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STRONG CONVERGENCE TO COMMON ATTRACTIVE POINTS OF UNIFORMLY ASYMPTOTICALLY REGULAR NONEXPANSIVE SEMIGROUPS

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Abstract. In this paper, we study Halpern's type iterations [11] for nonexpansive semigroups and prove strong convergence to common attractive points of uniformly asymptotically left regular nonexpansive semigroups in Hilbert spaces. Using this result, we obtain some strong convergence theorems in Hilbert spaces.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a nonempty subset of H. For a mapping $T : C \to C$, we denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [19] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;
(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \ \forall x \in C\}$.

A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Kocourek, Takahashi and Yao [12] introduced a broad class of nonlinear mappings called *generalized hybrid* which containing nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. They proved a mean convergence theorem for generalized hybrid mappings which generalizes Baillon's nonlinear ergodic theorem [8]. Motivated by Baillon [8], and Kocourek, Takahashi and Yao [12], Takahashi and Takeuchi [19] introduced the concept of attractive points of a nonlinear mapping in a Hilbert space and they proved a mean convergence theorem of Baillon's type without convexity for generalized hybrid mappings. Motivated by Takahashi and Takeuchi [19], author and Takahashi [7] introduced the concept of common attractive points of a nonexpansive semigroup in a Hilbert space and proved a nonlinear mean convergence theorem of Baillon's type without convexity for nonexpansive semigroups.

In 1992, Wittmann [20] proved the following strong convergence theorems of Halpern's type [11] in a Hilbert space;

Theorem 1.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. For any $x_1 = x \in C$, define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n, \ \forall n \geq 1,$$

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where $\{\alpha_n\} \subset [0,1]$ *satisfies*

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty.$$

Then, $\{x_n\}$ *converges strongly to* $P_{F(T)}x$*, where* $P_{F(T)}$ *is the metric projection from* H *onto* $F(T)$*.*

Motivated by Takahashi and Takeuchi [19], Akashi and Takahashi [2] proved a strong convergence theorem of Halpern's type [11] for nonexpansive mappings in a star-shapes subset of a Hilbert space. On the other hand, Domingues Benavides, Acedo and Xu [10] proved strong convergence theorems of Halpern's type [11] for uniformly asymptotically regular one-parameter nonexpansive semigroups. They [10] also proved Browder's type [9] strong convergence theorems for the semigroups. Acedo and Suzuki [1] generalized Domingues Benavides, Acedo and Xu's results which is Browder's type [9] concerning the condition of the sequences in real numbers. Atsushiba [4] studied Browder's type iterations for nonexpansive semigroups and proved strong convergence theorems for uniformly asymptotically regular nonexpansive semigroups in Hilbert spaces (see also [5, 17, 18]).

In this paper, we study Halpern's type iterations [11] for nonexpansive semigroups and prove strong convergence to common attractive points of uniformly asymptotically left regular nonexpansive semigroups in Hilbert spaces. Using this result, we obtain some strong convergence theorems in Hilbert spaces.

2. Preliminaries and notations

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all positive integers and the set of all real numbers, respectively. We also denote by $\mathbb{Z}^+$ and $\mathbb{R}^+$ the set of all nonnegative integers and the set of all nonnegative real numbers, respectively. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We know the following basic equality from [18]. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.1}$$

Furthermore, we obtain that for all $x, y, w \in H$,

$$\langle (x-y) + (x-w), y-w \rangle = \|x-w\|^2 - \|x-y\|^2. \tag{2.2}$$

In fact, we have that

$$\begin{aligned}\langle (x-y) + (x-w), y-w \rangle &= \langle (x-y) + (x-w), (y-x) + (x-w) \rangle \\ &= \|x-w\|^2 - \|x-y\|^2 + \langle x-y, x-w \rangle + \langle x-w, y-x \rangle \\ &= \|x-w\|^2 - \|x-y\|^2.\end{aligned}$$

Let C be a closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|$$

for all $y \in C$. The mapping P_C is called the *metric projection* of H onto C. It is characterized by

$$\langle P_C x - y, x - P_C x \rangle \ge 0$$

for all $y \in C$. See [18] for more details. The following result is well-known; see [18].

Lemma 2.1. *Let C be a nonempty, bounded, closed and convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, $F(T) \neq \emptyset$.*

We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges strongly to x. We also write $x_n \rightharpoonup x$ (or w-$\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges weakly to x. In a Hilbert space, it is well known that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$. We say that a Banach space E satisfies *Opial's condition* [15] if for each sequence $\{x_n\}$ in E which converges weakly to x,

$$\varliminf_{n\to\infty} \|x_n - x\| < \varliminf_{n\to\infty} \|x_n - y\| \tag{2.3}$$

for each $y \in E$ with $y \neq x$. In a reflexive Banach space, this condition is equivalent to the analogous condition for a bounded net which has been introduced in [13]. It is also known that this condition is equivalent to the analogous condition of $\varlimsup_{n\to\infty}$ (see [6]). It is known that Hilbert spaces satisfy Opial's condition (see [15, 18]).

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from S to S are continuous. S is called right reversible if any two closed left ideals of S has non-void intersection. If S is right reversible, $(S, \leq)$ is a directed system when the binary relation "$\leq$" on S is defined by $s \leq t$ if and only if $\{s\} \cup \overline{Ss} \supset \{t\} \cup \overline{St},\ s, t \in S$, where $\overline{A}$ is the closure of A. Right reversible semitopological semigroups include all commutative semigroups and all semitopological semigroups which are right amenable as discrete semigroups (see [14, p.335]). Left reversibility of S is defined similarly. S is called reversible if it is both left and right reversible.

Let C be a nonempty subset of a Hilbert space H and let S be a semigroup. A family $\mathcal{S} = \{T(t) : t \in S\}$ of mappings of C into itself is said to be a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) For each $t \in S$, $T(t)$ is nonexpansive;
(ii) $T(ts) = T(t)T(s)$ for each $t, s \in S$;
(iii) for each $x \in C$, $t \mapsto T(t)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of all common fixed points of a nonexpansive semigroup $\mathcal{S}$, i.e.,

$$F(\mathcal{S}) = \bigcap_{t\in S} F(T(t)).$$

Motivated by Takahashi and Takeuchi [19], the author and Takahashi [7] introduced the set $A(\mathcal{S})$ of all common attractive points of the family $\mathcal{S} = \{T(t) : t \in S\}$ of mappings on C, i.e.,

$$A(\mathcal{S}) = \{x \in H : \|T(t)y - x\| \leq \|y - x\|,\ \forall y \in C,\ t \in S\}.$$

3. Lemmas

In this section, we give some lemmas which are used in the proof of our main theorem. They are basic properties of common attractive points of nonexpansive semigroups in a Hilbert space. Let S be a semigroup. We get the following lemmas as in the proof of lemmas in the case of commutative semigroups ([7]). For the sake of completeness, we give the proof.

Lemma 3.1. *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H, and let S be a semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a family of mappings on C. If $A(\mathcal{S}) \neq \emptyset$, then $F(\mathcal{S}) \neq \emptyset$.*

Proof. Let $u \in A(\mathcal{S})$ and $y = P_C u \in C$. Then, we have $T(t)y \in C$ from $T(t)C \subset C$. Furthermore, we have

$$\|T(t)y - u\| \leq \|y - u\| = \|P_C u - u\|.$$

By the properties of P_C, we have $T(t)y = P_C u = y$. Therefore $y \in F(\mathcal{S})$. □

Lemma 3.2. *Let H be a Hilbert space, let C be a nonempty subset of H, and let S be a semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a family of mappings on C. Then, $A(\mathcal{S})$ is a closed and convex subset of H.*

Proof. We show that $A(\mathcal{S})$ is closed. Let $\{z_n\} \subset A(\mathcal{S})$ be a sequence which converges strongly to $z \in H$. Take $x \in C$ and $t \in S$. From $z_n \in A(\mathcal{S})$, we have

$$\begin{aligned} \|z - T(t)x\| &\leq \|z - z_n\| + \|z_n - T(t)x\| \\ &\leq \|z - z_n\| + \|z_n - x\|. \end{aligned}$$

Since $z_n \to z$, we have

$$\|z - T(t)x\| \leq \|z - x\|.$$

This implies that $z \in A(\mathcal{S})$. So, $A(\mathcal{S})$ is closed. We prove that $A(\mathcal{S})$ is convex. Let $z_1, z_2 \in A(\mathcal{S}), \alpha \in [0,1]$ and $z = \alpha z_1 + (1-\alpha)z_2$. We prove from (2.1) that for any $x \in C$,

$$\begin{aligned} \|z - T(t)x\|^2 &= \|\alpha z_1 + (1-\alpha)z_2 - T(t)x\|^2 \\ &= \alpha\|z_1 - T(t)x\|^2 + (1-\alpha)\|z_2 - T(t)x\|^2 - \alpha(1-\alpha)\|z_1 - z_2\|^2 \\ &\leq \alpha\|z_1 - x\|^2 + (1-\alpha)\|z_2 - x\|^2 - \alpha(1-\alpha)\|z_1 - z_2\|^2 \\ &= \|\alpha(z_1 - x) + (1-\alpha)(z_2 - x)\|^2 = \|z - x\|^2. \end{aligned}$$

This implies that $z \in A(\mathcal{S})$. So, $A(\mathcal{S})$ is convex. □

We also have the following lemma (see also [7, 19]).

Lemma 3.3. *Let H be a Hilbert space, let C be a nonempty subset of H, and let S be a semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a family of mappings on C. Let $\{u_n\}$ be a sequence in H such that*

$$\varlimsup_{n\to\infty} \langle (u_n - y) + (u_n - T(t)y), y - T(t)y \rangle \leq 0$$

for all $t \in S$ and $y \in C$. If a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ converges weakly to $u \in H$, then $u \in A(\mathcal{S})$.

Proof. Since $\{u_{n_i}\}$ converses weakly to $u \in H$, we have that for any $t \in S$ and $y \in C$,

$$\begin{aligned} &\langle (u - y) + (u - T(t)y), y - T(t)y \rangle \\ &\qquad = \lim_{i\to\infty} \langle (u_{n_i} - y) + (u_{n_i} - T(t)y), y - T(t)y \rangle \end{aligned}$$

$$\leq \varlimsup_{n\to\infty} \langle (u_n - y) + (u_n - T(t)y), y - T(t)y \rangle$$
$$\leq 0.$$

On the other hand, we know from (2.2) that

$$0 \geq \langle (u - y) + (u - T(t)y), y - T(t)y \rangle = \|u - T(t)y\|^2 - \|u - y\|^2.$$

Thus we have

$$\|u - T(t)y\| \leq \|u - y\|.$$

for all $t \in S$ and $y \in C$. This implies $u \in A(\mathcal{S})$. □

We get the following lemma by [7] (see also [19]).

Lemma 3.4. *Let H be a Hilbert space, let C be a nonempty subset of H, and let S be a semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Suppose that there exists an $x \in C$ such that $\{T(t)x : t \in S\}$ is bounded. Then, $A(\mathcal{S}) \neq \emptyset$.*

To prove our main result, we need the following lemma (see [3]; see also [21]).

Lemma 3.5. *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$ and let $\{\gamma_n\}$ be a sequence of real numbers with $\varlimsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} s_n = 0$.

4. Strong convergence theorems

In this section, we study Halpern's type iterations [11] for nonexpansive semigroups and prove strong convergence to common attractive points of uniformly asymptotically left regular nonexpansive semigroups in Hilbert spaces (see also [2, 4, 7, 10, 16–19]).

Let C be a nonempty subset of H. Then, C is called star-shaped if there exists $z \in C$ such that for any $x \in C$ and any $\lambda \in (0,1)$,

$$\lambda z + (1 - \lambda)x \in C.$$

Throughout the rest of this section, we assume that C is a nonempty subset of H, and S is a right reversible semitopological semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. We say that a nonexpansive semigroup $\mathcal{S} = \{T(t) : t \in S\}$ is asymptotically left regular if

$$\lim_{s\in S} \|T(h)T(s)x - T(s)x\| = 0$$

for all $h \in S$ and $x \in C$ (see also [17, 18]). We also say that a nonexpansive semigroup $\mathcal{S} = \{T(t) : t \in S\}$ is uniformly asymptotically left regular if for every $h \in S$ and for every bounded subset K of C,

$$\lim_{s\in S} \sup_{x\in K} \|T(h)T(s)x - T(s)x\| = 0$$

holds. We prove a Halpern's [11] strong convergence theorem for a uniformly asymptotically regular nonexpansive semigroup. We also generalize Domingues Benavides, Acedo and Xu's result of Halpern's type [11] concerning the conditions of the sequence $\{\alpha_n\}$ in real numbers.

Theorem 4.1. *Let H be a Hilbert space, let C be a star-shaped subset of H with center $z \in C$. Let S be a right reversible semitopological semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a uniformly asymptotically left regular nonexpansive semigroup on C such that $A(\mathcal{S}) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{Z}^+$ such that $m_n \to \infty$. Let $t \in S$. Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n z + (1-\alpha_n)(T(t))^{m_n} x_n$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then, $\{x_n\}$ converges strongly to $P_{A(\mathcal{S})}z$, where $P_{A(\mathcal{S})}$ is the metric projection from H onto $A(\mathcal{S})$.

Proof. Let $x_1 \in C$ and $u \in A(\mathcal{S})$. Define $M = \|x_1 - z\| + \|z - u\|$. It is obvious that $\|x_1 - u\| \leq M$. Since $A(\mathcal{S}) \neq \emptyset$, we have that $\{T(t)x : t \in S\}$ is bounded for $x \in C$ and $\{x_n\}$ is bounded. Indeed,

$$\begin{aligned}\|T(t)x_1\| &\leq \|T(t)x_1 - u\| + \|u\| \\ &\leq \|x_1 - u\| + \|u\| \\ &\leq M + \|u\|.\end{aligned}$$

So, we have that $\{T(t)x : t \in S\}$ is bounded for $x \in C$. Suppose that $\|x_k - u\| \leq M$ for some $k \in \mathbb{N}$. We have that

$$\begin{aligned}\|x_{k+1} - u\| &= \|\alpha_k z + (1-\alpha_k)(T(t))^{m_k} x_k - u\| \\ &\leq \alpha_k \|z - u\| + (1-\alpha_k)\|(T(t))^{m_k} x_k - u\| \\ &\leq \alpha_k \|z - u\| + (1-\alpha_k)\|x_k - u\| \\ &\leq \alpha_k M + (1-\alpha_k) M \\ &= M\end{aligned}$$

By mathematical induction, we have that $\|x_k - u\| \leq M$ for each $k \in \mathbb{N}$. Thus, $\{x_n\}$ is bounded. Since

$$\|T(t)x_k - u\| \leq \|x_k - u\|$$

for each $t \in S$ and $k \in \mathbb{N}$, $\{T(t)x_n\}$ is also bounded. Since $\mathcal{S} = \{T(t) : t \in S\}$ is a nonexpansive semigroup on C, we have that

$$\begin{aligned}&\langle ((T(t))^{m_n} x_n - y) + ((T(t))^{m_n} x_n - T(h)y), y - T(h)y \rangle \\ &\quad = \|(T(t))^{m_n} x_n - T(h)y\|^2 - \|(T(t))^{m_n} x_n - y\|^2 \\ &\quad \leq \|(T(t))^{m_n} x_n - T(h)y\|^2 - \|T(h)(T(t))^{m_n} x_n - T(h)y\|^2 \\ &\quad = (\|(T(t))^{m_n} x_n - T(h)y\| + \|T(h)(T(t))^{m_n} x_n - T(h)y\|) \\ &\qquad \times (\|(T(t))^{m_n} x_n - T(h)y\| - \|T(h)(T(t))^{m_n} x_n - T(h)y\|) \\ &\quad \leq L(\|(T(t))^{m_n} x_n - T(h)y\| - \|T(h)(T(t))^{m_n} x_n - T(h)y\|)\end{aligned}$$

$$\begin{aligned}
&\leq L\|(T(t))^{m_n}x_n - T(h)(T(t))^{m_n}x_n\| \\
&= L\|T(t^{m_n})x_n - T(h)T(t^{m_n})x_n\| \qquad (4.1)
\end{aligned}$$

for all $h \in S$ and $y \in C$, where $L = \sup_{s\in S}\sup_{n\in\mathbb{N}}\{\|T(s)x_n - T(h)y\| + \|T(s)x_n - y\|\}$. For uniformly asymptotically left regular, we have that

$$\overline{\lim_{n\to\infty}} \langle ((T(t))^{m_n}x_n - y) + ((T(t))^{m_n}x_n - T(h)y), y - T(h)y\rangle \leq 0 \qquad (4.2)$$

for all $h \in S$ and $y \in C$. Furthermore, we have from

$$x_{n+1} = \alpha_n z + (1-\alpha_n)(T(t))^{m_n}x_n$$

and $\lim_{n\to\infty}\alpha_n = 0$ that

$$\lim_{n\to\infty}(x_{n+1} - (T(t))^{m_n}x_n) = \lim_{n\to\infty}\alpha_n(z - (T(t))^{m_n}x_n) = 0. \qquad (4.3)$$

Set $K = \sup_{t\in S}\|y - T(t)y\|$. We have that

$$\begin{aligned}
&\langle (x_{n+1} - y) + (x_{n+1} - T(h)y), y - T(h)y\rangle \\
&= \langle (x_{n+1} - (T(t))^{m_n}x_n + (T(t))^{m_n}x_n - y), y - T(h)y\rangle \\
&\qquad + \langle (x_{n+1} - (T(t))^{m_n}x_n + (T(t))^{m_n}x_n - T(h)y), y - T(h)y\rangle \\
&= \langle 2(x_{n+1} - (T(t))^{m_n}x_n), y - T(h)y\rangle \\
&\qquad + \langle ((T(t))^{m_n}x_n - y) + ((T(t))^{m_n}x_n - T(h)y), y - T(h)y\rangle \\
&\leq 2\|x_{n+1} - (T(t))^{m_n}x_n\|\|y - T(h)y\| \\
&\qquad + \langle ((T(t))^{m_n}x_n - y) + ((T(t))^{m_n}x_n - T(h)y), y - T(h)y\rangle \qquad (4.4)
\end{aligned}$$

for all $h \in S$ and $y \in C$. By (4.2), (4.3) and (4.4), we have

$$\overline{\lim_{n\to\infty}} \langle (x_{n+1} - y) + (x_{n+1} - T(h)y), y - T(h)y\rangle \leq 0.$$

Since $\{x_{n+1}\}$ is bounded, there exists a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ which converges weakly to a point $b \in H$. By Lemma 3.3, we have $b \in A(\mathcal{S})$. By Lemma 3.2, we have that $A(\mathcal{S})$ is closed and convex. So, there exists $P_{A(\mathcal{S})}z \in A(\mathcal{S})$, where $P_{A(\mathcal{S})}$ is the metric projection of H onto $A(\mathcal{S})$.

Next, let us show that

$$\overline{\lim_{n\to\infty}} \langle x_{n+1} - P_{A(\mathcal{S})}z, z - P_{A(\mathcal{S})}z\rangle \leq 0.$$

Without loss of generality, we may assume that there exists a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ such that

$$\overline{\lim_{n\to\infty}} \langle x_{n+1} - P_{A(\mathcal{S})}z, z - P_{A(\mathcal{S})}z\rangle = \lim_{i\to\infty}\langle x_{n_i+1} - P_{A(\mathcal{S})}z, z - P_{A(\mathcal{S})}z\rangle$$

and $x_{n_i+1} \rightharpoonup w$. As in the above, we have $w \in A(\mathcal{S})$. Thus, we have that

$$\begin{aligned}
\overline{\lim_{n\to\infty}} \langle x_{n+1} - P_{A(\mathcal{S})}z, z - P_{A(\mathcal{S})}z\rangle &= \lim_{i\to\infty}\langle x_{n_i+1} - P_{A(\mathcal{S})}z, z - P_{A(\mathcal{S})}z\rangle \\
&= \langle w - P_{A(\mathcal{S})}z, z - P_{A(\mathcal{S})}z\rangle \\
&\leq 0. \qquad (4.5)
\end{aligned}$$

By $P_{A(\mathcal{S})}z \in A(\mathcal{S})$, we also have that

$$\|(T(t))^{m_n}x_n - P_{A(\mathcal{S})}z\| \leq \|x_n - P_{A(\mathcal{S})}z\|.$$

Finally, we will show that $x_n \to P_{A(\mathcal{S})}z$.

$$\begin{aligned}
&\|x_{n+1} - P_{A(\mathcal{S})}z\|^2 \\
&= \|\alpha_n z + (1-\alpha_n)(T(t))^{m_n}x_n - P_{A(\mathcal{S})}z\|^2 \\
&= \|\alpha_n(z - P_{A(\mathcal{S})}z) + (1-\alpha_n)((T(t))^{m_n}x_n - P_{A(\mathcal{S})}z)\|^2 \\
&\leq (1-\alpha_n)^2\|(T(t))^{m_n}x_n - P_{A(\mathcal{S})}z\|^2 + 2\langle \alpha_n(z - P_{A(\mathcal{S})}z), x_{n+1} - P_{A(\mathcal{S})}z\rangle \\
&\leq (1-\alpha_n)\|(T(t))^{m_n}x_n - P_{A(\mathcal{S})}z\|^2 + 2\alpha_n\langle z - P_{A(\mathcal{S})}z, x_{n+1} - P_{A(\mathcal{S})}z\rangle \\
&\leq (1-\alpha_n)\|x_n - P_{A(\mathcal{S})}z\|^2 + 2\alpha_n\langle z - P_{A(\mathcal{S})}z, x_{n+1} - P_{A(\mathcal{S})}z\rangle.
\end{aligned}$$

From $\sum_{n=1}^{\infty}\alpha_n = \infty$, (4.5) and Lemma 3.5, we have

$$\lim_{n\to\infty}\|x_{n+1} - P_{A(\mathcal{S})}z\| = 0.$$

This completes the proof. □

5. Deduced theorems

Since we use an abstract semigroup in our main result, we can deduce some theorems from them. We say that a mapping T on C is asymptotically regular if

$$\lim_{n\to\infty}\|T^{n+1}x - T^n x\| = 0$$

for all $x \in C$ (see also [18]). We also say that a mapping T on C is uniformly asymptotically regular if for every bounded subset K of C,

$$\lim_{n\to\infty}\sup_{x\in K}\|T^{n+1}x - T^n x\| = 0$$

holds. By Theorems 4.1, we get the following strong convergence theorem. We also generalize Wittmann's conditions (Theorem 1.1) of the sequence $\{\alpha_n\}$ in real numbers.

Theorem 5.1. *Let H be a Hilbert space, let C be a star-shaped subset of H with center $z \in C$. Let T be a uniformly asymptotically regular nonexpansive mapping on C such that $A(T) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{Z}^+$ such that $m_n \to \infty$. Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n z + (1-\alpha_n)T^{m_n}x_n$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ satisfies

$$\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty.$$

Then, $\{x_n\}$ converges strongly to $P_{A(T)}z$, where $P_{A(T)}$ is the metric projection from H onto $A(T)$.

A family $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ of mappings of C into itself satisfying the following conditions is said to be a one-parameter nonexpansive semigroup on C:

(i) For each $t \in \mathbb{R}^+$, $T(t)$ is nonexpansive;

(ii) $T(0) = I$;
(iii) $T(t+s) = T(t)T(s)$ for every $t, s \in \mathbb{R}^+$;
(iv) for each $x \in C$, $t \mapsto T(t)x$ is continuous.

We say that a one-parameter nonexpansive semigroup $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ is asymptotically regular if

$$\lim_{s\to\infty} \|T(h+s)x - T(s)x\| = 0$$

for all $h \in \mathbb{R}^+$ and $x \in C$ (see also [17, 18]). We also say that a one-parameter nonexpansive semigroup $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ is uniformly asymptotically regular if for every $h \in \mathbb{R}^+$ and for every bounded subset K of C,

$$\lim_{s\to\infty} \sup_{x\in K} \|T(h+s)x - T(s)x\| = 0.$$

holds.

By Theorems 4.1, we get the following strong convergence theorem for a uniformly asymptotically regular one-parameter nonexpansive semigroup. We also generalize Domingues Benavides, Acedo and Xu's result [10] of Halpern's type [11] concerning the conditions of the sequence $\{\alpha_n\}$ in real numbers.

Theorem 5.2. *Let H be a Hilbert space, let C be a star-shaped subset of H with center $z \in C$. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a uniformly asymptotically regular one-parameter nonexpansive semigroup on C such that $A(\mathcal{S}) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{Z}^+$ such that $m_n \to \infty$. Let $t \in \mathbb{R}^+$. Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n z + (1-\alpha_n)(T(t))^{m_n} x_n$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then, $\{x_n\}$ converges strongly to $P_{A(\mathcal{S})}z$, where $P_{A(\mathcal{S})}$ is the metric projection from H onto $A(\mathcal{S})$.

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