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BEST PROXIMITY POINTS FOR ASYMPTOTIC POINTWISE CONTRACTIONS

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Dedicated to Professor Sompong Dhompongsa on the occasion of his 65th birthday

Abstract. In this paper, we prove best proximity point results for weaker Meir-Keeler-type cyclic contractions and asymptotic pointwise cyclic contractions. We also provide some examples illustrating our results.

1. Introduction

Let A and B be nonempty subsets of a metric space (X, d). A map $T : A \cup B \to A \cup B$ is called a *cyclic* mapping if $T(A) \subset B$ and $T(B) \subset A$. Recently, Kirk et al. obtained the following interesting generalization of the Banach contraction principle:

Theorem 1.1 ([12]). *Let A and B be two nonempty closed subsets of a complete metric space X. Suppose that T is a cyclic map such that*

$$d(Tx, Ty) \leq \alpha\, d(x, y)$$

for some $\alpha \in (0, 1)$ and for all $x \in A, y \in B$. Then T has a unique fixed point in $A \cap B$.

If in the above theorem $A \cap B = \emptyset$, then the fixed point equation $Tx = x$ has no solution. Hence it is contemplated to find an approximate $x \in A$ such that the error $d(x, Tx)$ is minimum.

Definition 1.2. Let A and B be nonempty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ a cyclic mapping. A point $x^* \in A \cup B$ is called a best proximity point of T if $d(x^*, Tx^*) = dist(A, B)$, where

$$dist(A, B) = \inf\{d(x, y) : (x, y) \in A \times B\}.$$

Best proximity point theory has recently attracted the attention of number of authors (see for instance [2, 6, 7, 9, 15, 16, 17, 18]). For other related results, we refer to [3, 4].

In 2006, Eldred and Veeramani proved the following best proximity point theorem in uniformly convex Banach spaces.

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Theorem 1.3 ([9]). *Let A and B be two nonempty closed convex subsets of a uniformly convex Banach space X and $T : A \cup B \to A \cup B$ a cyclic contraction mapping, that is, T is cyclic and satisfies*

$$\|Tx - Ty\| \leq \alpha\|x - y\| + (1 - \alpha)dist(A, B)$$

for some $\alpha \in (0, 1)$ and for all $x \in A, y \in B$. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then there exists a unique $x^* \in A$ such that $x_{2n} \to x$ and $\|x^* - Tx^*\| = dist(A, B)$.

Di Bari et al. [7] extended Theorem 1.3 to cyclic Meir-Keeler contractions. Afterward, Suzuki et al. [18] introduced the property UC (see Definition 1.4) and extended the same result in metric spaces having the property UC.

Definition 1.4 ([18]). Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) is said to satisfy the property UC provided if $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_{n\to\infty} d(x_n, y_n) = dist(A, B)$ and $\lim_{n\to\infty} d(z_n, y_n) = dist(A, B)$, then

$$\lim_{n\to\infty} d(x_n, z_n) = 0.$$

Example 1.5 ([9]). Let A and B be two nonempty subsets of a uniformly convex Banach space X such that A is convex. Then (A, B) satisfies the property UC.

Other examples of pairs having the property UC can be found in [18].

The following lemma is needed in the sequel.

Lemma 1.6 ([18]). *Let A and B be two nonempty subsets of a metric space (X, d). Assume that (A, B) satisfies the property UC. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A and B, respectively, such that either of the following holds:*

$$\lim_{m\to\infty} \sup_{n\geq m} d(x_m, y_n) = d(A, B) \text{ or } \lim_{n\to\infty} \sup_{m\geq n} d(x_m, y_n) = d(A, B).$$

Then $\{x_n\}$ is a Cauchy sequence.

2. Preliminaries

In [10], Kirk introduced the notion of asymptotic contraction mappings as follows.

Definition 2.1 ([10]). A mapping $T : X \to X$ is said to be asymptotic contraction with $\varphi, \varphi_i : [0, \infty) \to [0, \infty)$ if φ, φ_i are continuous, $\varphi(s) < s$ for $s > 0$ and for all $x, y \in X$

$$d(T^i x, T^i y) \leq \varphi_i(d(x, y)), \tag{2.1}$$

where $\varphi_i \to \varphi$ uniformly on the range of d.

Theorem 2.2 ([10]). *Let (X, d) be a complete metric space and $T : X \to X$ a continuous asymptotic contraction such that (φ_i) in (2.1) are continuous. Assume also that some orbit of T is bounded. Then T has a fixed point $z \in X$, and moreover the Picard iterates $\{T^n(x)\}$ converges to z for each $x \in X$.*

Using of the notion of Meir-Keeler-type function [14], Chen in [5] defined the notion of the weaker Meir-Keeler-type function as follows.

Definition 2.3 ([5])**.** The function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a weaker Meir-Keeler-type function, if for each $\eta > 0$, there exists $\delta > \eta$ such that for $t \in \mathbb{R}^+$ with $\eta \leq t < \delta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(t) < \eta$.

Also, the notion of asymptotic pointwise weaker Meir-Keeler-type ψ-contractions was introduced in [5] as follows.

Definition 2.4 ([5])**.** Let X be a Banach space, and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ a weaker Meir-Keeler-type function. Then the mapping $T : X \to X$ is said to be an asymptotic pointwise weaker Meir-Keeler-type ψ-contraction, if for each $i \in \mathbb{N}$ and for each $x, y \in X$,

$$||T^i x - T^i y|| \leq \psi^i(||x||)||x - y||.$$

The following is the main result in [5].

Theorem 2.5 ([5])**.** *Let A be nonempty weakly compact convex subset of a Banach space X and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ a weaker Meir-Keeler-type function such that for each $t \in \mathbb{R}^+$, $\{\psi^i(t)\}_{i\in\mathbb{N}}$ is nonincreasing. Suppose that $T : A \to A$ is an asymptotic pointwise weaker Meir-Keeler-type ψ-contraction. Then, T has a unique fixed point $z \in A$, and for each $x \in A$, the sequence of Picard iterates, $\{T^n(x)\}$ converges in norm to z.*

The aim of this paper is to prove best proximity point theorems for weaker Meir-Keeler-type cyclic contractions and asymptotic pointwise cyclic contractions. Examples are also given to illustrate our main results.

3. Weaker Meir-Keeler-type cyclic contractions

Lemma 3.1. *Let (A, B) be a nonempty weakly compact convex pair of subsets of a Banach space X. Let $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a weaker Meir-Keeler-type function such that for each $t \in \mathbb{R}$, $\{\psi^i(t)\}_{i\in\mathbb{N}}$ is nonincreasing and for each $(x, y) \in A \times B$*

$$||T^{2i}x - T^{2i}y||^* \leq \psi^i(||x||)||x - y||^*, \ \forall y \in B, \tag{3.1}$$

$$||T^{2i}x - T^{2i}y||^* \leq \psi^i(||y||)||x - y||^*, \ \forall x \in A, \tag{3.2}$$

where $||x - y||^* := ||x - y|| - dist(A, B)$. Then for each $(x, y) \in A \times B$ there exits a pair $(v, w) \in A \times B$ such that

$$\limsup_{i\to\infty} ||T^{2i}x - w|| = dist(A, B) = \limsup_{i\to\infty} ||v - T^{2i}y||.$$

Proof. Assume that x is an arbitrary point in A and define $f : B \to [0, \infty)$ by

$$f(y) = \limsup_{i\to\infty} ||T^{2i}x - y||^*, \quad \forall y \in B.$$

Since B is weakly compact and convex, f attains its minimum at one point $w \in B$. On the other hand,

$$\begin{aligned} f(T^{2j}y) &= \limsup_{i\to\infty} ||T^{2i}x - T^{2j}y||^* \\ &= \limsup_{i\to\infty} ||T^{2i+2j}x - T^{2j}y||^* \\ &= \limsup_{i\to\infty} ||T^{2j}(T^{2i}x) - T^{2j}y||^* \\ &\leq \limsup_{i\to\infty} \psi^j(||y||)||T^{2i}x - y||^* \\ &= \psi^j(||y||)f(y), \end{aligned}$$

for all $y \in B$. Since $w \in B$ is a minimum of f, we conclude that

$$f(w) \le f(T^{2j}w) \le \psi^j(||w||)f(w), \quad \forall j \in \mathbb{N}. \tag{3.3}$$

It now follows from Theorem 3 of [5] that $\lim_{j\to\infty} \psi^j(||w||) = 0$. Hence, by (3.3), we conclude that $f(w) = 0$ and so $f(T^{2j}w) = 0$ for all $j \in \mathbb{N}$. Thus,

$$\limsup_{i\to\infty} ||T^{2i}x - w|| = dist(A, B) = \limsup_{i\to\infty} ||T^{2i}x - T^{2j}w||, \quad \forall j \in \mathbb{N}. \tag{3.4}$$

Since A is weakly compact convex subset of X, an argument similar to the above yields that for all $y \in B$ there exists $v \in A$ such that

$$\limsup_{i\to\infty} ||v - T^{2i}y|| = dist(A, B) = \limsup_{i\to\infty} ||T^{2j}v - T^{2i}y||, \ \forall j \in \mathbb{N}. \tag{3.5}$$

□

Here, we give an example illustrating Lemma 3.1.

Example 3.2. Consider $X = \mathbb{R}$ with the usual metric. Let $A = [1, 2]$, $B = [-2, -1]$. Define the mapping $T : A \cup B \to A \cup B$ by

$$T(x) = \begin{cases} -\frac{x}{2} - \frac{1}{2} & \text{if } \ x \in A, \\ -\frac{x}{2} + \frac{1}{2} & \text{if } \ x \in B. \end{cases}$$

Now, it is easy to see that T is cyclic on $A \cup B$ and $dist(A, B) = 2$. Moreover, for all $i \in \mathbb{N}$, we have $T^{2i}x = \frac{x}{2^{2i}} + \frac{2^{2i}-1}{2^{2i}}$, for $x \in A$ and $T^{2i}y = \frac{y}{2^{2i}} - \frac{2^{2i}-1}{2^{2i}}$, for $y \in B$. Also, for each $i \in \mathbb{N}$, we define the function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = \frac{t}{2}$. Obviously, ψ is a weaker Meir-Keeler-type function with for each $t \in \mathbb{R}$, $\{\psi^i(t)\}_{i\in\mathbb{N}}$ is nonincreasing; also for each $x \in A$, $y \in B$,

$$\begin{aligned} ||T^{2i}x - T^{2i}y||^* &= \left\| \frac{1}{2^{2i}}(x-y) + 2 - \frac{1}{2^{2i-1}} \right\| - 2 \\ &\le \left\| \frac{1}{2^{2i}}(x-y) - \frac{1}{2^{2i-1}} \right\| \\ &= \frac{1}{2^{2i}}||x - y - 2|| \\ &\le \frac{||x||}{2^i}||x-y||^* \\ &= \psi^i(||x||)||x-y||^*. \end{aligned} \tag{3.6}$$

Similarly, we can see that $||T^{2i}x - T^{2i}y||^* \le \psi^i(||y||)||x - y||^*$ for all $(x, y) \in A \times B$. Therefore, T satisfies all conditions of Lemma 3.1.

The following is a generalization of Theorem 2.5 due to Chen.

Theorem 3.3. *Let (A, B) be a nonempty weakly compact convex pair of subsets of a Banach space X. Let $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a weaker Meir-Keeler-type function such that for each $t \in \mathbb{R}$, $\{\psi^i(t)\}_{i\in\mathbb{N}}$ is nonincreasing and for each $(x, y) \in A \times B$*

$$||T^{2i}x - T^{2i}y|| \le \psi^i(||x||)||x - y||, \ \forall y \in B,$$

$$||T^{2i}x - T^{2i}y|| \le \psi^i(||y||)||x - y||, \ \forall x \in A.$$

Then $A \cap B$ is nonempty. Moreover, if

$$\|T^2x - Tx\| < \|Tx - x\|, \forall x \in A \cap B \text{ with } x \neq Tx,$$

then T has a unique fixed point in $A \cap B$. Further, for each $x \in A \cap B$ if $x_{2i} = T^{2i}x$, then $\{x_{2i}\}$ converges strongly to the fixed point of T.

Proof. If $x \in A$ and we define the function $f : B \to [0, \infty)$ with $f(y) = \limsup_{i\to\infty} \|T^{2i}x - y\|$, then Lemma 3.1 implies that there exists $w \in B$ such that

$$\limsup_{i\to\infty} \|T^{2i}x - w\| = \limsup_{i\to\infty} \|T^{2i}x - T^2w\| = 0.$$

Therefore, $w \in B$ is a fixed point of the mapping $T^2|_B$. Again, by using Lemma 3.1 there exists an element $v \in A$ such that

$$\|v - w\| = \limsup_{i\to\infty} \|v - T^{2i}w\| = 0,$$

which deduce that $v = w \in A \cap B$. That is $A \cap B$ is nonempty. We note that $T(A \cap B) \subseteq A \cap B$ and $A \cap B$ is weakly compact convex subset of X. Also,

$$||T^{2i}x - T^{2i}y|| \leq \psi^i(||x||)||x - y||,$$

for each $x, y \in A \cap B$. It now follows from Theorem 2.5 that the mapping T^2 has a unique fixed point in $A \cap B$. Suppose that $x^* \in A \cap B$ is a unique fixed point of T^2. If x^* is not a fixed point of T, then we must have

$$\|x^* - Tx^*\| = \|T^2x^* - Tx^*\| < \|Tx^* - x^*\|,$$

which is a contradiction, that is, x^* is a unique fixed point of T in $A \cap B$. □

The following example shows that Theorem 3.3 is a real generalization of Theorem 2.5.

Example 3.4. Let $X = \mathbb{R}$ and $A = B = [0, 2]$. Define the mapping $T : A \cup B \to A \cup B$ by

$$T(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1, \\ \frac{x}{2} & \text{if } 1 < x \leq 2. \end{cases}$$

Now, it is easy to see that T is cyclic on $A \cup B$ and $T^{2i}x = T^{2i}y = 0$ for any $i \in \mathbb{N}$, $x \in A$ and $y \in B$. Let ψ be an arbitrary weaker Meir-Keeler-type function. Then all hypothesis of Theorem 3.3 are satisfied and 0 is a unique fixed point for T. But, if we set $\psi(t) = \frac{t}{4}$ for all $t \in \mathbb{R}$, then, it is easy to see that T is not an asymptotic pointwise weaker Meir-Keeler-type ψ-contraction with $x = \frac{3}{2}$, $y = \frac{1}{2}$ and $i = 1$. So Theorem 2.5 cannot be applied.

In the next theorem, we prove the existence and uniqueness of a best proximity point for a class of cyclic mappings.

Theorem 3.5. *Let (A, B) be a nonempty, closed and convex pair in a uniformly convex Banach space X such that A is bounded. Suppose that the cyclic mapping $T : A \cup B \to A \cup B$ satisfies the condition* (3.1) *of Lemma 3.1 and*

$$\|T^2x - Tx\| < \|x - Tx\|, \quad \text{for all } x \in A \text{ with } dist(A, B) < ||x - Tx||. \tag{3.7}$$

Then T has a unique best proximity point in A.

Proof. Since A is a bounded closed convex subset of a uniformly convex Banach space X, the function $f : A \to [0,\infty)$ defined by $f(x) = \limsup_{i\to\infty} \|x - T^{2i}y\|^*$, where $y \in B$, attains its minimum at one point $v \in A$. By using Lemma 3.1 we have $f(v) = f(T^2v)$, that is, $v \in A$ is a fixed point of $T^2|_A$. We claim that v is a unique best proximity point of T in A. If $\|v - Tv\| > dist(A,B)$, by (3.7) we conclude that

$$\|v - Tv\| = \|T^2v - Tv\| < \|v - Tv\|,$$

which is a contradiction. Now, let $\acute{v} \in A$ be another best proximity point of T in A. By the strict convexity of X and convexity of A and B, we obtain

$$\begin{aligned} dist(A,B) &\leq \Big\|\frac{v+\acute{v}}{2} - \frac{Tv+T\acute{v}}{2}\Big\| = \Big\|\frac{v-Tv}{2} + \frac{\acute{v}-T\acute{v}}{2}\Big\| \\ &< \Big\|\frac{v-Tv}{2}\Big\| + \Big\|\frac{\acute{v}-T\acute{v}}{2}\Big\| = dist(A,B), \end{aligned}$$

which is a contradiction. Hence, $v = \acute{v}$ and this completes the proof. □

Definition 3.6 ([2]). Let (A,B) be a nonempty pair of subsets of a normed linear space X. The cyclic mapping $T : A\cup B \to A\cup B$ is said to satisfy the proximal property if

$$x_n \rightharpoonup x \in A\cup B \;\;\&\;\; \|x_n - Tx_n\| \to dist(A,B) \Rightarrow \|x - Tx\| = dist(A,B),$$

where "$\rightharpoonup$" denotes the weak convergence in X.

We note that, if $dist(A,B) = 0$, the proximal property reduces to the usual demiclosedness property of the mapping $I - T$ at 0, where I is the identity map on $A \cup B$.

Theorem 3.7. *Let (A,B) be a nonempty pair in a reflexive Banach space X such that A is bounded and weakly closed. Suppose that $T : A\cup B \to A\cup B$ is a cyclic mapping such that for each $(x,y) \in A\times B$*

$$||T^{2i}x - T^{2i}y||^* \leq \varphi^i(||x||)||x-y||^*, \;\forall y \in B, \tag{3.8}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is function satisfying $\varphi^i(t) \to 0$ for $t \geq 0$. Then, T has a best proximity point in A provided one of the following conditions is satisfied.

(i) *T is weakly continuous on A.*
(ii) *T satisfies the proximal property.*

Proof. At first, we note that for each $v \in A$

$$0 \leq \limsup_{i\to\infty} \|T^{2i}v - T^{2i+1}v\| \leq \limsup_{i\to\infty} \varphi^i(\|v\|)\|v - Tv\| = 0.$$

Since A is a bounded and X is reflexive, we may assume that $T^{2i}v \rightharpoonup v^* \in A$.

(i) Since T is weakly continuous on A, $T^{2i+1}v \rightharpoonup Tv^* \in B$. Hence,

$$\|v^* - Tv^*\| \leq \liminf_{i\to\infty} \|T^{2i}v - T^{2i+1}v\| = dist(A,B).$$

That is, v^* is a best proximity point of T in A.

(ii) Since $\|T^{2i}v - T^{2i+1}v\| \to dist(A,B)$ and $T^{2i}v \rightharpoonup v^* \in A$ and T satisfies the proximal property, we conclude that $||v^* - Tv^*\| = dist(A,B)$.

□

Here, we recall the notion of relatively nonexpansive mappings.

Definition 3.8. Let (A, B) be a nonempty pair of subsets of a Banach space X and $T : A \cup B \to A \cup B$ be a cyclic mapping. T is said to be a relatively nonexpansive mapping if $\|Tx - Ty\| \le \|x - y\|$ for all $(x, y) \in A \times B$.

We note that the class of relatively nonexpansive mappings contains the class of nonexpansive mapping as a subclass.

The next theorem guarantees the existence of a best proximity point for relatively nonexpansive mappings.

Theorem 3.9. *Let (A, B) be a nonempty pair of subsets of a Banach space X and $T : A \cup B \to A \cup B$ be a relatively nonexpansive map. Let $x_0 \in A$ be given. Define an iterative sequence $\{x_n\}_{n \in \mathbb{N}}$ by $x_{n+1} = Tx_n$ for $n \in \mathbb{N} \cup \{0\}$. Suppose that*

(i) *$\{x_{2n}\}$ has a convergent subsequence in A;*
(ii) *$\lim_{n\to\infty} ||T^{2n}x - T^{2n+1}x|| = dist(A, B)$.*

Then T has a best proximity point in A.

Proof. Suppose that $\{x_{2n_k}\}$ is a subsequence of the sequence $\{x_{2n}\}$ such that $x_{2n_k} \to z \in A$. We now have

$$dist(A, B) \le ||z - x_{2n_k+1}|| \le ||z - x_{2n_k}|| + ||x_{2n_k} - x_{2n_k+1}|| \quad \text{for all } k \in \mathbb{N},$$

which implies that $\lim_{k\to\infty} ||z - x_{2n_k+1}|| = dist(A, B)$. Thus,

$$dist(A, B) \le ||x_{2n_k+2} - Tz|| \le ||z - x_{2n_k+1}|| \quad \text{for all } k \in \mathbb{N},$$

and hence $||z - Tz|| = dist(A, B)$, that is z is a best proximity point of the mapping T in A. □

4. Asymptotic pointwise cyclic contractions

In [11] W. A. Kirk introduced the notion of an asymptotic pointwise contraction map:

Definition 4.1. Suppose that (X, d) is a metric space. Let $T : X \to X$ and for each $n \in \mathbb{N}$ let $\alpha_n : X \to \mathbb{R}^+$ such that

$$d(T^n x, T^n y) \le \alpha_n(x) d(x, y) \quad \forall x, y \in X.$$

If the sequence $\{\alpha_n\}$ converges pointwise to the function $\alpha : X \to [0, 1)$, then T is called an asymptotic pointwise contraction.

It was announced in [11] that any asymptotic pointwise contraction defined on a bounded closed convex subset of a superreflexive Banach space has a fixed point. In [13], Kirk and Xu proved the following theorem for asymptotic pointwise contractions.

Theorem 4.2. *Let K be a weakly compact convex subset of a Banach space X and let $T : K \to K$ be an asymptotic pointwise contraction. Then T has a unique fixed point $z \in K$, and for each $x \in K$, the sequence of Picard iterates, $\{T^n x\}$, converges in norm to z.*

In 2005, Eldred et al. (see [8]) introduced the notion of proximal normal structure as follows.

Definition 4.3. A pair (A, B) of subsets of a normed linear space is said to be a proximal pair if for each $(x, y) \in A \times B$ there exists $(\acute{x}, \acute{y}) \in A \times B$ such that

$$\|x - \acute{y}\| = \|\acute{x} - y\| = dist(A, B).$$

Definition 4.4. A convex pair (K_1, K_2) in a Banach space X is said to have proximal normal structure if for any closed, bounded, convex proximal pair $(H_1, H_2) \subseteq (K_1, K_2)$ for which $dist(H_1, H_2) = dist(K_1, K_2)$ and $\delta(H_1, H_2) > dist(H_1, H_2)$, there exists $(x_1, x_2) \in H_1 \times H_2$ such that

$$\delta(x_1, H_2) < \delta(H_1, H_2), \quad \delta(x_2, H_1) < \delta(H_1, H_2).$$

They used this geometric property to study mappings that are relatively nonexpansive in the sense that they are defined on the union of two subsets A and B of a Banach space X and satisfy $\|Tx - Ty\| \leq \|x - y\|$ for all $x \in A$, $y \in B$. It was shown that if A and B are weakly compact and convex, and if the pair (A, B) has proximal normal structure, then a cyclic relatively nonexpansive mapping $T : A \cup B \to A \cup B$ has at least one best proximity point.

In this section we study the existence and convergence of best proximity points for cyclic relatively nonexpansive mappings, which are asymptotic pointwise cyclic contraction in the following sense.

Definition 4.5. Let (A, B) be a nonempty pair in a Banach space X. A mapping $T : A \cup B \to A \cup B$ is said to be an asymptotic pointwise cyclic contraction if T is cyclic and there exists a function $\alpha : A \cup B \to [0, 1)$ such that for any integer $n \geq 1$ and $(x, y) \in A \times B$,

$$\|T^{2n}x - T^{2n}y\| \leq \alpha_n(x)\|x - y\| + (1 - \alpha_n(x))\, dist(A, B) \quad \text{for all} \quad y \in B, \tag{4.1}$$

$$\|T^{2n}x - T^{2n}y\| \leq \alpha_n(y)\|x - y\| + (1 - \alpha_n(y))\, dist(A, B) \quad \text{for all} \quad x \in A. \tag{4.2}$$

where $\alpha_n \to \alpha$ pointwise on $A \cup B$.

It is easy to see that the class of mappings which was introduced in previous definition, generalizes the class of mappings which was introduced by Abkar and Gabeleh in Definition 3.5 of [1].

Theorem 4.6. *Let (A, B) be a nonempty bounded closed convex pair in a uniformly convex Banach space X and $T : A \cup B \to A \cup B$ an asymptotic pointwise cyclic contraction map. If T is a relatively nonexpansive mapping, then there exits a unique pair $(v^*, u^*) \in A \times B$ such that*

$$\|v^* - Tv^*\| = \|Tu^* - u^*\| = dist(A, B).$$

Further, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges in norm to v^* and $\{x_{2n+1}\}$ converges in norm to u^*.

Proof. Let $x_0 \in A$ and define $f : B \to [0, \infty)$ by $f(u) = \limsup_{n\to\infty} \|T^{2n}x_0 - u\|$. Since X is uniformly convex, and B is bounded closed and convex, it follows that f attains its minimum at exactly one point in B namely u^*. We note that for all integers $m \geq 1$ and $u \in B$,

$$f(T^{2m}u) = \limsup_{n\to\infty} \|T^{2n}x_0 - T^{2m}u\| = \limsup_{n\to\infty} \|T^{2n+2m}x_0 - T^{2m}u\|$$

$$\begin{aligned} &= \limsup_{n\to\infty} \|T^{2m}(T^{2n}x_0) - T^{2m}u\| \\ &\le \limsup_{n\to\infty} [\alpha_m(u)\|T^{2n}x_0 - u\| + (1-\alpha_m(u))dist(A,B)] \\ &= \alpha_m(u)f(u) + (1-\alpha_m(u))dist(A,B). \end{aligned}$$

Since u^* is the minimum of f, we must have

$$f(u^*) \le f(T^{2m}u^*) \le \alpha_m(u^*)f(u^*) + (1-\alpha_m(u^*))dist(A,B), \text{ for all } m \ge 1. \tag{4.3}$$

Now by $\alpha_m(u^*) \to \alpha(u^*) < 1$, we have

$$f(u^*) \le \alpha(u^*)f(u^*) + (1-\alpha(u^*))dist(A,B).$$

This shows that $f(u^*) = dist(A,B)$. On the other hand,

$$f(T^2u^*) = \limsup_{n\to\infty} \|T^{2n}x_0 - T^2u^*\| \le \limsup_{n\to\infty} \|T^{2n-2}x_0 - u^*\| = f(u^*).$$

This implies that $T^2u^* = u^*$, by the uniqueness of minimum of f. Then u^* is a fixed point of T^2 in B. We also note that,

$$\lim_{m\to\infty} \sup_{n\ge m} \|T^{2m}x_0 - T^{2n}u^*\| = \lim_{m\to\infty} \|T^{2m}x_0 - u^*\| = f(u^*) = dist(A,B).$$

Since (A,B) has the property UC, it follows from Lemma 1.6 that the sequence $\{T^{2n}x_0\}$ is a Cauchy sequence and then there exists $\tilde{x}$ in A such that $x_{2n} \to \tilde{x}$. By a similar argument, if $y_0 \in B$ and $g : A \to [0,\infty)$ is given by $g(v) = \limsup_{n\to\infty} \|T^{2n}y_0 - v\|$, then g takes it's minimum at exactly one point, v^*, which is a fixed point of T^2 in A, moreover $T^{2n}y_0 \to \tilde{y} \in B$. Hence we obtain $u^* = T^{2n}u^* \to \tilde{y}$ and $v^* = T^{2n}v^* \to \tilde{x}$. This shows that $(v^*, u^*) = (\tilde{x}, \tilde{y})$, and $T^{2n}x_0 \to v^*$, $T^{2n}y_0 \to u^*$. Further

$$\|v^* - u^*\| = \|T^{2n}v^* - T^{2n}u^*\| \le \alpha_n(v^*)\|v^* - u^*\| + (1-\alpha_n(v^*))dist(A,B).$$

Now if $n \to \infty$ then we have $\|v^* - u^*\| = dist(A,B)$. It follows from the uniform convexity of X that there is a unique pair $(v^*, u^*) \in A \times B$ such that $\|v^* - u^*\| = dist(A,B)$. Since T is a relatively nonexpansive mapping, $\|Tv^* - Tu^*\| \le \|v^* - u^*\| = dist(A,B)$, therefore $Tv^* = u^*$ and $Tu^* = v^*$. This implies that $\|v^* - Tv^*\| = \|Tu^* - u^*\| = dist(A,B)$. □

Let us illustrate the above theorem with the following example.

Example 4.7. Consider $X = \mathbb{R}$ with the usual metric. Let $A = [1,2]$, $B = [-2,-1]$. Define the mapping $T : A \cup B \to A \cup B$ by

$$T(x) = \begin{cases} -\sqrt{x} & \text{if } x \in A, \\ \sqrt{-x} & \text{if } x \in B. \end{cases}$$

Now, it is easy to see that T is cyclic on $A \cup B$ and $dist(A,B) = 2$. Moreover, for all $n \in \mathbb{N}$, we have $T^{2n}x = \sqrt[2n]{x}$, for $x \in A$ and $T^{2n}y = -\sqrt[2n]{-y}$, for $y \in B$. Also, for each $n \in \mathbb{N}$, we define the function $\alpha_n : A \cup B \to \mathbb{R}$ as follows:

$$\alpha_n(x) = \begin{cases} \frac{nx}{3n+1} & \text{if } x \in A, \\ \frac{-nx}{3n+1} & \text{if } x \in B. \end{cases}$$

Obviously, $\alpha_n(x) \to \alpha(x)$, where

$$\alpha(x) = \begin{cases} \frac{x}{3} & \text{if } \ x \in A, \\ \frac{-x}{3} & \text{if } \ x \in B. \end{cases}$$

We show that T satisfy the relations (4.1), (4.2). Indeed, if $(x, y) \in A \times B$, then

$$\begin{aligned} \alpha_n(x)\|x-y\| + (1-\alpha_n(x))\ dist(A,B) &= \Big(\frac{nx}{3n+1}\Big)(x-y) + \Big(1 - \frac{nx}{3n+1}\Big) \\ &\geq \Big(\frac{nx}{3n+1}\Big)(x-y) + \Big(\frac{3n+1-nx}{3n+1}\Big)(x-y) \\ &= x - y \geq \sqrt[2n]{x} + \sqrt[2n]{-y} \\ &= \|T^{2n}x - T^{2n}y\|. \end{aligned}$$

Similarly, we can see that (4.2) holds. Therefore, all conditions of Theorem 4.6 are satisfied and hence there exists a unique point $(v^*, u^*) = (1, -1) \in A \times B$ such that

$$\|v^* - Tv^*\| = \|Tu^* - u^*\| = dist(A, B).$$

It is interesting to note that $T^{2n}(x) = \sqrt[2n]{x} \to v^* = 1$, for all $x \in A$ and $T^{2n}(y) = -\sqrt[2n]{-y} \to -1 = u^*$, for all $y \in B$.

Remark 4.8. In [8], Eldred et al. proved that every nonempty, bounded, closed and convex pair of subsets of a uniformly convex Banach space X has the proximal normal structure and, by using this geometric property, established the existence of a best proximity point for cyclic relatively nonexpansive mappings. In Theorem 4.6, we have proved directly the existence of a best proximity point for cyclic relatively nonexpansive mappings which are asymptotic pointwise contraction, and further our assumptions on T have enabled us to approximate the best proximity point of T.

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