Journal of Nonlinear and Convex Analysis
Volume 16, Number 1, 2015, 95–107

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL


# ON SPLIT SYSTEM OF VARIATIONAL PROBLEMS

ZENN-TSUN YU AND LAI-JIU LIN

Abstract. In this paper, we study the convergence theorems of the following problems: the general system of split monotonic variational inclusion problems; the general system of split equilibrium problems; the split multiply equilibrium problems; the general system of split variational inequality problems; split bilevel equilibrium problem; the mathematical programming with fixed point, zero points and split systems of variational constraints and the quadratic programming with fixed point, zero points and split systems of variational inequalities constraints. We establish iteration processes and prove strong convergence theorems of these problems. Our result on split bilevel equilibrium problem improves recent result of Moudafi [9].

## 1. Introduction

The split feasibility problem (**SFP**) in finite dimensional Hilbert spaces was first introduced by Censor and Elfving [5] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. Since then, the split feasibility problem (**SFP**) has received much attention due to its applications in signal processing, image reconstruction, with particular progress in intensity-modulated radiation therapy, approximation theory, control theory, biomedical engineering, communications, and geophysics. For examples, one can refer to [2, 4, 5, 8, 11, 15].

Variational inequality theory has been studied quite extensively and has emerged as an essential tool in the study of a wide class of obstacle, free moving, equilibrium problem and optimization theory. Recently, Cai and Bu [3] considered the following systems of variational inequalities in the smooth Banach space X, which involves finding

$$\begin{cases} \text{Find } \bar{x} \in C,\ \bar{y} \in C \text{ such that} \\ \langle r\Upsilon_2\bar{x} + \bar{y} - \bar{x}, J(x-\bar{y})\rangle \geq 0, \\ \langle \lambda\Upsilon_1\bar{y} + \bar{x} - \bar{y}, J(x-\bar{x})\rangle \geq 0 \end{cases} \tag{1.1}$$

for all $x \in C$, where $r$ and $\lambda$ are two positive constants, $C$ is a nonempty closed convex subset of $X$, $\Upsilon_1, \Upsilon_2 : C \to X$ are two nonlinear mappings, $J$ is the normalized duality mappings. For the recent trends and developments as problem (1.1) and its special cases, one can see [13, 10] and references therein.

Let $C$ and $Q$ be nonempty closed convex subsets of real Hilbert spaces $H_1$ and $H_2$, respectively. For each $i = 1, 2$, let $\varepsilon_i > 0$, let $\Upsilon_i$ be a $\varepsilon_i-$inverse-strongly monotone mapping of $C$ into $H_1$, let $\delta > 0, \delta^{'} > 0$, let $B$ be a $\delta-$inverse-strongly monotone

2010 *Mathematics Subject Classification.* 47J20, 47J25, 47H05, 47H09, 65K05.

*Key words and phrases.* System of split monotonic variational inclusion problem, system of split equilibrium problem, the general system of split variational inequality problem, quadratic function programming.

mapping of $Q$ into $H_2$, let $B^{'}$ be a $\delta^{'}-$inverse-strongly monotone mapping of $Q$ into $H_2$. For each $i = 1, 2$, let $\Phi_i$ be a maximal monotone mapping on $H_1$ such that the domain of $\Phi_i$ is included in $C$. Let $G, G^{'}$ be maximal monotone mapping on $H_2$ such that the domain of $G, G^{'}$ are included in $Q$. Throughout this paper, we use these notations and assumptions unless specify otherwise.

We know that the equilibrium problem is to find $z \in C$ such that

$$\text{(EP)} \qquad h(z, y) \geq 0 \text{ for each } y \in C,$$

where $h : C \times C \to \mathbb{R}$ is a bifunction. This problem includes fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, minimax inequalities, and saddle point problems as special cases (see [1]). The solution set of equilibrium problem (**EP**) is denoted by $EP(C, h)$.

To the best of our knowledge, there is no result on the systems of split variational inequalities problem.

For $i = 1, 2$, let $f_i : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1)-(A4) and let $g_i : Q \times Q \to \mathbb{R}$ be a bifunction satisfying the conditions (A1)-(A4). Throughout this paper, we use these notions and assumptions unless specified otherwise.

Recently Moudafi [9] gave an iteration to find solution to the bilevel equilibrium problem: Find $\bar{x} \in C$ such that $\bar{x} \in EP(EP(C, f_1), f_2)$.

This problem contains many problems, while Moudafi [9] only proved a weak convergence theorem to the solution of his problem.

Motivated by the the above problems, in this paper, iterations are used to find solutions to the following problems.

(i) general system of split variational inclusion problem (GSSMVIP):

$$\text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in Fix(J_{\lambda}^{\Phi_1}(I - \lambda\Upsilon_1)J_r^{\Phi_2}(I - r\Upsilon_2)),$$

and

$$\bar{u} = A\bar{x} \in H_2 \text{ such that } \bar{u} \in Fix(J_{\sigma}^{G}(I - \sigma B)J_{\rho}^{G'}(I - \rho B')).$$

(ii) general system of split equilibrium problem(GSSEP): Find $\bar{x} \in H_1,\ \bar{y} \in H_1$ such that

$$\begin{cases} f_2(\bar{y}, x) + \frac{1}{r}\langle \bar{y} - x, \bar{x} - \bar{y}\rangle - \langle \bar{y} - x, \Upsilon_2\bar{x}\rangle \geq 0, \\ f_1(\bar{x}, x) + \frac{1}{\lambda}\langle \bar{x} - x, \bar{y} - \bar{x}\rangle - \langle \bar{x} - x, \Upsilon_1\bar{y}\rangle \geq 0 \end{cases}$$

for all $x \in C$, and $\bar{u} = A\bar{x} \in H_2,\ \bar{v} \in H_2$ such that

$$\begin{cases} g_2(\bar{v}, u) + \frac{1}{\rho}\langle \bar{v} - u, \bar{u} - \bar{v}\rangle - \langle \bar{v} - u, B^{'}\bar{u}\rangle \geq 0, \\ g_1(\bar{u}, u) + \frac{1}{\sigma}\langle \bar{u} - u, \bar{v} - \bar{u}\rangle - \langle \bar{u} - u, B\bar{v}\rangle \geq 0 \end{cases}$$

for all $u \in Q$.

(iii) split multiple equilibrium problem(SMEP): Find $\bar{x} \in H_1$, $\bar{u} = A\bar{x} \in H_2$ such that $\bar{x} \in EP(C, f_1) \bigcap EP(C, f_2)$, and $\bar{u} \in EP(Q, g_1) \bigcap EP(Q, g_2)$.

(iv) split bilevel equilibrium problem: Find $\bar{x} \in C, \bar{u} = A\bar{x}$ such that $\bar{x} \in EP(EP(C, f_1), f_2)$ and $\bar{u} \in EP(EP(Q, f_1), f_2)$.

(v) general system of split variational inequality problem (GSSVIP): Find $\bar{x} \in H_1$, $\bar{y} \in H_1$ such that

$$\begin{cases} \langle r\Upsilon_2\bar{x} + \bar{y} - \bar{x}, x - \bar{y}\rangle \geq 0, \\ \langle \lambda\Upsilon_1\bar{y} + \bar{x} - \bar{y}, x - \bar{x}\rangle \geq 0 \end{cases}$$

for all $x \in C$, and $\bar{u} = A\bar{x} \in H_2$, $\bar{v} \in H_2$ such that

$$\begin{cases} \langle \rho B'\bar{u} + \bar{v} - \bar{u}, u - \bar{v}\rangle \geq 0, \\ \langle \sigma B\bar{v} + \bar{u} - \bar{v}, u - \bar{u}\rangle \geq 0 \end{cases}$$

for all $u \in Q$.

In this paper, we apply the convergence theorem of the multiply sets split feasibility problem to study the strong convergence theorems of the above problems. We apply our results to mathematical programming and quadratic programming. Our result improve recent result of Moudafi [9].

## 2. Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of positive integers and let $\mathbb{R}$ be the set of real numbers, $H_1$ be a (real) Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $||\cdot||$, and $C$ be a nonempty closed convex subset of $H_1$. We denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ .

Let $T : C \to H_1$ be a mapping, and let $Fix(T) := \{x \in C : Tx = x\}$ denote the set of fixed points of $T$. A mapping $T : C \to H_1$ is called

(i) firmly nonexpansive if $||Tx - Ty||^2 \leq ||x - y||^2 - ||(I - T)x - (I - T)y||^2$ for every $x, y \in C$.

(ii) quasi-nonexpansive if $Fix(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$ for all $x \in C$, $p \in Fix(T)$.

(iii) strongly monotone if there exists $\bar{\gamma} > 0$ such that $\langle x-y, Tx-Ty\rangle \geq \bar{\gamma}\|x-y\|^2$ for all $x, y \in C$.

(iv) Lipschitzian continuous if there exists $L > 0$ such that $\|Tx-Ty\| \leq L\|x-y\|$ for all $x, y \in C$.

(v) $\alpha$-inverse-strongly monotone if $\langle x - y, Vx - Vy\rangle \geq \alpha\|Tx - Ty\|^2$ for all $x, y \in C$ and $\alpha > 0$.

(vi) demiclosed if $\{x_n\}$ is any sequence in $C$ such that $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$ and $x_n \rightharpoonup w \in C$, then $Tw = w$.

Let $B$ be a mapping of $H_1$ into $2^{H_1}$. The effective domain of $B$ is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping $B$ is said to be a monotone operator on $H_1$ if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in D(B), u \in Bx$, and $v \in By$. A monotone operator $B$ on $H_1$ is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on $H_1$. For a maximal monotone operator $B$ on $H_1$ and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1} : H_1 \to D(B)$, which is called the resolvent of $B$ for $r$, and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$.

A mapping $T : C \to C$ is said to be averaged if $T = (1 - \alpha)I + \alpha S$, where $\alpha \in (0, 1)$ and $S : C \to C$ is a nonexpansive mapping. In this case, we also say that $T$ is $\alpha$-averaged. A firmly nonexpansive mapping is $\frac{1}{2}$-averaged. The follows lemmas are needed in this paper.

**Lemma 2.1** ([6]). *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H_1$, and let $T : C \to C$ be a mapping. Then the following satisfied:*

(i) *$T$ is nonexpansive if and only if the complement $(I - T)$ is $1/2$-ism.*
(ii) *If $S$ is $\upsilon$-ism, then for $\gamma > 0$, $\gamma S$ is $\upsilon/\gamma$-ism.*
(iii) *$S$ is averaged if and only if the complement $I - S$ is $\upsilon$-ism for some $\upsilon > 1/2$.*
(iv) *If $S$ and $T$ are both averaged, then the product (composite) $ST$ is averaged.*
(v) *If the mappings $\{T_i\}_{i=1}^n$ are averaged and have a common fixed point, then $\bigcap_{i=1}^n Fix(T_i) = Fix(T_1 \cdots T_n)$.*

For solving the equilibrium problem, let us assume that the bifunction $g : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $g(x, x) = 0$ for each $x \in C$;
(A2) $g$ is monotone, i.e., $g(x, y) + g(y, x) \leq 0$ for any $x, y \in C$;
(A3) for each $x, y, z \in C$, $\limsup_{t \downarrow 0} g(tz + (1 - t)x, y) \leq g(x, y)$;
(A4) for each $x \in C$, the scalar function $y \to g(x, y)$ is convex and lower semicontinuous.

**Theorem 2.2** ([1]). *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H_1$. Let $g : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4). Then for each $r > 0$ and each $x \in H$, there exists $z \in C$ such that*

$$g(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$$

*for all $y \in C$.*

**Theorem 2.3** ([7]). *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H_1$ and let $g : C \times C \to \mathbb{R}$ be a function satisfying conditions (A1)–(A4). For $r > 0$, define $T_r^g : H_1 \to C$ by*

$$T_r^g x = \left\{ z \in C : g(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall \ y \in C \right\}$$

*for all $x \in H_1$. Then the following hold:*

(i) *$T_r^g$ is single-valued;*
(ii) *$T_r^g$ is firmly nonexpansive, that is, $||T_r^g x - T_r^g y||^2 \leq \langle x - y, T_r^g x - T_r^g y\rangle$ for all $x, y \in H_1$;*
(iii) *$\{x \in H_1 : \ T_r^g x = x\} = \{x \in C : \ g(x, y) \geq 0, \ \forall y \in C\}$;*
(iv) *$\{x \in C : \ g(x, y) \geq 0, \ \forall y \in C\}$ is a closed and convex subset of $C$.*

We call such $T_r^g$ the resolvent of $g$ for $r > 0$.

**Lemma 2.4** ([12]). *Let $C$ be a nonempty closed convex subset of a Hilbert space $H_1$ and let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1)-(A4). Define $A_g$ as follows:*

$$A_g x = \begin{cases} \{z \in H : g(x, y) \geq \langle y - x, z\rangle, \forall y \in C\}, \forall x \in C \\ \emptyset, \forall x \notin C. \end{cases} \tag{2.1}$$

*Then, $EP(g) = A_g^{-1}0$ and $A_g$ is a maximal monotone operator with the domain of $A_g \subset C$. Furthermore, for any $x \in H_1$ and $r > 0$, the resolvent $T_r^g$ of $g$ coincides with the resolvent of $A_g$, i.e., $T_r^g x = (I + rA_g)^{-1}x$.*

## 3. Convergence theorems of hierarchical problems

For each $i = 1, 2, 3$, let $H_i$ be a real Hilbert space, $G_i$ be a maximal monotone mapping on $H_1$ such that the domain of $G_i$ is included in $C$. Let $J_\lambda^{G_i} = (I + \lambda G_i)^{-1}$ for each $\lambda > 0$. Let $\{\theta_n\} \subset H_1$ be a sequence. Let $V$ be a $\bar{\gamma}-$ strongly monotone and $L-$ Lipschitzian continuous operator with $\bar{\gamma} > 0$ and $L > 0$. Let $T : C \to H_1$ be a quasi-nonexpansive mapping with demiclosed. Let $C$ and $Q$ be nonempty closed convex subsets of real Hilbert spaces $H_1$ and $H_2$, respectively. Let $F1 : H_2 \to H_2$ and $F_2 : H_3 \to H_3$ be firmly nonexpansive mappings. Let $A_1 : H_1 \to H_2$ and $A_2 : H_1 \to H_3$ be bounded linear operators. Let $A_1^*$ be the adjoint of $A_1$ and $A_2^*$ be the adjoint of $A_2$. Let $I : H_1 \to H_1$ be a identity mapping, and let $I_i : H_{i+1} \to H_{i+1}$ be a identity mapping for $i = 1, 2$. Throughout this paper, we use these notations and assumptions unless specify otherwise. In the following, we say that conditions (D) hold if

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim\limits_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^\infty \beta_n = \infty$;
(iii) $0 < a \le \lambda_n \le b < \frac{2}{\|A_1\|^2+2}$, and $0 < a \le r_n \le b < \frac{2}{\|A_2\|^2+2}$;
(iv) $\lim_{n\to\infty} \theta_n = \theta$ for some $\theta \in H$.

Now, we recall the following multiple sets split feasibility problem (**MSSFP – firmily**):

$$\text{Find } \bar{x} \in H_1 \text{ such that } A_1\bar{x} \in Fix(F_1) \text{ and } A_2\bar{x} \in Fix(F_2).$$

Let $\Omega$ is a solution of (**MSSFP – firmily**).

With the same proof as Theorem 3.3 in [14], we have the following theorem which is slightly different from Theorem 3.3 in [14] is an important tool in this paper.

**Theorem 3.1** ([14])**.** *Suppose that $\Delta =: Fix(T) \bigcap \Omega \bigcap Fix(J_{\lambda_n}^{G_1}) \bigcap Fix(J_{r_n}^{G_2}) \neq \emptyset$. A sequence $\{x_n\} \subset H$ is defined as follows: $x_1 \in C$ chosen arbitrarily and*

$$\begin{cases} y_n = J_{\lambda_n}^{G_1}(I - \lambda_n A_1^*(I_1 - F_1)A_1)J_{r_n}^{G_2}(I - r_n A_2^*(I_2 - F_2)A_2)x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n\theta_n + (I - \beta_n V)s_n) \end{cases} \tag{3.1}$$

*for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$, and $\{r_n\} \subset (0, \infty)$. Assume that conditions (D) hold.*

*Then $\lim\limits_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_\Delta(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also the unique solution to the following hierarchical problem: Find $\bar{x} \in \Delta$ such that*

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \ge 0 \text{ for all } q \in \Delta.$$

**Remark 3.2.** Theorem 3.3 [14] assumes that $F_2$ is a firmly nonexpansive on $H_2$ and $A_2 : H_1 \to H_2$ ia a bounded linear operator, but Theorem 3.1 assumes that $F_2$ is a firmly nonexpansive on $H_3$ and $A_2 : H_1 \to H_3$ is a bounded linear operator.

Now, we recall the following split fixed point problem
(**SFP − nonexpansive**):

$$\text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in Fix(\Psi) \text{ and } A_1\bar{x} \in Fix(\Psi_1).$$

where $\Psi_1$ is a nonexpansive mapping of $H_2$ into $H_2$ and $\Psi$ is a nonexpansive mapping of $H_1$ into $H_1$. Let $\Omega_1$ be a solution set of (**SFP − nonexpansive**).

**Theorem 3.3.** Let $\Psi_1$ be a nonexpansive mapping of $H_2$ into $H_2$, let $\Psi$ be a nonexpansive mapping of $H_1$ into $H_1$. Suppose that

$$\Delta_1 =: Fix(T)\bigcap \Omega_1 \bigcap Fix(J^{G_1}_{\lambda_n})\bigcap Fix(J^{G_2}_{r_n}) \neq \emptyset.$$

A sequence $\{x_n\} \subset H$ is defined as follows: $x_1 \in C$ chosen arbitrarily, and

$$\begin{cases} y_n = J^{G_1}_{\lambda_n}(I - \lambda_n A_1^*(I_1 - \Psi_1)A_1)J^{G_2}_{r_n}(I - r_n(I - \Psi))x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n\theta_n + (1 - \beta_n V)s_n), \end{cases} \tag{3.2}$$

for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$, and $\{r_n\} \subset (0, \infty)$. Assume that conditions (D) hold.

Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Delta_1}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also the unique solution to the following hierarchical problem: Find $\bar{x} \in \Delta_1$ such that

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \text{ for all } q \in \Delta_1.$$

*Proof.* Put $H_3 = H_1$, $A_2 = I$, $F_1 = \frac{I_1+\Psi_1}{2}$ and $F_2 = \frac{I+\Psi}{2}$ in Theorem 3.1. Since $\Psi_1$, $\Psi$ are nonexpansive mappings, it easy see that $F_1 = \frac{I_1+\Psi_1}{2} : H_2 \to H_2$ is a firmly nonexpansive mapping and $F_2 = \frac{I+\Psi}{2} : H_1 \to H_1$ is a firmly nonexpansive mapping. This implies that $I_1 - F_1 = \frac{I_1-\Psi_1}{2}$ and $I - F_2 = \frac{I-\Psi}{2}$. Then algorithm (3.1) in Theorem 3.1 follows immediately from algorithm (3.2) in Theorem 3.3.

Since $F_1 = \frac{I_1+\Psi_1}{2}$ and $F_2 = \frac{I+\Psi}{2}$, it easy see that $Fix(F_1) = Fix(\Psi_1)$ and $Fix(F_2) = Fix(\Psi)$. This implies that $\Omega = \Omega_1$. Since

$$\Delta_1 =: Fix(T)\bigcap \Omega_1 \bigcap Fix(J^{G_1}_{\lambda_n})\bigcap Fix(J^{G_2}_{r_n}) \neq \emptyset$$

and $\Omega = \Omega_1$, it follows that

$$\Delta =: Fix(T)\bigcap \Omega \bigcap Fix(J^{G_1}_{\lambda_n})\bigcap Fix(J^{G_2}_{r_n}) \neq \emptyset.$$

It follow from Theorem 3.1 that $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_\Delta(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the following hierarchical variational inequality:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \text{ for all } q \in \Delta.$$

Since $\Omega = \Omega_1$, we have $\Delta = \Delta_1$ and the proof is completed. □

## 4. Applications to general system of split monotonic variational inclusion problems

In this following theorem, an iteration is used to find solution to the problem (**GSSMVIP**). Let $GSSMVI(\Phi_1, \Phi_2, G, G')$ be the solution set of the problem (**GSSMVIP**).

**Theorem 4.1.** *Suppose that*

$$\Pi_1 =: Fix(T)\bigcap GSSMVI(\Phi_1,\Phi_2,G,G^{'})\bigcap Fix(J^{G_1}_{\lambda_n})\bigcap Fix(J^{G_2}_{r_n}) \neq \emptyset.$$

*A sequence* $\{x_n\}\subset H$ *is defined as follows:* $x_1 \in C$ *chosen arbitrarily and*

$$\begin{cases} y_n = J^{G_1}_{\lambda_n}(I-\lambda_n A_1^*(I_1-\Psi_1)A_1)J^{G_2}_{r_n}(I-r_n(I-\Psi))x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n\theta_n + (1-\beta_n V)s_n), \end{cases}$$

*where* $\Psi_1 = J^G_\sigma(I_1-\sigma B)J^{G'}_\rho(I_1-\rho B')$, $\Psi = J^{\Phi_1}_\lambda(I-\lambda\Upsilon_1)J^{\Phi_2}_r(I-r\Upsilon_2)$ *for each* $n\in\mathbb{N}$, $\{\lambda_n\}\subset(0,\infty)$, $\{\alpha_n\}\subset(0,1)$, $\{\beta_n\}\subset(0,1)$, *and* $\{r_n\}\subset(0,\infty)$. *Assume that*

(i) *conditions (D) hold;*

(ii) $0<\lambda<2\varepsilon_1,\ 0<r<2\varepsilon_2,\ 0<\sigma<2\delta,$ *and* $0<\rho<2\delta^{'}$.

*Then* $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\Pi_1}(\bar{x}-V\bar{x}+\theta)$. *This point* $\bar{x}$ *is also the unique solution to the following hierarchical problem: Find* $\bar{x}\in\Pi_1$ *such that*

$$\langle V\bar{x}-\theta, q-\bar{x}\rangle \geq 0 \text{ for all } q\in\Pi_1.$$

*Proof.* Since $\Upsilon_i$ is $\varepsilon_i-$ism, $0<\lambda<2\varepsilon_1,\ 0<r<2\varepsilon_2$ for each $i=1,2$, it follows from lemma 2.1(ii),(iii), we know that

$$(I-\lambda\Upsilon_1) \text{ and } (I-r\Upsilon_2) \text{ are averaged.} \tag{4.1}$$

On the other hand, since $J^{\Phi_1}_\lambda$ and $J^{\Phi_2}_r$ are firmly nonexpansive mappings,

$$J^{\Phi_1}_\lambda \text{ and } J^{\Phi_2}_r \text{ are } \frac{1}{2} \text{ averaged.} \tag{4.2}$$

By Lemma 2.1(iv), we see that

$$J^{\Phi_1}_\lambda(I-\lambda\Upsilon_1)J^{\Phi_2}_r(I-r\Upsilon_2) \text{ is averaged.} \tag{4.3}$$

This implies that,

$$\Psi = J^{\Phi_1}_\lambda(I-\lambda\Upsilon_1)J^{\Phi_2}_r(I-r\Upsilon_2) \text{ is a nonexpansive mapping of } H_1 \text{ into } H_1. \tag{4.4}$$

Following the same argument as (4.1), (4.2), (4.3) and (4.4), we know that

$$\Psi_1 = J^G_\sigma(I_1-\sigma B)J^{G'}_\rho(I_1-\rho B') \text{ is a nonexpansive mapping of } H_2 \text{ into } H_2.$$

Since $\Psi = J^{\Phi_1}_\lambda(I-\lambda\Upsilon_1)J^{\Phi_2}_r(I-r\Upsilon_2)$ and $\Psi_1 = J^G_\sigma(I_1-\sigma B)J^{G'}_\rho(I_1-\rho B')$, it is easy to see that $Fix(\Psi) = Fix(J^{\Phi_1}_\lambda(I-\lambda\Upsilon_1)J^{\Phi_2}_r(I-r\Upsilon_2))$ and $Fix(\Psi_1) = Fix(J^G_\sigma(I_1-\sigma B)J^{G'}_\rho(I_1-\rho B'))$. This implies that $GSSMVI(\Phi_1,\Phi_2,G,G^{'}) = \Omega_1$.

Since $\Pi_1 =: Fix(T)\bigcap GSSMVI(\Phi_1,\Phi_2,G,G^{'})\bigcap Fix(J^{G_1}_{\lambda_n})\bigcap Fix(J^{G_2}_{r_n}) \neq \emptyset$, it follows that

$$\Delta_1 = Fix(T)\bigcap \Omega_1 \bigcap Fix(J^{G_1}_{\lambda_n})\bigcap Fix(J^{G_2}_{r_n}) \neq \emptyset.$$

It follow from Theorem 3.3 that $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Delta_1}(\bar{x}-V\bar{x}+\theta)$. This point $\bar{x}$ is also the unique solution to the following hierarchical problem: Find $\bar{x}\in\Delta_1$ such that $\langle V\bar{x}-\theta, q-\bar{x}\rangle \geq 0$ for all $q\in\Delta_1$. Since $GSSMVI(\Phi_1,\Phi_2,G,G^{'}) = \Omega_1$, we have $\Pi_1 = \Delta_1$ and the proof is completed. □

In the following theorem, an iteration is used to find solution to the problem (**GSSEP**). Let $GSSEP(f_1, f_2, \Upsilon_1, \Upsilon_2, g_1, g_2, B, B^{'})$ be the solution set to the problem (**GSSEP**).

**Theorem 4.2.** *For each $i = 1, 2$, let $A_{f_i}, A_{g_i}$ be defined as (L4.1) in Lemma 2.4. Suppose that $\Pi_2 =: Fix(T) \bigcap GSSEP(f_1, f_2, \Upsilon_1, \Upsilon_2, g_1, g_2, B, B^{'}) \bigcap Fix(J_{\lambda_n}^{G_1}) \bigcap Fix(J_{r_n}^{G_2}) \neq \emptyset$. A sequence $\{x_n\} \subset H$ is defined as follows: $x_1 \in C$ chosen arbitrarily and*

$$\begin{cases} y_n = J_{\lambda_n}^{G_1}(I - \lambda_n A_1^*(I_1 - \Psi_1)A_1)J_{r_n}^{G_2}(I - r_n(I - \Psi))x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n \theta_n + (1 - \beta_n V)s_n), \end{cases}$$

*where $\Psi_1 = J_\sigma^{A_{g_1}}(I_1 - \sigma B)J_\rho^{A_{g_2}}(I_1 - \rho B')$, $\Psi = J_\lambda^{A_{f_1}}(I - \lambda\Upsilon_1)J_r^{A_{f_2}}(I - r\Upsilon_2)$ for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$, and $\{r_n\} \subset (0, \infty)$. Assume that*

(i) *conditions (D) hold;*

(ii) $0 < \lambda < 2\varepsilon_1,\ 0 < r < 2\varepsilon_2,\ 0 < \sigma < 2\delta,$ *and* $0 < \rho < 2\delta^{'}$.

*Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Pi_2}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also the unique solution to the following hierarchical problem: Find $\bar{x} \in \Pi_2$ such that*

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0 \text{ for all } q \in \Pi_2.$$

*Proof.* For each $i = 1, 2$, by Lemma 2.4, we know that $EP(f_i) = A_{f_i}^{-1}0$, $EP(g_i) = A_{g_i}^{-1}0$ , $A_{f_i}$ and $A_{g_i}$ are maximal monotone operators with the domain of $A_{f_i} \subset C$ and the domain of $A_{g_i} \subset Q$.

Put $G = A_{g_1}$, $G^{'} = A_{g_2}$, $\Phi_1 = A_{f_1}$ and $\Phi_2 = A_{f_2}$ in Theorem 4.2. Since $\Pi_2 \neq \emptyset$, there exist $\bar{x}_1 \in H_1,\ \bar{y}_1 \in H_1$ such that

$$\begin{cases} f_2(\bar{y}_1, x) + \frac{1}{r}\langle \bar{y}_1 - x, \bar{x}_1 - \bar{y}_1\rangle - \langle \bar{y}_1 - x, \Upsilon_2\bar{x}_1\rangle \geq 0, \\ f_1(\bar{x}_1, x) + \frac{1}{\lambda}\langle \bar{x}_1 - x, \bar{y}_1 - \bar{x}_1\rangle - \langle \bar{x}_1 - x, \Upsilon_1\bar{y}_1\rangle \geq 0 \end{cases}$$

for all $x \in C$, and $\bar{u}_1 = A\bar{x}_1 \in H_2,\ \bar{v}_1 \in H_2$ such that

$$\begin{cases} g_2(\bar{v}_1, u) + \frac{1}{\rho}\langle \bar{v}_1 - u, \bar{u}_1 - \bar{v}_1\rangle - \langle \bar{v}_1 - u, B^{'}\bar{u}_1\rangle \geq 0, \\ g_1(\bar{u}_1, u) + \frac{1}{\sigma}\langle \bar{u}_1 - u, \bar{v}_1 - \bar{u}_1\rangle - \langle \bar{u}_1 - u, B\bar{v}_1\rangle \geq 0 \end{cases}$$

for all $u \in Q$. Hence, there exist $\bar{x}_1 \in H_1,\ \bar{y}_1 \in H_1$ such that

$$\bar{x}_1 = J_\lambda^{A_{f_1}}(I - \lambda\Upsilon_1)\bar{y}_1, \bar{y}_1 = J_r^{A_{f_2}}(I - r\Upsilon_2)\bar{x}_1,$$

and $\bar{u}_1 = A\bar{x}_1 \in H_2,\ \bar{v}_1 \in H_2$ such that

$$\bar{u}_1 = J_\sigma^{A_{g_1}}(I_1 - \sigma B)\bar{v}_1, \bar{v}_1 = J_\rho^{A_{g_2}}(I_1 - \rho B')\bar{u}_1.$$

That is, there exist $\bar{x}_1 \in H_1$ such that

$$\bar{x}_1 \in Fix(J_\lambda^{A_{f_1}}(I - \lambda\Upsilon_1)J_r^{A_{f_2}}(I - r\Upsilon_2)),$$

and $\bar{u}_1 = A\bar{x}_1 \in H_2$ such that

$$\bar{u}_1 \in Fix(J_\sigma^{A_{g_1}}(I_1 - \sigma B)J_\rho^{A_{g_2}}(I_1 - \rho B')).$$

This implies that

$$GSSEP(f_1, f_2, \Upsilon_1, \Upsilon_2, g_1, g_2, B, B^{'}) = GSSMVI(\Phi_1, \Phi_2, G, G^{'})$$

and

$$\Pi_1 =: Fix(T) \bigcap GSSMVI(\Phi_1, \Phi_2, G, G^{'}) \bigcap Fix(J_{\lambda_n}^{G_1}) \bigcap Fix(J_{r_n}^{G_2}) \neq \emptyset.$$

It follow from Theorem 4.1 that $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Pi_1}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also the unique solution to the following hierarchical variational inequality: $\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0$ for all $q \in \Pi_1$.

Since $GSSEP(f_1, f_2, \Upsilon_1, \Upsilon_2, g_1, g_2, B, B^{'}) = GSSMVI(\Phi_1, \Phi_2, G, G^{'})$, it follows that $\Pi_1 = \Pi_2$ and the proof is completed. □

In the following theorem, an iteration is used to find solution to the problem (**SMEP**). Let $SMEP(f_1, f_2, g_1, g_2)$ be the solution set of the problem (**SMEP**).

**Theorem 4.3.** *For each $i = 1, 2$, let $A_{f_i}, A_{g_i}$ be defined as (L4.1) in Lemma 2.4. Suppose that*

$$\Pi_3 =: Fix(T) \bigcap SMEP(f_1, f_2, g_1, g_2) \bigcap Fix(J_{\lambda_n}^{G_1}) \bigcap Fix(J_{r_n}^{G_2}) \neq \emptyset.$$

*a sequence $\{x_n\} \subset H$ is defined as follows:$x_1 \in C$ chosen arbitrarily,and*

$$\begin{cases} y_n = J_{\lambda_n}^{G_1}(I - \lambda_n A_1^*(I_1 - \Psi_1)A_1)J_{r_n}^{G_2}(I - r_n(I - \Psi))x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n\theta_n + (1 - \beta_n V)s_n), \end{cases}$$

*where $\Psi_1 = J_\sigma^{A_{g_1}} J_\rho^{A_{g_2}}$, $\Psi = J_\lambda^{A_{f_1}} J_r^{A_{f_2}}$ for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$, and $\{r_n\} \subset (0, \infty)$. Assume that conditions (D) hold.*

*Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Pi_3}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also the unique solution to the following hierarchical problem: Find $\bar{x} \in \Pi_3$ such that*

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0 \textit{ for all } q \in \Pi_3.$$

*Proof.* By assumption , $\Pi_3 \neq \emptyset$. This implies hat

$$\Pi_2 =: Fix(T) \bigcap GSSEP(f_1, f_2, 0, 0, g_1, g_2, 0, 0) \bigcap Fix(J_{\lambda_n}^{G_1}) \bigcap Fix(J_{r_n}^{G_2}) \neq \emptyset.$$

It follow from Theorem 4.2 that $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Pi_2}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also the unique solution to the following hierarchical variational inequality: $\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0$ for all $q \in \Pi_2$.

On the other hand, by assumption, there exists $w \in \Pi_3$ and $w \in SMEP(f_1, f_2, g_1, g_2)$. This implies that

$$(4.5) \quad w \in Fix(J_\lambda^{A_{f_1}}) \bigcap Fix(J_r^{A_{f_2}}) \neq \emptyset \text{ and } A_1 w \in Fix(J_\sigma^{A_{g_1}}) \bigcap Fix(J_\rho^{A_{g_2}}) \neq \emptyset.$$

Since $J_\lambda^{A_{f_1}}$ and $J_r^{A_{f_2}}$ are $\frac{1}{2}$−averaged, by (4.5) and Lemma 2.1(v), we have that $Fix(J_\lambda^{A_{f_1}} J_r^{A_{f_2}}) = Fix(J_\lambda^{A_{f_1}}) \bigcap Fix(J_r^{A_{f_2}})$ and $Fix(J_\sigma^{A_{g_1}} J_\rho^{A_{g_2}}) = Fix(J_\sigma^{A_{g_1}}) \bigcap Fix(J_\rho^{A_{g_2}})$. That is, $SMEP(f_1, f_2, g_1, g_2) = SSEP(f_1, f_2, g_1, g_2)$ and therefore, $\Pi_3 = \Pi_2$. The proof is completed. □

By Theorem 4.3, we study a strong convergence convergence to the solution of split bilevel equilibrium problem.

**Theorem 4.4.** *Under the assumptions of Theorem 4.3, then there exists* $\bar{x} \in C$, $\bar{u} = A\bar{x}$ *such that* $\bar{x} \in EP(EP(C, f_1), f_2)$ *and* $\bar{u} \in EP(EP(Q, g_1), g_2)$.

*Proof.* By Theorem 4.3, there exists $\bar{x} \in C$, $\bar{u} = A\bar{x} \in H_2$ such that $\bar{x} \in EP(C, f_1) \bigcap EP(C, f_2)$, and $\bar{u} \in EP(Q, g_1) \bigcap EP(Q, g_2)$. Therefore,

$$\bar{x} \in EP(C, f_1) \subset C \text{ and } f_2(\bar{x}, y) \geq 0 \text{ for all } y \in C.$$

This shows that

$$\bar{x} \in EP(C, f_1) \subset C \text{ and } f_2(\bar{x}, y) \geq 0 \text{ for all } y \in EP(C, f_1).$$

Hence, $\bar{x} \in EP(EP(C, f_1), f_2)$. Similarly, we can show that $\bar{u} \in EP(EP(Q, g_1), g_2)$ and the proof is completed. □

**Remark 4.5.** Mondafi [9] gave an iteration to find the solution to the bilevel equilibrium problem, he proved a weak convergence theorem of this problem, while Theorem 4.4 is a strong convergence theorem for split bilevel equilibrium problem.

In the following theorem, an iteration is used to find solution to the problem (**GSSVIP**).

Let $GSSVI(\Upsilon_1, \Upsilon_2, B, B')$ be the solution set of the problem (**GSSVIP**).

**Theorem 4.6.** *Suppose that*

$$\Pi_5 =: Fix(T) \bigcap GSSVI(\Upsilon_1, \Upsilon_2, B, B') \bigcap Fix(J^{G_1}_{\lambda_n}) \bigcap Fix(J^{G_2}_{r_n}) \neq \emptyset.$$

*A sequence* $\{x_n\} \subset H$ *is defined as follows:*$x_1 \in C$ *chosen arbitrarily, and*

$$\begin{cases} y_n = J^{G_1}_{\lambda_n}(I - \lambda_n A_1^*(I_1 - \Psi_1)A_1) J^{G_2}_{r_n}(I - r_n(I - \Psi))x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n \theta_n + (1 - \beta_n V)s_n), \end{cases}$$

*where* $\Psi_1 = P_Q(I_1 - \sigma B)P_Q(I_1 - \rho B')$, $\Psi = P_C(I - \lambda\Upsilon_1)P_C(I - r\Upsilon_2)$ *for each* $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$, *and* $\{r_n\} \subset (0, \infty)$. *Assume that*

(i) *conditions (D) hold;*
(ii) $0 < \lambda < 2\varepsilon_1$, $0 < r < 2\varepsilon_2$, $0 < \sigma < 2\delta$, *and* $0 < \rho < 2\delta'$;

*Then* $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\Pi_5}(\bar{x} - V\bar{x} + \theta)$. *This point* $\bar{x}$ *is also the unique solution to the following hierarchical problem: Find* $\bar{x} \in \Pi_5$ *such that*

$$\langle V\bar{x} - \theta, q - \bar{x} \rangle \geq 0 \text{ for all } q \in \Pi_5.$$

*Proof.* For $i = 1, 2$, let $f_i : C \times C \to \mathbb{R}$ be a bifunction defined by $f_i(x, y) = 0, \forall x, y \in C$, then $f_i$ satisfies the conditions (A1)-(A4). Let $g_i : Q \times Q \to \mathbb{R}$ be a bifunction defined by $g_i(x, y) = 0, \forall x, y \in Q$, then $g_i$ satisfies the conditions (A1)-(A4). It easy see that $J^{A_{g_i}}_\sigma = P_Q$ and $J^{A_{f_i}}_\sigma = P_C$ for $i = 1, 2$. Then Theorem 4.6 follows immediately from Theorem 4.2. □

By Theorem 4.1, we obtain that mathematical programming with fixed point, zero points and the general system of split monotonic variational inclusion problem (**GSSMVIP**) constraints.

**Theorem 4.7.** *In Theorem 4.1, let $h : C \to \mathbb{R}$ be a convex Gâteaux differential function with Gâteaux derivative $V$. Let*

$$\Pi_1 =: Fix(T) \bigcap GSSMVI(\Phi_1, \Phi_2, G, G^{'}) \bigcap Fix(J_{\lambda_n}^{G_1}) \bigcap Fix(J_{r_n}^{G_2}) \neq \emptyset.$$

*Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Pi_1}(\bar{x} - V\bar{x})$. This point $\bar{x}$ is also the unique solution to the mathematical programming with fixed point, zero points and multiple sets split feasibility constraints:*

$$\min_{q\in\Pi_1} h(q).$$

*Proof.* Apply Theorem 4.1 and argue as in the proof of Theorem 4.1 in [14], we can prove Theorem 4.7. □

**Corollary 4.8.** *In Theorem 4.1, replace $x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n\theta_n + (I-\beta_n V)s_n)$ in algorithm (4.1) by $x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n\theta_n + (1-\beta_n)s_n)$. Let*

$$\Pi =: Fix(T) \bigcap GSSMVI(\Phi_1, \Phi_2, G, G^{'}) \bigcap Fix(J_{\lambda_n}^{G_1}) \bigcap Fix(J_{r_n}^{G_2}) \neq \emptyset.$$

*Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Pi_1}(0)$. This point $\bar{x}$ is also the unique solution to the mathematical programming with fixed point, zero points and multiple sets split feasibility constraints:* $\min_{q\in\Pi_1} \|q\|$.

*Proof.* Let $h(x) = \frac{1}{2}\|x\|^2$, and let $V$ be the Gâteaux derivative of $h$. It is easy to see $V(x) = x$ for each $x \in H$. Then Corollary 4.8 follows immediately from Theorem 4.7. □

We can apply Theorem 4.7 to study the mathematical programming of quadratic function with fixed point, zero points and the general system of split monotonic variational inclusion problem (**GSSMVIP**) constraints.

**Theorem 4.9.** *In Theorem 4.1, let $V_1 : C \to C$ be a strongly positive self adjoint bounded linear operator and $a \in H$ . Let*

$$\Pi_1 =: Fix(T) \bigcap GSSMVI(\Phi_1, \Phi_2, G, G^{'}) \bigcap Fix(J_{\lambda_n}^{G_1}) \bigcap Fix(J_{r_n}^{G_2}) \neq \emptyset.$$

*Let $\{x_n\} \subset H$ be defined by*

$$\text{(4.6)} \qquad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = J_{\lambda_n}^{G_1}(I - \lambda_n A_1^*(I_1 - \Psi_1)A_1)J_{r_n}^{G_2}(I - r_n(I - \Psi))x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n\theta_n + (s_n - \beta_n(V_1(s_n) - a))), \end{cases}$$

*where $\Psi_1 = J_\sigma^G(I_1 - \sigma B)J_\rho^{G'}(I_1 - \rho B')$, $\Psi = J_\lambda^{\Phi_1}(I - \lambda\Upsilon_1)J_r^{\Phi_2}(I - r\Upsilon_2)$ for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, and $\{r_n\} \subset (0,\infty)$. Assume that*

(i) *conditions (D) hold;*

(ii) $0 < \lambda < 2\varepsilon_1,\ 0 < r < 2\varepsilon_2,\ 0 < \sigma < 2\delta,$ *and* $0 < \rho < 2\delta^{'}$;

*Then* $\lim_{n\to\infty} x_n = \bar{x}$*, This point* $\bar{x}$ *is also the unique solution to the mathematical programming of quadratic function with fixed point, zero points and multiple sets split feasibility constraints:*

$$\min_{q\in\Pi} \frac{1}{2}\langle V_1 q, q\rangle - \langle a, q\rangle.$$

*Proof.* Apply Theorem 4.7 and argue as in the proof of Theorem 4.2 in [14], we can prove Theorem 4.9. □

## References


[1] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Mathematics student **63** (1994), 123–146.

[2] C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Problems **18** (2002), 441–453.

[3] G. Cai and S. Q. Bu, *Convergence analysis for variational inequalities problems and fixed point problems in 2-uniformly smooth and uniformly convex Banach spaces*, Math Comput.Modeling, **55** (2012), 538–546.

[4] Y. Censor, T. Bortfeld and B. Martin, *A. Trofimov, A unified approach for inversion problems in intensitymodulated radiation therapy*, Phys. Med. Biol. **51** (2003), 2353–2365.

[5] Y. Censor and T. Elfving, *A multiprojection algorithm using Bregman projection in a product space*, J. Numer. Algorithm **8** (1994), 221–239.

[6] P. L. Combettes, *Solving monotone inclusions via compositions of nonexpansive averaged operators*, Optimization **53** (2004), 475–504.

[7] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), 117–136.

[8] G. López, V. Martín-Márquez and H. K. Xu, *Iterative algorithms for the multiple sets split feasibility problem Biomedical Mathematics*, in Promising Directions in Imaging, Therapy Planning and Inverse Problems, Y Censor, M. Jiang, and G. Wang (eds.), Madison, WI: Medical Physics Publishing, 2010, pp. 243–279.

[9] A. Moudafi, *Proximal methods for a class of bilevel monotone equilibrium problems*, J.Global Optimization **47** (2010), 289–292.

[10] N. H. Nie, Z. Liu, K. H. Kim and S. M. Kang, *A system of nonlinear variational inequalities involving strong monotone and pseudocontractive mappings*, Adv. Nonlinear Var. Inequal. **6** (2003), 91–99.

[11] H. Stark, *Image Recovery: Theory and Applications*, Academic Press, Orlando, 1987.

[12] S. Takahashi, W. Takahashi and M. Toyoda, *Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces*, J. Optim. Theory Appl. **147** (2010), 27–41.

[13] R. U. Verma, *Projection methods, algorithms, and a new system of nonlinear variational inequalities*, Comput. Math. Appl. **41** (2001), 1025–1031.

[14] Z. T. Yu and L. J. Lin, *Hierarchical problems with applications to mathematical programming with multiple sets split feasibility constraints*, Fixed Point Theory Appl., 2013:238 (2013).

[15] Z. T. Yu, L. J. Lin and C. S. Chuang, *A unified study of the split feasible problems With applications*, J. Nonlinear Convex Anal. **15** (2014), 604–622.


*Manuscript received December 3, 2013*
*revised May 5, 2014*

Z. T. Yu
Department of Electronic Engineering, Nan Kai University of Technology, Nantou 542, Taiwan
*E-mail address*: t106@nkut.edu.tw

L. J. Lin
Department of Mathematics, National Changhua University of Education, Changhua, 50058, Taiwan
*E-mail address*: maljlin@cc.ncue.edu.tw