langle x_m-p, x_{m+1}-p\rangle - (1-\beta_m)\alpha_m\langle x_m, x_{m+1}-p\rangle \\
&\quad + \beta_m\langle Tx_m-p, x_{m+1}-p\rangle \\
&= (1-\beta_m)\langle y_m-(1-\alpha_m)x_m, x_{m+1}-p\rangle \\
&\quad + \langle x_m-p, x_{m+1}-p\rangle - (1-\beta_m)\alpha_m\langle x_{m+1}-p, x_{m+1}-p\rangle \\
&\quad - (1-\beta_m)\alpha_m\langle x_m-x_{m+1}, x_{m+1}-p\rangle \\
&\quad - (1-\beta_m)\alpha_m\langle p, x_{m+1}-p\rangle + \beta_m\langle Tx_m-Tx_{m+1}, x_{m+1}-p\rangle \\
&\quad + \beta_m\langle x_{m+1}-x_m, x_{m+1}-p\rangle - \beta_m\langle x_{m+1}-Tx_{m+1}, x_{m+1}-p\rangle.
\end{aligned}$$

Then, from (3.11), we get

$$\begin{aligned}
\|x_{m+1}-p\|^2 &\le (1-\beta_m)\|y_m-(1-\alpha_m)x_m\|\|x_{m+1}-p\| + \|x_m-p\|\|x_{m+1}-p\| \\
&\quad - (1-\beta_m)\alpha_m\|x_{m+1}-p\|^2 + (1-\beta_m)\alpha_m\|p\|\|x_{m+1}-p\| \\
&\quad + (1-\beta_m)\alpha_m(\|x_{m+1}-x_m\| + \|p\|)\|x_{m+1}-p\| \\
&\quad + \beta_m(\|Tx_m-Tx_{m+1}\| + \|x_{m+1}-x_m\|)\|x_{m+1}-p\| \\
&\le \|x_m-p\|\|x_{m+1}-p\| + 2(1-\beta_m)\alpha_m(\|x_m-p\| + \|p\|)\|x_{m+1}-p\| \\
&\quad - (1-\beta_m)\alpha_m\|x_{m+1}-p\|^2 + (1-\beta_m)\alpha_m\|p\|\|x_{m+1}-p\| \\
&\quad + (1-\beta_m)\alpha_m(\|x_{m+1}-x_m\| + \|p\|)\|x_{m+1}-p\| \\
&\quad + \beta_m(L+1)\|x_{m+1}-x_m\|\|x_{m+1}-p\| \\
&\le \|x_m-p\|\|x_{m+1}-p\| + (1-\beta_m)\alpha_m(2\|x_m-p\| + 4\|p\|)\|x_{m+1}-p\| \\
&\quad - (1-\beta_m)\alpha_m\|x_{m+1}-p\|^2 \\
&\quad + (\alpha_m+\beta_m)(L+1)\|x_{m+1}-x_m\|\|x_{m+1}-p\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
[1+(1-\beta_m)\alpha_m]\|x_{m+1}-p\| \quad &\le \quad \|x_m-p\| + \alpha_m(2\|x_m-p\| + 4\|p\|) \\
&\quad\; +(L+1)(\alpha_m+\beta_m)\|x_{m+1}-x_m\|.
\end{aligned} \tag{3.12}$$

By (3.8), we have

$$\begin{aligned}
\|x_{m+1}-x_m\| \quad &\le \quad (1-\beta_m)\|P_C[(1-\alpha_m)x_m] - P_C[x_m]\| + \beta_m\|Tx_m-x_m\| \\
&\le \quad (1-\beta_m)\alpha_m(\|x_m-p\| + \|p\|) + \beta_m(\|Tx_m-p\| + \|p-x_m\|) \\
&\le \quad \alpha_m(\|x_m-p\| + \|p\|) + \beta_m(L+1)\|x_m-p\| \\
&\le \quad (L+1)(\alpha_m+\beta_m)\|x_m-p\| + \alpha_m\|p\| \\
&\le \quad (L+2)(\alpha_m+\beta_m)M_1.
\end{aligned} \tag{3.13}$$

Substitute (3.13) into (3.12) to obtain

$$\begin{aligned}
&[1+(1-\beta_m)\alpha_m]\|x_{m+1}-p\| \\
\leq\ & \|x_m-p\|+\alpha_m(2\|x_m-p\|+4\|p\|)+(L+1)(L+2)(\alpha_m+\beta_m)^2M_1 \\
\leq\ & (1+\frac{1}{2}\alpha_m)M_1+(L+1)(L+2)(\alpha_m+\beta_m)^2M_1.
\end{aligned}$$

This together with (3.9) and (3.10) imply that

$$\begin{aligned}
\|x_{m+1}-p\| &\leq \left[1-\frac{(1/2-\beta_m)\alpha_m-(L+1)(L+2)(\alpha_m+\beta_m)^2}{1+(1-\beta_m)\alpha_m}\right]M_1 \\
&= \left\{1-\frac{(1/2-\beta_m)\alpha_m[1-\frac{1}{1/2-\beta_m}(L+1)(L+2)\left(\alpha_m+2\beta_m+(\beta_m^2/\alpha_m)\right)]}{1+(1-\beta_m)\alpha_m}\right\}M_1 \\
&\leq M_1.
\end{aligned}$$

By induction, we get

$$\|x_n-p\|\leq M_1,\quad \forall n\geq 0,$$

which implies that $\{x_n\}$ is bounded and so is $\{Tx_n\}$.

By (3.8), we have

$$\begin{aligned}
\|x_n-Tx_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-Tx_n\| \\
&\leq \|x_n-x_{n+1}\|+(1-\beta_n)\|P_C[(1-\alpha_n)x_n]-Tx_n\| \\
&\leq \|x_n-x_{n+1}\|+(1-\beta_n)\|x_n-Tx_n\|+\alpha_n\|x_n\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_n-Tx_n\| &\leq \frac{1}{\beta_n}\|x_n-x_{n+1}\|+\frac{\alpha_n}{\beta_n}\|x_n\| \\
&\leq \frac{1}{\beta_n}\|x_n-x_{n+1}\|+\frac{\alpha_n}{\beta_n}\|x_n\|.
\end{aligned}$$

By the assumptions, we have

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{3.14}$$

Next, we prove that

$$\limsup_{n\to\infty}\langle x^*,x^*-y_n\rangle\leq 0.$$

where $x^*=\lim_{t\to 0}z_t$ and $\{z_t\}$ is a net defined by $z_t=(1-\beta)P_C[(1-t)z_t]+\beta Tz_t$. From the definition of $\{z_t\}$, we obtain

$$z_t-x_n=(1-\beta)(P_C[(1-t)z_t]-x_n)+\beta(Tz_t-Tx_n)+\beta(Tx_n-x_n).$$

It follows that

$$\begin{aligned}
\|z_t-x_n\|^2 &= (1-\beta)\langle P_C[(1-t)z_t]-x_n,z_t-x_n\rangle+\beta\langle Tz_t-Tx_n,z_t-x_n\rangle \\
&\quad+\beta\langle Tx_n-x_n,z_t-x_n\rangle \\
&= (1-\beta)\langle P_C[(1-t)z_t]-(1-t)z_t,z_t-x_n\rangle \\
&\quad+(1-\beta)\langle(1-t)z_t-x_n,z_t-x_n\rangle+\beta\langle Tz_t-Tx_n,z_t-x_n\rangle \\
&\quad+\beta\langle Tx_n-x_n,z_t-x_n\rangle.
\end{aligned}$$

Noting that $x_n \in C$ and by using the property of the metric projection P_C, we have

$$\langle P_C[(1-t)z_t] - (1-t)z_t, z_t - x_n\rangle \le 0.$$

So,

$$\begin{aligned}\|z_t - x_n\|^2 &\le (1-\beta)\langle (1-t)z_t - x_n, z_t - x_n\rangle + \beta\|z_t - x_n\|^2 \\ &\quad + \beta\|Tx_n - x_n\|\|z_t - x_n\| \\ &= (1-\beta)\|z_t - x_n\|^2 - (1-\beta)t\langle z_t, z_t - x_n\rangle + \beta\|z_t - x_n\|^2 \\ &\quad + \beta\|Tx_n - x_n\|\|z_t - x_n\|.\end{aligned}$$

It follows that

$$\langle z_t, z_t - x_n\rangle \le \frac{\beta}{(1-\beta)t}\|Tx_n - x_n\|\|z_t - x_n\|.$$

By (3.14), we deduce

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle z_t, z_t - x_n\rangle \le 0. \tag{3.15}$$

Note the fact that the two limits $\limsup_{t\to 0}$ and $\limsup_{n\to\infty}$ are interchangeable. As a matter of fact, we have

$$\begin{aligned}\langle x^*, x^* - x_n\rangle &= \langle x^*, x^* - z_t\rangle + \langle x^* - z_t, z_t - x_n\rangle + \langle z_t, z_t - x_n\rangle \\ &\le \langle x^*, x^* - z_t\rangle + \|x^* - z_t\|\|z_t - x_n\| + \langle z_t, z_t - x_n\rangle \\ &\le (\|x^*\| + \|z_t - x_n\|)\|x^* - z_t\| + \langle z_t, z_t - x_n\rangle.\end{aligned}$$

This together with $z_t \to x^*$ and (3.15) imply that

$$\limsup_{n\to\infty}\langle x^*, x^* - x_n\rangle \le 0.$$

Note that $\|y_n - x_n\| \to 0$. We derive that

$$\limsup_{n\to\infty}\langle x^*, x^* - y_n\rangle \le 0. \tag{3.16}$$

Finally, we will show that $x_n \to x^*$. First, we have

$$\begin{aligned}\langle Tx_n - x^*, x_{n+1} - x^*\rangle &= \langle Tx_n - x^*, x_n - x^*\rangle + \langle Tx_n - x^*, x_{n+1} - x_n\rangle \\ &\le \|x_n - x^*\|^2 + \|Tx_n - x^*\|\|x_{n+1} - x_n\|, \qquad (3.17)\end{aligned}$$

and

$$\begin{aligned}\|y_n - x^*\|^2 &= \langle y_n - (1-\alpha_n)x_n, y_n - x^*\rangle + \langle (1-\alpha_n)x_n - x^*, y_n - x^*\rangle \\ &\le \langle (1-\alpha_n)x_n - x^*, y_n - x^*\rangle \\ &= (1-\alpha_n)\langle x_n - x^*, y_n - x^*\rangle - \alpha_n\langle x^*, y_n - x^*\rangle \\ &\le \frac{(1-\alpha_n)}{2}\|x_n - x^*\|^2 + \frac{1}{2}\|y_n - x^*\|^2 - \alpha_n\langle x^*, y_n - x^*\rangle.\end{aligned}$$

Thus,

$$\|y_n - x^*\|^2 \le (1-\alpha_n)\|x_n - x^*\|^2 - 2\alpha_n\langle x^*, y_n - x^*\rangle. \tag{3.18}$$

Therefore, from (3.8), (3.13), (3.17) and (3.18), we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1-\beta_n)(y_n - x^*) + \beta_n(Tx_n - x^*)\|^2 \\ &\le \|(1-\beta_n)(y_n - x^*)\|^2 + 2\beta_n\langle Tx_n - x^*, x_{n+1} - x^*\rangle\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\beta_n)^2(1-\alpha_n)\|x_n-x^*\|^2 - 2\alpha_n(1-\beta_n)^2\langle x^*, y_n - x^*\rangle \\
&\quad +2\beta_n\|x_n-x^*\|^2 + 2\beta_n\|Tx_n - x^*\|\|x_{n+1}-x_n\| \\
&\leq [1-(1-2\beta_n)\alpha_n]\|x_n-x^*\|^2 + 2\alpha_n(1-\beta_n)^2\langle x^*, x^*-y_n\rangle \\
&\quad +\beta_n^2\|x_n-x^*\|^2 + 2\beta_n\|Tx_n-x^*\|(L+2)(\alpha_n+\beta_n)M_1 \\
&= (1-\gamma_n)\|x_n-x^*\|^2 + \gamma_n\delta_n, \qquad (3.19)
\end{aligned}
$$

where $\gamma_n = (1-2\beta_n)\alpha_n$ and

$$
\begin{aligned}
\delta_n &= \frac{2(1-\beta_n)^2}{1-2\beta_n}\langle x^*, x^*-y_n\rangle + \frac{\beta_n^2}{(1-2\beta_n)\alpha_n}\|x_n-x^*\|^2 \\
&\quad + \frac{2\beta_n}{1-2\beta_n}\|Tx_n-x^*\|(L+2)M_1 + \frac{2\beta_n^2}{(1-2\beta_n)\alpha_n}\|Tx_n-x^*\|(L+2)M_1.
\end{aligned}
$$

It is clear that $\sum_{n=0}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\delta_n \leq 0$. We can therefore apply Lemma 2.2 to (3.19) and conclude that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

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