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ON GENERALIZATION OF RICCERI'S THEOREM FOR FAN-TAKAHASHI MINIMAX INEQUALITY INTO SET-VALUED MAPS VIA SCALARIZATION

YUTAKA SAITO, TAMAKI TANAKA, AND SYUUJI YAMADA

Dedicated to Professor Sompong Dhompongsa on the occasion of his 65th birthday

Abstract. In the paper, we propose Ricceri's theorem on Fan-Takahashi minimax inequality for set-valued maps by using the scalarization method proposed by Kuwano, Tanaka and Yamada, and we give a characterization of a certain set which plays an important role in Ricceri's theorem.

1. Introduction

In functional analysis, nonlinear analysis, convex analysis as well as optimization, many inequality theorems related to minimality or maximality have been studied. The Fan-Takahashi minimax inequality theorem (see [1] in 1972 and [12] in 1976) is one of important results in the areas above with many applications to other mathematical areas. Then, in [10], Ricceri proposed a reasonable substitute of assumptions for the Fan-Takahashi minimax inequality for real-valued functions, that is, he showed the same conclusion on the inequality under the different assumption which contains a certain mutually exclusive condition to the assumption in [12].

On the other hand, Kuwano, Tanaka and Yamada in [7] proposed the Fan-Takahashi minimax inequality for set-valued maps. They use certain scalarization methods for set-valued maps, proposed in [6], based on set-relations in [4].

The aim of this paper is to generalize Ricceri's theorem for the Fan-Takahashi minimax inequality into its results for set-valued maps by a similar method to the approach above. By the modification with a certain scalarized target map, we contrive the proof of the main theorem.

The organization of this paper is as follows. In Section 2, we recall set-relations and scalarization methods for sets. In Section 3, we introduce Ricceri's theorem (Theorem 3.2) on the Fan-Takahashi minimax inequality for real-valued functions. Also, we show an outline of the proof of a key theorem (Theorem 3.7) proved by Ricceri in order to help the reader follow easily the proof of the main theorem (Theorem 4.1). In Section 4, we propose generalizations of the two Ricceri's theorems above for set-valued maps. Finally, we notice that a certain set which plays an important role in condition (3) of Theorem 3.2 coincides with the relative boundary of a given set.

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2. PRELIMINARIES

Throughout the paper, let E be a real topological vector space, V a linear subspace of E, D a non-empty subset of V, Y an ordered topological vector space, C an ordering cone in Y with $\operatorname{int} C \neq \emptyset$, θ_E (resp., θ_Y) the zero vector of E (resp., Y), $\mathcal{V}(x)$ the open neighborhood system of a point x and F a set-valued map from E into $2^Y \setminus \{\emptyset\}$.

Moreover, if S, T, U are three non-empty subsets of E, we put

$$I_{S,T,U} := \{x \in S \mid T \subseteq \cup_{\lambda>0} \lambda(x-U)\}.$$

Futhermore, we denote the algebraic sum and difference of any subsets A and B in Y by $A+B := \{a+b \mid a \in A, b \in B\}$ and $A-B := \{a-b \mid a \in A, b \in B\}$, respectively. Also, given $A \subset Y$, we write $tA := \{ta \mid a \in A\}$ for $t \in \mathbb{R}$ and $A+x := A+\{x\}$ for $x \in Y$.

At first, we introduce some set-relations by Kuroiwa, Tanaka and Ha.

Definition 2.1 (set-relation, [4])**.** For any nonempty sets A, $B \subset Y$, we write

$A \leq_C^{(1)} B$ by $A \subset \bigcap_{b \in B}(b - C)$, equivalently $B \subset \bigcap_{a \in A}(a + C)$;
$A \leq_C^{(2)} B$ by $A \cap \left(\bigcap_{b \in B}(b - C)\right) \neq \emptyset$;
$A \leq_C^{(3)} B$ by $B \subset (A + C)$;
$A \leq_C^{(4)} B$ by $\left(\bigcap_{a \in A}(a + C)\right) \cap B \neq \emptyset$;
$A \leq_C^{(5)} B$ by $A \subset (B - C)$;
$A \leq_C^{(6)} B$ by $A \cap (B - C) \neq \emptyset$, equivalently $(A + C) \cap B \neq \emptyset$.

Proposition 2.2 ([6])**.** *For any nonempty sets $A, B \subset Y$, the following statements hold.*

(i) *For each $j = 1, \ldots, 6$,*
$A \leq_C^{(j)} B$ implies $(A + y) \leq_C^{(j)} (B + y)$ for $y \in Y$, and
$A \leq_C^{(j)} B$ implies $\alpha A \leq_C^{(j)} \alpha B$ for $\alpha > 0$;

(ii) *For each $j = 1, \ldots, 5$, $\leq_C^{(j)}$ is transitive;*

(iii) *For each $j = 3, 5, 6$, $\leq_C^{(j)}$ is reflexive;*

(iv) *For each $j = 1, \ldots, 6$, $A \leq_C^{(j)} B$ and $y_1 \leq_C y_2$ for $y_1, y_2 \in Y$ imply $A + y_1 \leq_C^{(j)} B + y_2$.*

We recall some definitions of C–notions which are referred in [8]. A subset A in Y is said to be C–convex (resp., C–closed) if $A + C$ is convex (resp., closed); C–proper if $A + C \neq Y$. Moreover, A is said to be C–bounded if for each $U \in \mathcal{V}(\theta_Y)$ there exists $t \geq 0$ such that $A \subset tU + C$. Furthermore, we say that F is each C–notion mentioned above if the set $F(x)$ for each $x \in E$ has the property of the corresponding C–notion.

Definition 2.3 (type (j) C–convexity)**.** For each $j = 1, \ldots, 6$, a set-valued map F is called a *type (j) C–convex function* if for each $x, y \in E$ and $\lambda \in (0, 1)$,

$$F(\lambda x + (1-\lambda)y) \leq_C^{(j)} \lambda F(x) + (1-\lambda)F(y).$$

Definition 2.4 (type (j) C–concavity)**.** For each $j = 1, \dots, 6$, a set-valued map F is called a *type* (j) C*–concave function* if for each $x, y \in E$ and $\lambda \in (0, 1)$,

$$\lambda F(x) + (1 - \lambda)F(y) \leq_C^{(j)} F(\lambda x + (1 - \lambda)y).$$

Definition 2.5 (C–continuity, [8])**.**

(i) F is called a C*–lower continuous function* if for each $\bar{x} \in E$ and open set W with $F(\bar{x}) \cap W \neq \emptyset$, there exists $U \in \mathcal{V}(\bar{x})$ such that $F(y) \cap (W + C) \neq \emptyset$ for all $y \in U$.

(ii) F is called a C*–upper continuous function* if for each $\bar{x} \in E$ and open set W with $F(\bar{x}) \subset W$, there exists $U \in \mathcal{V}(\bar{x})$ such that $F(y) \subset W + C$ for all $y \in U$.

Next, we introduce the definition of two types of nonlinear scalarizing functions for sets proposed by [6].

Definition 2.6 (unified scalarization for sets, [6])**.** Let A and V' be nonempty subsets in Y and direction $k \in \operatorname{int} C$. For each $j = 1, \dots, 6$, we define scalarizing functions $I_{k,V'}^{(j)}$ and $S_{k,V'}^{(j)} : 2^Y \setminus \{\emptyset\} \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\begin{aligned} I_{k,V'}^{(j)}(A) &:= \inf \left\{ t \in \mathbb{R} \;\middle|\; A \leq_C^{(j)} (tk + V') \right\} \quad \text{and} \\ S_{k,V'}^{(j)}(A) &:= \sup \left\{ t \in \mathbb{R} \;\middle|\; (tk + V') \leq_C^{(j)} A \right\}, \end{aligned}$$

respectively. They are called unified scalarizing functions for sets.

Proposition 2.7 ([6])**.** *Let A, B and V' be nonempty subsets in Y and $k \in \operatorname{int} C$. Then, the following statements hold:*

(i) *For each $j = 1, \dots, 6$ and $\alpha \in \mathbb{R}$,*

$$\begin{aligned} I_{k,V'}^{(j)}(A + \alpha k) &= I_{k,V'}^{(j)}(A) + \alpha, \\ S_{k,V'}^{(j)}(A + \alpha k) &= S_{k,V'}^{(j)}(A) + \alpha; \end{aligned}$$

(ii) *For each $j = 1, \dots, 5$,*

$$A \leq_C^{(j)} B \quad \textit{implies} \quad I_{k,V'}^{(j)}(A) \leq I_{k,V'}^{(j)}(B) \quad \textit{and} \quad S_{k,V'}^{(j)}(A) \leq S_{k,V'}^{(j)}(B).$$

Proposition 2.8 ([7])**.** *Let A and V' be nonempty subsets in Y and $k \in \operatorname{int} C$. Then, the following statements hold:*

(i) *If A is C–bounded and V' is $(-C)$–bounded then $S_{k,V'}^{(1)}(A) \in \mathbb{R}$;*

(ii) *For each $j = 2, 3$, if A is C–bounded then $S_{k,V'}^{(j)}(A) > -\infty$. If V' is C–proper then $S_{k,V'}^{(j)}(A) < +\infty$;*

(iii) *For each $j = 4, 5$, if A is $(-C)$–proper then $S_{k,V'}^{(j)}(A) < +\infty$. If V' is $(-C)$–bounded then $S_{k,V'}^{(j)}(A) > -\infty$;*

(iv) *If A is $(-C)$–proper and V' is C–bounded then $S_{k,V'}^{(6)}(A) \in \mathbb{R}$.*

Proposition 2.9 ([5])**.** *For nonempty subsets A, B and V' in Y, $\lambda \in (0, 1)$ and $k \in \operatorname{int} C$, the following statements hold:*

(i) $I_{k,V'}^{(3)}(\lambda A + (1 - \lambda)B) \leq \lambda I_{k,V'}^{(3)}(A) + (1 - \lambda) I_{k,V'}^{(3)}(B)$*;*

(ii) *If V' is $(-C)$–convex then*
$I^{(5)}_{k,V'}(\lambda A + (1-\lambda)B) \le \lambda I^{(5)}_{k,V'}(A) + (1-\lambda) I^{(5)}_{k,V'}(B)$;

(iii) *If V' is C–convex then*
$\lambda S^{(3)}_{k,V'}(A) + (1-\lambda) S^{(3)}_{k,V'}(B) \le S^{(3)}_{k,V'}(\lambda A + (1-\lambda)B)$;

(iv) $\lambda S^{(5)}_{k,V'}(A) + (1-\lambda) S^{(5)}_{k,V'}(B) \le S^{(5)}_{k,V'}(\lambda A + (1-\lambda)B)$,

with the agreement that $-\infty + \infty = +\infty$.

Remark 2.10. The statements (i) and (iv) in Proposition 2.9 hold without any cone convexity for V'.

Proof. We prove statement (iv) only. Let $t_1 = S^{(5)}_{k,V'}(A)$, $t_2 = S^{(5)}_{k,V'}(B)$. In the three cases of (a) $t_1 = t_2 = -\infty$, (b) $t_1 = -\infty$ and $t_2 \in \mathbb{R}$, (c) $t_1 = \mathbb{R}$ and $t_2 \in -\infty$, (iv) is clearly true. Hence, we assume that $t_1, t_2 \in \mathbb{R}$. For any $\epsilon > 0$, we have

$$\begin{aligned} &(t_1 - \epsilon)k + V' \subset A - C \quad \text{and} \\ &(t_2 - \epsilon)k + V' \subset B - C. \end{aligned}$$

Then, we get $\{\lambda t_1 + (1-\lambda)t_2 - \epsilon\}k + \lambda V' + (1-\lambda)V' \subset \lambda A + (1-\lambda)B - \{\lambda C + (1-\lambda)C\}$. By $V' \subset \lambda V' + (1-\lambda)V'$ and $\lambda C + (1-\lambda)C = C$,

$$\{\lambda t_1 + (1-\lambda)t_2 - \epsilon\}k + V' \subset \lambda A + (1-\lambda)B - C.$$

Since ϵ is an arbitrary positive real number, we get

$$\lambda t_1 + (1-\lambda)t_2 \le S^{(5)}_{k,V'}(\lambda A + (1-\lambda)B).$$

Next, we assume that $t_1 = +\infty$, that is, $tk + V' \subset A - C$ for any $t \in \mathbb{R}$. We prove $S^{(5)}_{k,V'}(\lambda A + (1-\lambda)B) = +\infty$, that is,

$$tk + V' \subset \lambda A + (1-\lambda)B - C \text{ for any } t \in \mathbb{R}.$$

We have $uk + V' \in \lambda A - C$ for any $u \in \mathbb{R}$. Indeed, we take any $v \in V'$. Since $k \in \operatorname{int} C$, there exists $r_v > 0$ such that $k - r_v v \in C$, and hence $-\frac{1-\lambda}{r_v}k + (1-\lambda)v \in -C$. For this result and $tk + \lambda v \in \lambda A - C$ for any $t \in \mathbb{R}$, we get $sk + v \in \lambda A - C$ for any $s \in \mathbb{R}$. Moreover, for any $b \in B$, there exist $r_b > 0$ such that $-\frac{1-\lambda}{r_b}k - (1-\lambda)b \in -C$. Hence, we get $tk + V' \in \lambda A + (1-\lambda)b - C \subset \lambda A + (1-\lambda)B - C$ for any $t \in \mathbb{R}$. $\square$

For each $x \in E$ and $j = 1, \ldots, 6$, we consider the following composite functions:

$$(I^{(j)}_{k,V'} \circ F)(x) := I^{(j)}_{k,V'}(F(x));$$

$$(S^{(j)}_{k,V'} \circ F)(x) := S^{(j)}_{k,V'}(F(x)).$$

Then, we can get the following properties between a set-valued map F and the composite function $S^{(j)}_{k,V'} \circ F$.

Proposition 2.11 ([5])**.** *If F is type (5) C–concave, then for each fixed $(k, V') \in (\operatorname{int} C) \times (2^Y \setminus \{\emptyset\})$, $S^{(5)}_{k,V'} \circ F$ is concave on E.*

Proposition 2.12 ([11])**.** *For each fixed $(k, V') \in (\operatorname{int} C) \times (2^Y \setminus \{\emptyset\})$, the following statements hold:*

(i) *For each $j = 1, 2, 3$,*

  (a) *if F is $(-C)$–lower continuous on E then $I^{(j)}_{k,V'} \circ F$ is upper semicontinuous in E,*
  (b) *if F is C–upper continuous on E then $I^{(j)}_{k,V'} \circ F$ is lower semicontinuous in E;*

(ii) *For each $j = 4, 5, 6$,*
  (a) *if F is C–lower continuous on E then $S^{(j)}_{k,V'} \circ F$ is lower semicontinuous in E,*
  (b) *if F is $(-C)$–upper continuous on E then $S^{(j)}_{k,V'} \circ F$ is upper semicontinuous in E.*

3. Ricceri's theorems on the Fan-Takahashi minimax inequality

At first, we recall the following two theorems.

Theorem 3.1 (The Fan-Takahashi minimax inequality, [12])**.** *Let E be a real Hausdorff topological vector space, X a non-empty compact convex subset of E and f a real function on $X \times X$ satisfying the following conditions:*

(1) *for every $x \in X$, the function $f(x, \cdot)$ is concave in X;*
(2) *for every $y \in X$, the function $f(\cdot, y)$ is lower semicontinuous in X;*
(3) *for every $x \in X$ such that one has $f(x, x) \leq 0$.*

Then, there exists $\hat{x} \in X$ such that $f(\hat{x}, y) \leq 0$ for all $y \in X$.

Theorem 3.2 (Ricceri's theorem for the Fan-Takahashi minimax inequality, [10])**.** *Let E be a real topological vector space, X a non-empty compact convex subset of E, $\theta_E \in X$ and f a real function on $X \times E$ satisfying the following conditions:*

(1) *for every $x \in X$, the function $f(x, \cdot)$ is concave in E and $f(x, \theta_E) = 0$;*
(2) *for every $y \in E$, the function $f(\cdot, y)$ is lower semicontinuous in X;*
(3) *for every $x \in X$ such that $X \setminus \cup_{\lambda > 0} \lambda(x - X) \neq \emptyset$, one has $f(x, x) > 0$.*

Then, there exists $\hat{x} \in X$ such that $f(\hat{x}, y) \leq 0$ for all $y \in X$.

Ricceri proposed Theorem 3.2 which is a reasonable substitute of Theorem 3.1. Clearly, both the third conditions in the theorems cannot occur at the same time. However the two theorems have the same result, and so they are mutually exclusive. Also, a set-valued version of the Fan-Takahashi minimax inequality theorem is proposed by the scalarization method in [7]. On the other hand, a set-valued version of Ricceri's theorem hasn't been proposed by the same approach yet. To achieve it, we introduce Theorem 3.7, whose corollary is Theorem 3.2. The detailed proof is shown in [10] but we shall give an outline of the proof so that the reader can follow the proof of Theorem 4.1, which is the main result of the paper.

In this section, let $\mathcal{F}_D$ be the family of all finite-dimensional linear subspaces of V meeting D, $\mathcal{U}_D$ the collection of all families $\mathcal{F}$ of finite-dimensional linear subspaces of V meeting D such that $\mathcal{F}$ is directed by (set-theoretic) inclusion and $D \subset \cup_{S \in \mathcal{F}} S$. Moreover we define some symbols for certain classes of several functions as follows: M_V is the set of all real-valued functions on V; C_V is the set of all $\psi \in M_V$ such that $\psi(\theta_E) \leq 0$, the set $\psi^{-1}(]0, +\infty[)$ is convex and finitely open and

$$\cup_{\lambda \in]0,1[} \lambda \psi^{-1}(]0, +\infty[) \subset \psi^{-1}(]0, +\infty[);$$

$\hat{C}_V$ is the set of all concave real-valued functions ψ on V such that $\psi(\theta_E) = 0$ (clearly, $\hat{C}_V \subset C_V$); if $\Gamma \subset M_V$, $\overline{(\gamma)}_{\tau_D}$ is the closure of γ with respect to the topology τ_D; $\mathcal{A}_D$ is the family of all sets $\Gamma \subset M_V$ for which there exists $\psi \in \Gamma$ such that $\sup_{x\in D}\psi(x) \leq 0$; $\mathcal{G}_D$ is the family of all sets $\Gamma \subset M_V$ for which there exists $S \in \mathcal{F}_D$ such that $\sup_{x\in D\cap S}\psi(x) > 0$ for all $\psi \in \Gamma$;

$$\mathcal{K}_{D,M_V} := \{\Gamma \subset M_V \mid \Gamma - \psi \in \mathcal{A}_D \cup \mathcal{G}_D, \forall \psi \in M_V\}.$$

Definition 3.3 (D-regular, [10])**.** Let K be a non-empty subset of E and A an operator from K into M_V. A is called *D-regular* in K if one of the two following conditions is satisfied:

(i) $A(K) \in \mathcal{A}_D$ where $A(K) := \cup_{x\in K}\{A(x)\}$;
(ii) there exists $S_0 \in \mathcal{F}_D$ such that, for every $S \in \mathcal{F}_D$, with $S_0 \subset S$, one has $\sup_{y\in D\cap S}(A(x))(y) > 0$ for all $x \in K \cap S$.

For an operator A from X into M_V, we often consider it as two-variable real-valued function on $X \times V$. We write $A(x, y)$ by $(A(x))(y)$ for each $x \in X$ and $y \in V$.

Theorem 3.4 ([10])**.** *Let $\mathcal{F} \in \mathcal{U}_D$, with $V = \cup_{S\in\mathcal{F}}S$, and A an operator from $X \subset E$ into C_V. Moreover, for each $S \in \mathcal{F}$, let K_S and X_S be two non-empty subsets of $X \cap S$, with $K_S \subset X_S$, satisfying the following conditions:*

(1) *K_S is compact in S and X_S is convex and closed in S;*
(2) *for every $y \in X_S - X_S$, the set $\{x \in X_S \mid A(x, y) \leq 0\}$ is closed in S;*
(3) *for every $x \in X_S \setminus I_{K_S,D\cap S,X_S}$, one has $\sup_{y\in K_S} A(x, x - y) > 0$.*

Under such hypotheses, the following conclusions hold:

(i) *$\theta_{M_V} \in \overline{(A(\cup_{S\in\mathcal{F}}K_S))}_{\tau_D}$;*
(ii) *for every set $K \subset X$, with $\cup_{S\in\mathcal{F}}K_S \subset K$, such that the operator A is D-regular in K, one has $A(K) \in \mathcal{A}_D$;*
(iii) *if $\Gamma \in \mathcal{A}_D \cup \mathcal{G}_D$ and $A(\cup_{S\in\mathcal{F}}K_S) \subset \Gamma$, then $\Gamma \in \mathcal{A}_D$.*

Lemma 3.5 ([10])**.** *If $A(K) \in \mathcal{A}_D \cup \mathcal{G}_D$, then A is D-regular in K.*

Lemma 3.6 ([10])**.** *Let K be a compact topological space such that, for every $y \in D$, the function $A(\cdot, y)$ is lower semicontinuous. Then, $A(K) \in \mathcal{K}_{D,M_V}$.*

Theorem 3.7 ([10])**.** *Let E be a real topological vector space, X a non-empty finitely closed and convex subset of E, K a finitely compact subset of X with $\theta_E \in K$, $\tilde{\tau}$ a topology on K with respect to which K is compact, f a real-valued function on $X \times V$. We assume that f satisfies the following conditions:*

(1) *for every $x \in X$, the function $f(x, \cdot)$ is concave in V;*
(2) *the function $f(\cdot, y)$ is finitely lower semicontinuous in X for every $y \in (X - X) \cap V$, is $\tilde{\tau}$-lower semicontinuous in K for every $y \in D$, is finitely continuous in X and $\tilde{\tau}$-continuous in K for $y = \theta_E$.*

Then, for any convex real-valued function ψ on V with $\psi(\theta_E) = 0$ and

$$f(x, x) > f(x, \theta_E) + \psi(x) \qquad \text{for all} \quad x \in (X \cap V) \setminus I_{K,D,X},$$

there exists $\hat{x} \in K$ such that

$$f(\hat{x}, y) \leq f(\hat{x}, \theta_E) + \psi(y) \qquad \text{for all} \quad y \in D.$$

Proof. For each $S \in \mathcal{F}_D$, put $X_S = X \cap S$ and $K_S = K \cap S$. Of course, $X_S \setminus I_{K_S, D\cap S, X_S} \subset X \setminus I_{K,D,X}$. Now, we define $A : X \times V \to \mathbb{R}$ by putting

$$A(x, y) := f(x, y) - f(x, \theta_E) - \psi(y)$$

for all $x \in X$, $y \in V$. At first, we fix the first variable of A for any $x \in X$ as

$$A(x, \cdot) = f(x, \cdot) - f(x, \theta_E) - \psi(\cdot).$$

Then, the function $A(x, \cdot)$ is concave in V and $A(x, \theta_E) = 0$. Thus, $A(X, \cdot) \subset \hat{C}_V$, and hence, $A(X, \cdot) \subset C_V$.

Next, we fix the second variable of A for any $y \in (X - X) \cap V$. Thanks to condition (2), the function $A(\cdot, y)$ is finitely lower semicontinuous in X. This is condition (2) of Theorem 3.4 because a level set of lower semicontinous function is closed.

Since $\theta_E \in K$, $\sup_{y\in K_s} A(x, x - y) \geq A(x, x) > 0$ for all $x \in (X \cap V) \setminus I_{K,D,X}$. This is condition (3) of Theorem 3.4. Therefore, all hypotheses of Theorem 3.4 are satisfied.

On the other hand, by $\tilde{\tau}$-lower semicontinuity of $A(\cdot, y)$ for every $y \in D$ and Lemmas 3.5 and 3.6, the operator A is D-regular in K. Therefore, by conclusion (ii) of Theorem 3.4, there exists $\hat{x} \in K$ such that $\sup_{y\in D} A(\hat{x}, y) \leq 0$. □

4. Ricceri's theorem for set-valued maps

In this secton, we propose Ricceri's theorems for set-valued maps. After that, we replace the set $X \setminus I_{X,X,X}$ with $X \setminus \operatorname{ri} X$. These sets are same set with the assumptions of Theorem 3.2. In this section, we assume that $\operatorname{int} C \neq \emptyset$.

Theorem 4.1. *Let E be a real topological vector space, Y an ordered topological vector space with ordering cone C, X a non-empty finitely closed and convex subset of E, K a finitely compact subset of X with $\theta_E \in K$, $\tilde{\tau}$ a topology on K with respect to which K is compact, F a set-valued map from $X \times V$ to $2^Y \setminus \{\emptyset\}$. We assume that F satisfies the following conditions:*

1. *F is $(-C)$–proper;*
2. *for every $x \in X$, the map $F(x, \cdot)$ is type (5) C–concave in V and $F(x, \theta_E)$ is singleton;*
3. *the map $F(\cdot, y)$ is finitely C–lower continuous in X for every $y \in (X - X) \cap V$, is $\tilde{\tau}$-C–lower continuous in K for every $y \in D$, is finitely $(-C)$–lower continuous in X and $\tilde{\tau}$-$(-C)$–lower continuous in K for $y = \theta_E$.*

Then, for any C–convex vector-valued map ψ from V to Y with $\psi(\theta_E) = \theta_Y$ and

$$F(x, \theta_E) + \psi(x) \leq^{(5)}_{\operatorname{int} C} F(x, x) \qquad \text{for all} \quad x \in (X \cap V) \setminus I_{K,D,X},$$

there exists $\hat{x} \in K$ such that

$$F(\hat{x}, \theta_E) + \psi(y) \not\leq^{(5)}_{\operatorname{int} C} F(\hat{x}, y) \qquad \text{for all} \quad y \in D.$$

Proof. Let $V' := \{\theta_Y\}, k \in \operatorname{int} C$ be fixed. We consider the set-valued map B from $X \times V$ to $2^Y \setminus \{\emptyset\}$ defined by

$$B(x, y) := F(x, y) - F(x, \theta_E) - \psi(y).$$

Now, we consider that the composite function $S^{(5)}_{k,V'} \circ B$ corresponds to the function "A" in the proof of Theorem 3.7. Then, there exists $\hat{x} \in K$ such that $(S^{(5)}_{k,V'} \circ B)(\hat{x}, y) \leq 0$ for all $y \in D$ if $S^{(5)}_{k,V'} \circ B$ satisfies the following conditions:

(a) $S^{(5)}_{k,V'} \circ B(x, \cdot)$ is concave for any $x \in X$;

(b) $(S^{(5)}_{k,V'} \circ B)(x, \theta_E) = 0$;

(c) $S^{(5)}_{k,V'} \circ B(\cdot, y)$ is finitely lower semicontinuous in X for any $y \in (X - X) \cap V$;

(d) $S^{(5)}_{k,V'} \circ B(\cdot, y)$ is $\tilde{\tau}$-lower semicontinuous in K for any $y \in D$;

(e) $(S^{(5)}_{k,V'} \circ B)(x, x) > 0$ for all $x \in (X \cap V) \setminus I_{K,D,X}$.

We show each proof of the five statements above.

(a) By assumption (2) and the C–concavity of vector-valued function $-\psi$, it follows from (iv) of Proposition 2.2 that $B(x, \cdot)$ is type (5) C–concave in V. From Proposition 2.11, it follows that $S^{(5)}_{k,V'} \circ B(x, \cdot)$ is concave.

(b) Since $F(x, \theta_E)$ is singleton and $\psi(\theta_E) = \theta_Y$, we get $B(x, \theta_E) = \{\theta_Y\} = V'$. Clearly, $S^{(5)}_{k,V'}(V') = 0$ is always true.

(c) Let $y \in (X - X) \cap V$ be fixed. For each finite dimensional subspace S in E, we take $x \in X \cap S$ and an open subset W of Y with $(F(x, y) - F(x, \theta_E)) \cap W \neq \emptyset$. Hence, there exist $w_1 \in F(x, y)$ and $w_2 \in (-F(x, \theta_E))$ such that $w_1 + w_2 \in W$, and there exists U_{θ_Y} such that U_{θ_Y} is an open neighborhood of θ_Y and $2U_{\theta_Y} + w_1 + w_2 \subset W$ (see the first lemma in section 9 of [2]). We put $W_1 := U_{\theta_Y} + w_1$ and $W_2 := U_{\theta_Y} + w_2$. Both W_1 and W_2 are open, and they satisfy $w_1 \in (F(x, y) \cap W_1) \neq \emptyset$ and $w_2 \in (-F(x, \theta_E) \cap W_2) \neq \emptyset$, respectively. By the C–lower continuity of $F(\cdot, y)$ and $F(\cdot, \theta_E)$, there exist open neighborhoods $U^{(1)}_x$ and $U^{(2)}_x$ of x such that $F(z_1, y) \cap (W_1 + C) \neq \emptyset$ and $(-F(z_2, \theta_E)) \cap (W_2 + C) \neq \emptyset$ for any $z_1 \in U^{(1)}_x$ and $z_2 \in U^{(2)}_x$. We put $U_x := U^{(1)}_x \cap U^{(2)}_x$, then U_x is an open neighborhood of x and

$$(F(z, y) - F(z, \theta_E)) \cap (W_1 + W_2 + C) \neq \emptyset \qquad \text{for all } z \in U_x.$$

We know $(W_1 + W_2) \subseteq W$, so we obtain $F(\cdot, y) - F(\cdot, \theta_E)$ is finitely C–lower continuous. Thus, $B(\cdot, y)$ is (finitely) C–lower continuous. By (ii)-(a) of Proposition 2.12, $S^{(5)}_{k,V'} \circ B(\cdot, y)$ is (finitely) lower semicontinuous.

(d) It can be proved in a similar way to the proof of (c).

(e) For each $x \in (X \cap V) \setminus I_{K,D,X}$, by assumption and (i) of Proposition 2.2, we have

$$\{\theta_Y\} = V' \leq^{(5)}_{\operatorname{int} C} B(x, x).$$

Thus, $V' = \{\theta_Y\} \subset B(x, x) - \operatorname{int} C$. Since $B(x, x) - \operatorname{int} C$ is open, there exists $t > 0$ such that $(tk + V') \subset B(x, x) - \operatorname{int} C$, which implies that $0 < t \leq (S^{(5)}_{k,V'} \circ B)(x, x)$.

Therefore, in the same way as the proof of Theorem 3.7, we can see that there exists $\hat{x} \in K$ such that $(S^{(5)}_{k,V'} \circ B)(\hat{x}, y) \leq 0$ for all $y \in D$. By the definition of $S^{(5)}_{k,V'}$, for each $y \in D$ and $s > 0$,

$$\{\theta_Y\} \quad \not\subseteq \quad B(\hat{x}, y) - sk - C.$$

By $\cup_{s>0}(-sk - C) = -\mathrm{int}\, C$, we obtain

$$\{\theta_Y\} \quad \not\leq^{(5)}_{\mathrm{int}\, C} \quad B(\hat{x}, y).$$

Now, since $F(x, \theta_E)$ is singleton, by (i) of proposition 2.2, we obtain

$$F(\hat{x}, \theta_E) + \psi(y) \quad \not\leq^{(5)}_{\mathrm{int}\, C} \quad F(\hat{x}, y).$$

□

Corollary 4.2. *Let E be a real topological vector space, Y an ordered topological vector space with ordering cone C, X a non-empty compact convex subset of E, $\theta_E \in X$ and F a set-valued map from $X \times E$ to $2^Y \setminus \{\emptyset\}$ satisfying the following conditions:*

(1) *F is $(-C)$–proper;*
(2) *for every $x \in X$, $F(x, \cdot)$ is type (5) C–concave in E and $F(x, \theta_E) = \{\theta_Y\}$;*
(3) *for every $y \in E$, $F(\cdot, y)$ is C–lower continuous in X;*
(4) *for every $x \in X$ such that $X \setminus \cup_{\lambda>0}\lambda(x - X) \neq \emptyset$, one has $\{\theta_Y\} \leq^{(5)}_{\mathrm{int}\, C} F(x, x)$.*

Then, there exists $\hat{x} \in X$ such that $\{\theta_Y\} \not\leq^{(5)}_{\mathrm{int}\, C} F(\hat{x}, y)$ for all $y \in X$.

Proof. In Theorem 4.1, take $V = E$, $X = K = D$, $\hat{\tau}$ being the relativization to K of the given Hausdorff vector topology on E, $\psi(\cdot) = \theta_Y$ and then observe that $X \setminus I_{X,X,X}$ coincides with $\{x \in X \mid X \setminus \cup_{\lambda>0}\lambda(x - X) \neq \emptyset\}$. □

In the rest of the paper, we shall discuss the set $X' := \{x \in X \mid X \setminus \cup_{\lambda>0}\lambda(x - X) \neq \emptyset\}$, which is used in Theorem 3.2 and Corollary 4.2. We define the affine hull of X by $\mathrm{aff}\, X$ as

$$\mathrm{aff}\, X := \{ax + by \mid a, b \in \mathbb{R}, x, y \in X, a + b = 1\}$$

and the relative interior of X by $\mathrm{ri}\, X$ as

$$\mathrm{ri}\, X := \{x \in X \mid U \cap \mathrm{aff}\, X \subset X \text{ for some } U \in \mathcal{V}(x)\}.$$

Proposition 4.3 ([9]). *Let E be a topological vector space and X a nonempty convex subset of E. Then $x \in \mathrm{ri}\, X$ and $y \in X$ imply $\lambda x + (1 - \lambda)y \in \mathrm{ri}\, X$ for all $0 < \lambda \leq 1$.*

Proposition 4.4. *Let E be a topological vector space and X a convex subset of E. If $\theta_E \in \mathrm{ri}\, X$ then*

$$X' = X \setminus \mathrm{ri}\, X.$$

Proof. First, we prove $X' \subset X \setminus \mathrm{ri}\, X$. For every $x \in X'$, we know $x \in X$ and $X \setminus \cup_{\lambda>0}\lambda(x - X) \neq \emptyset$. The conclusion can be seen by the method of reduction to absurdity. We suppose that $x \in \mathrm{ri}\, X$. By the definition of relative interior, there exists $U \in \mathcal{V}(x)$ such that $U \cap \mathrm{aff}\, X \subset X$, that is, $(x - U) \cap \mathrm{aff}\, X \subset (x - X)$. Since $\theta_E \in \mathrm{int}\,(x - U)$, we get $\cup_{\lambda>0}\lambda(x - U) = E$. Moreover, since $\theta_E \in \mathrm{ri}\, X$, $\mathrm{aff}\, X$ is a subspace and hence $\cup_{\lambda>0}\lambda(\mathrm{aff}\, X) = \mathrm{aff}\, X$. Then,

$$X \subset \mathrm{aff}\, X \subset (\cup_{\lambda>0}\lambda(x - U)) \cap \mathrm{aff}\, X \subset \cup_{\lambda>0}\lambda(x - X).$$

Hence, we have $X \setminus \cup_{\lambda>0}\lambda(x - X) = \emptyset$. This is a contradiction to the assumption $x \in X'$. Therefore, $x \in X \setminus \mathrm{ri}\, X$, that is to say $X' \subset X \setminus \mathrm{ri}\, X$.

Next, we show $X' \supset X \setminus \mathrm{ri}\, X$. Let $x \in X \setminus \mathrm{ri}\, X$. If there exists $\lambda \in]0,1[$ such that $x + \lambda x \in X$ then

$$x = \frac{\lambda}{1+\lambda}\theta_E + \frac{1}{1+\lambda}(x+\lambda x) \in \mathrm{ri}\, X.$$

This is a contradiction to the assumption $x \in X \setminus \mathrm{ri}\, X$. Hence, $x + \lambda x \notin X$ for each $\lambda \in]0,1[$, that is, $-x \notin \lambda(x - X)$ for each $\lambda > 0$. Since $\theta_E \in \mathrm{ri}\, X$, there exists $\mu \in]0,1[$ such that $\mu(-x) \in X$. Then, we obtain

$$\begin{array}{rrcll} & \mu(-x) & \notin & \mu\lambda(x-X) & \text{for each } \lambda > 0, \\ \Rightarrow & \mu(-x) & \notin & \lambda(x-X) & \text{for each } \lambda > 0, \\ \Rightarrow & \mu(-x) & \notin & \cup_{\lambda>0}\lambda(x-X), & \\ \Rightarrow & \mu(-x) & \in & X \setminus \cup_{\lambda>0}\lambda(x-X). & \end{array}$$

Thus, $X \setminus \cup_{\lambda>0}\lambda(x - X) \neq \emptyset$. Therefore, we get $x \in X'$. □

Proposition 4.5. *Let E be a topological vector space, X a convex subset of E and $\theta_E \in X$. If there exists a real-valued function f on $X \times E$ such that:*

(1) *for each $x \in X$, $f(x, \theta_E) = 0$;*
(2) *for each $x \in X'$, $f(x, x) > 0$.*

Then $\theta_E \in \mathrm{ri}\, X$.

Proof. This conclusion can be seen by the method reduction to absurdity. We suppose that $\theta_E \in X \setminus \mathrm{ri}\, X$. By the convexity of X, there exists $d \in X$ such that $\lambda d \notin -X$ for any $\lambda \in]0,1[$, that is, $d \notin \cup_{\lambda>0}\lambda(\theta_E - X)$. Then $d \in X \setminus \cup_{\lambda>0}\lambda(\theta_E - X)$. So we have $X \setminus \cup_{\lambda>0}\lambda(\theta_E - X) \neq \emptyset$, that is, $\theta_E \in X'$. Since condition (2), $f(\theta_E, \theta_E) > 0$. This is a contradiction to condition (1), which implies that $\theta_E \in \mathrm{ri}\, X$. □

Proposition 4.6. *Let E be a topological vector space, Y an ordered topological vector space with ordering cone C, X a convex subset of E and $\theta_E \in X$. If there exists a set-valued map F from $X \times E$ to $2^Y \setminus \{\emptyset\}$ such that:*

(1) *for each $x \in X$, $F(x, \theta_E) \cap (C \setminus \{\theta_Y\}) = \emptyset$ and $\theta_Y \in F(x, \theta_E)$*
(2) *for each $x \in X'$, $\{\theta_Y\} \leq^{(5)}_{\mathrm{int}\, C} F(x, x)$.*

Then $\theta_E \in \mathrm{ri}\, X$.

Proof. Take $k \in \mathrm{int}\, C$. First, for each $x \in X$, we consider the value of $(S^{(5)}_{k,\{\theta_Y\}} \circ F)(x, \theta_E)$. It is not positive because $F(x, \theta_E) \cap (C \setminus \{\theta_Y\}) = \emptyset$. Since $0 \cdot k = \theta_Y \in F(x, \theta_E)$, $(S^{(5)}_{k,\{\theta_Y\}} \circ F)(x, \theta_E) = 0$. Next, for each $x \in X'$, we consider the value of $(S^{(5)}_{k,\{\theta_Y\}} \circ F)(x, x)$. By condition (2), open set $F(x, x) - \mathrm{int}\, C$ contains θ_Y. Hence, there exists an open neighborhood U of θ_Y such that $U \subset F(x, x) - \mathrm{int}\, C$. Also, there exists $\alpha > 0$ such that $\alpha k \in U$. Then, $\alpha k \in F(x, x) - C$. Thus, $(S^{(5)}_{k,\{\theta_Y\}} \circ F)(x, x) \geq \alpha > 0$. By Proposition 4.5, we get $\theta_E \in \mathrm{ri}\, X$. □

Remark 4.7. By Propositions 4.4 and 4.6, condition (4) of Corollary 4.2 can be replaced by

(4)' for every $x \in X \setminus \mathrm{ri}\, X$, one has $\{\theta_Y\} \leq^{(5)}_{\mathrm{int}\, C} F(x, x)$.

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Y. Saito
Graduate School of Science and Technology, Niigata University, Niigata 950–2181, Japan
E-mail address: ysaito@m.sc.niigata-u.ac.jp

T. Tanaka
Graduate School of Science and Technology, Niigata University, Niigata 950–2181, Japan
E-mail address: tamaki@math.sc.niigata-u.ac.jp

S. Yamada
Faculty of Science, Niigata University, Niigata 950–2181, Japan
E-mail address: yamada@math.sc.niigata-u.ac.jp