Journal of Nonlinear and Convex Analysis
Volume 16, Number 1, 2015, 79–82

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

A CHARACTERIZATION RELATED TO A TWO-POINT BOUNDARY VALUE PROBLEM

BIAGIO RICCERI

Dedicated to Professor Sompong Dhompongsa on his 65th birthday

Abstract. In this short note, we establish the following result: Let $f : [0, +\infty[\to [0, +\infty[$, $\alpha : [0, 1] \to]0, +\infty[$ be two continuous functions, with $f(0) = 0$. Assume that, for some $a > 0$, the function $\xi \to \frac{\int_0^\xi f(t)dt}{\xi^2}$ is non-increasing in $]0, a]$.

Then, the following assertions are equivalent:

(i) for each $b > 0$, the function $\xi \to \frac{\int_0^\xi f(t)dt}{\xi^2}$ is not constant in $]0, b]$;

(ii) for each $r > 0$, there exists an open interval $I \subseteq]0, +\infty[$ such that, for every $\lambda \in I$, the problem

$$\begin{cases} -u'' = \lambda\alpha(t)f(u) & \text{in } [0,1] \\ u > 0 & \text{in }]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

has a solution u satisfying

$$\int_0^1 |u'(t)|^2 dt < r \ .$$

The aim of this very short note is to establish a characterization concerning the problem

$$(D) \begin{cases} -u'' = \lambda\alpha(t)f(u) & \text{in } [0,1] \\ u > 0 & \text{in }]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

where $f : [0, +\infty[\to [0, +\infty[$, $\alpha : [0, 1] \to]0, +\infty[$ are continuous functions, with $f(0) = 0$, and $\lambda > 0$.

For each $\xi \geq 0$, set

$$F(\xi) = \int_0^\xi f(t)dt \ .$$

Here is our result:

Theorem 1. *Assume that, for some $a > 0$, the function $\xi \to \frac{F(\xi)}{\xi^2}$ is non-increasing in $]0, a]$.*

Then, the following assertions are equivalent:

(i) *for each $b > 0$, the function $\xi \to \frac{F(\xi)}{\xi^2}$ is not constant in $]0, b]$;*

2010 *Mathematics Subject Classification.* 34B09, 34B18, 47J30.

Key words and phrases. Positive solutions, two-point boundary value problem, variational methods.

The author has been supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

(ii) for each $r > 0$, there exists an open interval $I \subseteq]0, +\infty[$ such that, for every $\lambda \in I$, problem (D) has a solution u satisfying

$$\int_0^1 |u'(t)|^2 dt < r \ .$$

Let $(X, \langle \cdot, \cdot \rangle)$ be a real Hilbert space. For each $r > 0$, set

$$B_r = \{x \in X : \|x\|^2 \leq r\} \ .$$

The key tool in our proof of Theorem 1 is provided by the following result which is entirely based on the very recent [1]:

Theorem 2. *Let $J : X \to \mathbf{R}$ be a sequentially weakly upper semicontinuos and Gâteaux differentiable functional, with $J(0) = 0$. Assume that, for some $r > 0$, there exists a global maximum $\hat{x}$ of $J_{|B_r}$ such that*

$$\langle J'(\hat{x}), \hat{x} \rangle < 2J(\hat{x}) \ .$$

Then, there exists an open interval $I \subseteq]0, +\infty[$ such that, for every $\lambda \in I$, the equation

$$x = \lambda J'(x)$$

has a non-zero solution lying in $\mathrm{int}(B_r)$.

Proof. Set

$$\beta_r = \sup_{B_r} J \ ,$$

$$\delta_r = \sup_{x \in B_r \setminus \{0\}} \frac{J(x)}{\|x\|^2}$$

and

$$\eta(s) = \sup_{y \in B_r} \frac{r - \|y\|^2}{s - J(y)}$$

for all $s \in]\beta_r, +\infty[$. From Proposition 2 of [1], it follows that

$$\frac{\beta_r}{r} < \delta_r \ .$$

As a consequence, by Theorem 1 of [1], for each $s \in]\beta_r, r\delta_r[$, the equation

$$x = \frac{\eta(s)}{2} J'(x)$$

has a non-zero solution lying in $\mathrm{int}(B_r)$. From Theorem 1 of [1] again, we know that the function η is convex and decreasing in $]\beta_r, +\infty[$. As a consequence, the set $\eta(]\beta_r, r\delta_r[)$ is an open interval. So, the conclusion is satisfied taking

$$I = \frac{1}{2}\eta(]\beta_r, r\delta_r[)$$

and the proof is complete. □

Now, we are able to prove Theorem 1.

Proof of Theorem 1. We adopt the variational point of view. So, let X be the space $H^1_0(0,1)$ with the usual inner product

$$\langle u, v\rangle = \int_0^1 u'(t)v'(t)dt \ .$$

Extend the definition of f (and of F as well) putting it zero in $]-\infty, 0[$. Let $J : X \to \mathbf{R}$ be the functional defined by setting

$$J(u) = \int_0^1 \alpha(t)F(u(t))dt$$

for all $u \in X$. By classical results, J is C^1 and sequentially weakly continuous, and (since $f \geq 0$) the solutions of problem (D) are exactly the non-zero solutions in X of the equation

$$u = \lambda J'(u) \ .$$

Let us prove that $(i) \to (ii)$. First of all, observe that, since $\xi \to \frac{F(\xi)}{\xi^2}$ is non-increasing in $]0, a]$, we have

$$f(\xi)\xi \leq 2F(\xi) \tag{1}$$

for all $\xi \in]0, a]$. Now, fix $r \in]0, a^2]$. Since

$$\sup_{u\in X} \frac{\max_{[0,1]} |u|}{\|u\|} \leq \frac{1}{2} \tag{2}$$

from (1) it follows that

$$f(u(t))u(t) \leq 2F(u(t)) \tag{3}$$

for all $u \in B_r$ and for all $t \in [0, 1]$. Now, let $u \in B_r$, with $\sup_{[0,1]} u > 0$. Observe that

$$\{t \in [0, 1] : f(u(t))u(t) < 2F(u(t))\} \neq \emptyset \ . \tag{4}$$

Indeed, otherwise, in view of (3) we would have

$$f(u(t))u(t) = 2F(u(t))$$

for all $t \in [0, 1]$ and so the function $\xi \to \frac{F(\xi)}{\xi^2}$ would be constant in the interval $]0, \sup_{[0,1]} u]$, against (i). Then, since α is positive in $[0, 1]$, from (4) we infer that

$$\int_0^1 \alpha(t)f(u(t))u(t)dt < 2\int_0^1 \alpha(t)F(u(t))dt \ .$$

This inequality can be rewritten as

$$\langle J'(u), u\rangle < 2J(u) \ .$$

Therefore, all the assumptions of Theorem 2 are satisfied and (ii) follows directly from it.

Now, let us prove that $(ii) \to (i)$. Arguing by contradiction, assume that there are $b, c > 0$ such that

$$F(\xi) = c\xi^2$$

and hence

$$f(\xi) = 2c\xi$$

for all $\xi \in [0, b]$. Fix $r \in]0, b^2]$. By (ii), there exists an open interval I such that, for every $\lambda \in I$, problem (D) has a solution u satisfying

$$\int_0^1 |u'(t)|^2 dt < r \ .$$

In view of (2), we have

$$\max_{[0,1]} u \leq b$$

and so

$$f(u(t)) = 2cu(t)$$

for all $t \in [0, 1]$. In other words, for every $\lambda \in I$, the problem

$$\begin{cases} -u'' = 2\lambda c\alpha(t)u & \text{in } [0,1] \\ u > 0 & \text{in }]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

would have a solution. This contradicts the classical fact that the above problem has a solution only for countably many $\lambda > 0$. □

Remark 3. *It is worth noticing the following wide class of functions f for which Theorem 1 applies. Namely, assume that f is $2k + 1$ times derivable (in a right neighbourhood of 0) and that $f^{(2k)}(0) < 0$ and $f^{(2m)}(0) = 0$ for all $m = 1, \ldots, k-1$ if $k \geq 2$. Then, there exists some $a > 0$ such that the function $\xi \to \frac{F(\xi)}{\xi^2}$ is decreasing in $]0, a]$. Indeed, if we put*

$$\varphi(\xi) = 2F(\xi) - \xi f(\xi) \ ,$$

we have $\varphi^{(2m)}(\xi) = -\xi f^{(2m)}(\xi)$ and $\varphi^{(2m+1)}(\xi) = -f^{(2m)}(\xi) - \xi f^{(2m+1)}(\xi)$ for all $m = 1, \ldots, k$. Hence, $\varphi(0) = \varphi^{(m)}(0) = 0$ for all $m = 1, \ldots, 2k$ and $\varphi^{(2k+1)}(0) > 0$. This clearly implies that, for some $a > 0$, one has $\varphi(\xi) > 0$ for all $\xi \in]0, a]$, as claimed.

References

[1] B. Ricceri, *A note on spherical maxima sharing the same Lagrange multiplier*, Fixed Point Theory Appl. **2014**, 2014: 25.

Manuscript received November 30, 2013
revised March 8, 2014

Biagio Ricceri
Department of Mathematics, University of Catania, Viale A. Doria 6, 95125 Catania, Italy
E-mail address: ricceri@dmi.unict.it