

STRONG CONVERGENCE THEOREMS FOR COMMUTATIVE FAMILIES OF LINEAR CONTRACTIVE OPERATORS IN BANACH SPACES

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ABSTRACT. In this paper, we study nonlinear analytic methods for linear contractive operators in Banach spaces. Using these results, we obtain some new strong convergence theorems for commutative families of linear contractive operators in Banach spaces. In the results, the limit points are characterized by sunny generalized nonexpansive retractions.

1. INTRODUCTION

Let E be a real Banach space and let C be a closed convex subset of E . For a mapping $T : C \rightarrow C$, we denote by $F(T)$ the set of fixed points of T . A mapping $T : C \rightarrow C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. In particular, a nonexpansive mapping $T : E \rightarrow E$ is called contractive if it is linear, that is, a linear contractive mapping $T : E \rightarrow E$ is a linear operator satisfying $\|T\| \leq 1$. From [33] we know a weak convergence theorem by Mann's iteration for nonexpansive mappings in a Hilbert space: Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let $T : C \rightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ such that

$$\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty.$$

Then, $\{x_n\}$ converges weakly to an element z of $F(T)$, where $z = \lim_{n \rightarrow \infty} Px_n$ and P is the metric projection of H onto $F(T)$. By Reich [26], such a theorem was extended to a uniformly convex Banach space with a Fréchet differentiable norm. However, we have not known whether the fixed point z is characterized under any projections in a Banach space. Recently, using nonlinear analytic methods obtained

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by [16], [17] and [11], Takahashi and Yao [37] solved such a problem for positively homogeneous nonexpansive mappings in a uniformly convex Banach space. Furthermore, Takahashi, Wong and Yao [35] extended such a result for commutative families of positively homogeneous nonexpansive mappings in a uniformly convex Banach space. In 1938, Yosida [39] also proved the following mean ergodic theorem for linear bounded operators: Let E be a real Banach space and let T be a linear operator of E into itself such that there exists a constant C with $\|T^n\| \leq C$ for $n \in \mathbb{N}$, and T is weakly completely continuous, i.e., T maps the closed unit ball of E into a weakly compact subset of E . Then, for each $x \in E$, the Cesàro means

$$S_n x = \frac{1}{n} \sum_{k=1}^n T^k x$$

converge strongly as $n \rightarrow \infty$ to a fixed point of T ; see also Kido and Takahashi [19]. Such a mean convergence theorem was also discussed in Takahashi, Wong and Yao [36] for commutative families of positively homogeneous nonexpansive mappings in a uniformly convex Banach space; see also Takahashi, Wong and Yao [34].

In this paper, motivated by these theorems, we study nonlinear analytic methods for linear contractive operators in Banach spaces and obtain some new strong convergence theorems for commutative families of linear contractive operators in Banach spaces. For example, we extend Bauschke, Deutsch, Hundal and Park's theorem [4] for linear contractive operators in Hilbert spaces to commutative families of linear contractive operators in Banach spaces. In our results, the limit points are characterized by sunny generalized nonexpansive retractions.

2. PRELIMINARIES

Throughout this paper, we assume that a Banach space E with the dual space E^* is real. We denote by \mathbb{N} and \mathbb{R} the sets of all positive integers and all real numbers, respectively. We also denote by $\langle x, x^* \rangle$ the dual pair of $x \in E$ and $x^* \in E^*$. A Banach space E is said to be strictly convex if $\|x + y\| < 2$ for $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $x \neq y$. A Banach space E is said to be smooth provided

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in E$ with $\|x\| = \|y\| = 1$. Let E be a Banach space. With each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

The multivalued operator $J : E \rightarrow E^*$ is called the normalized duality mapping of E . From the Hahn-Banach theorem, $Jx \neq \emptyset$ for each $x \in E$. We know that E is smooth if and only if J is single-valued. If E is strictly convex, then J is one-to-one, i.e., $x \neq y \Rightarrow J(x) \cap J(y) = \emptyset$. If E is reflexive, then J is a mapping of E onto E^* . So, if E is reflexive, strictly convex and smooth, then J is single-valued, one-to-one and onto. In this case, the normalized duality mapping J_* from E^* into E is the inverse of J , that is, $J_* = J^{-1}$; see [30] for more details. Let E be a smooth Banach space and let J be the normalized duality mapping of E . We define the function

$\phi : E \times E \rightarrow \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$.

It is easy to see that $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$ for all $x, y \in E$. Thus, in particular, $\phi(x, y) \geq 0$ for all $x, y \in E$. We also know the following:

$$(2.1) \quad \phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$$

for all $x, y, z \in E$. Further, we have

$$(2.2) \quad 2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w)$$

for all $x, y, z, w \in E$. If E is additionally assumed to be strictly convex, then

$$(2.3) \quad \phi(x, y) = 0 \Leftrightarrow x = y.$$

For a smooth, strictly convex and reflexive Banach space E , we also define the function $\phi_* : E^* \times E^* \rightarrow \mathbb{R}$ by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle x^*, J^{-1}y^* \rangle + \|y^*\|^2$$

for all $x^*, y^* \in E^*$. It is easy to see that

$$(2.4) \quad \phi(x, y) = \phi_*(Jy, Jx)$$

for all $x, y \in E$. Let E be a Banach space and let K be a closed convex cone of E . Then, $T : K \rightarrow K$ is called a positively homogeneous mapping if $T(\alpha x) = \alpha Tx$ for all $\alpha \geq 0$ and $x \in K$. Let M be a closed linear subspace of E . Then, $S : M \rightarrow M$ is called a homogeneous mapping if $T(\beta x) = \beta Tx$ for all $\beta \in \mathbb{R}$ and $x \in M$. Let E be a smooth Banach space and let C be a nonempty subset of E . A mapping $T : C \rightarrow C$ is called *generalized nonexpansive* [13] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y), \quad \forall x \in C, y \in F(T).$$

The following theorem was proved by Takahashi, Yao and Honda [38].

Theorem 2.1 (Takahashi, Yao and Honda [38]). *Let E be a smooth Banach space and let K be a closed convex cone of E . Then, a positively homogeneous mapping $T : K \rightarrow K$ is generalized nonexpansive if and only if for any $x \in K$ and $u \in F(T)$,*

$$\|Tx\| \leq \|x\| \text{ and } \langle x - Tx, Ju \rangle \leq 0.$$

Furthermore, let M be a closed linear subspace of E . Then, a homogeneous mapping $S : M \rightarrow M$ is generalized nonexpansive if and only if for any $x \in M$ and $v \in F(T)$,

$$\|Sx\| \leq \|x\| \text{ and } \langle x - Sx, Jv \rangle = 0.$$

We also know the following theorem from Takahashi and Yao [37]; see also Honda, Takahashi and Yao [11].

Theorem 2.2 (Takahashi and Yao [37]). *Let E be a smooth Banach space and let K be a closed convex cone in E . If $T : K \rightarrow K$ is a positively homogeneous nonexpansive mapping, then T is generalized nonexpansive. In particular, if $T : E \rightarrow E$ is a linear contractive mapping, then T is generalized nonexpansive.*

From Theorems 2.2 and 2.1, we have the following corollary.

Corollary 2.3. *Let E be a smooth Banach space and let K be a closed convex cone of E . If a mapping $T : K \rightarrow K$ is positively homogeneous nonexpansive, then for any $x \in K$ and $u \in F(T)$,*

$$\|Tx\| \leq \|x\| \text{ and } \langle x - Tx, Ju \rangle \leq 0.$$

Furthermore, let M be a closed linear subspace of E . If a mapping $S : M \rightarrow M$ is homogeneous nonexpansive, then for any $x \in M$ and $v \in F(T)$,

$$\|Sx\| \leq \|x\| \text{ and } \langle x - Sx, Jv \rangle = 0.$$

From Theorem 2.1, Takahashi, Yao and Honda [38] introduced the following concept.

Definition 2.4. Let E be a smooth Banach space, let $x \in E$ and let F be a nonempty subset of E . The Sizihwan region between x and F is the set

$$R(x; F) = \{z \in E : \langle x - z, Ju \rangle = 0 \text{ for all } u \in F \text{ and } \|z\| \leq \|x\|\}.$$

The following result is in Takahashi, Yao and Honda [38].

Lemma 2.5. *Let E be a strictly convex and smooth Banach space, let $x \in E$ and let F be a nonempty subset of E . Then $R(x; F)$ is nonempty, closed, convex and bounded, and $F \cap R(x; F)$ consists of at most one point.*

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E . For an arbitrary point x of E , the set

$$\{z \in C : \phi(z, x) = \min_{y \in C} \phi(y, x)\}$$

is always a singleton. Let us define the mapping Π_C of E onto C by $z = \Pi_C x$ for every $x \in E$, i.e.,

$$\phi(\Pi_C x, x) = \min_{y \in C} \phi(y, x)$$

for every $x \in E$. Such Π_C is called the generalized projection of E onto C ; see Alber [1]. The following lemma is due to Alber [1] and Kamimura and Takahashi [18].

Lemma 2.6 ([1, 18]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $(x, z) \in E \times C$. Then, the following hold:*

- (a) $z = \Pi_C x$ if and only if $\langle y - z, Jx - Jz \rangle \leq 0$ for all $y \in C$;
- (b) $\phi(z, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(z, x)$.

Let D be a nonempty closed subset of a smooth Banach space E , let T be a mapping from D into itself and let $F(T)$ be the set of fixed points of T . Then, T is said to be generalized nonexpansive [13] if $F(T)$ is nonempty and $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in D$ and $u \in F(T)$. Let C be a nonempty subset of E and let R be a mapping from E onto C . Then R is said to be a retraction, or a projection if $Rx = x$ for all $x \in C$. It is known that if a mapping P of E into E satisfies $P^2 = P$, then P is a projection of E onto $\{Px : x \in E\}$. A mapping $T : E \rightarrow E$ with $F(T) \neq \emptyset$ is a retraction if and only if $F(T) = R(T)$, where $R(T)$ is the range of T . The mapping R is also said to be sunny if $R(Rx + t(x - Rx)) = Rx$ whenever $x \in E$ and

$t \geq 0$. A nonempty subset C of a smooth Banach space E is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto C . The following lemmas were proved by Ibaraki and Takahashi [13].

Lemma 2.7 ([13]). *Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E and let R be a retraction from E onto C . Then, the following are equivalent:*

- (a) R is sunny and generalized nonexpansive;
- (b) $\langle x - Rx, Jy - JRx \rangle \leq 0$ for all $(x, y) \in E \times C$.

Lemma 2.8 ([13]). *Let C be a nonempty closed sunny and generalized nonexpansive retract of a smooth and strictly convex Banach space E . Then, the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.9 ([13]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then, the following hold:*

- (a) $z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$;
- (b) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

The following theorems were proved by Kohsaka and Takahashi [21].

Theorem 2.10 ([21]). *Let E be a smooth, strictly convex and reflexive Banach space, let C^* be a nonempty closed convex subset of E^* and let Π_{C^*} be the generalized projection of E^* onto C^* . Then the mapping R defined by $R = J^{-1}\Pi_{C^*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C^*$.*

Theorem 2.11 ([21]). *Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty subset of E . Then, the following are equivalent:*

- (1) D is a sunny generalized nonexpansive retract of E ;
- (2) D is a generalized nonexpansive retract of E ;
- (3) JD is closed and convex.

In this case, D is closed.

Let E be a smooth, strictly convex and reflexive Banach space, let J be the normalized duality mapping from E onto E^* and let C be a closed subset of E such that JC is closed and convex. Then, we can define a unique sunny generalized nonexpansive retraction R_C of E onto C as follows:

$$R_C = J^{-1}\Pi_{JC}J,$$

where Π_{JC} is the generalized projection from E^* onto JC .

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E . For an arbitrary point x of E , the set

$$\{z \in C : \|z - x\| = \min_{y \in C} \|y - x\|\}$$

is always nonempty and a singleton. Let us define the mapping P_C of E onto C by $z = P_C x$ for every $x \in E$, i.e.,

$$\|P_C x - x\| = \min_{y \in C} \|y - x\|$$

for every $x \in E$. Such P_C is called the metric projection of E onto C ; see [30]. The following lemma is in [30].

Lemma 2.12 ([30]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $(x, z) \in E \times C$. Then, $z = P_C x$ if and only if $\langle y - z, J(x - z) \rangle \leq 0$ for all $y \in C$.*

An operator $A \subset E \times E^*$ with domain $D(A) = \{x \in E : Ax \neq \emptyset\}$ and range $R(A) = \cup\{Ax : x \in D(A)\}$ is said to be monotone if $\langle x - y, x^* - y^* \rangle \geq 0$ for any $(x, x^*), (y, y^*) \in A$. An operator A is said to be strictly monotone if $\langle x - y, x^* - y^* \rangle > 0$ for any $(x, x^*), (y, y^*) \in A$ ($x \neq y$). Let J be the normalized duality mapping from E into E^* . Then, J is monotone. If E is strictly convex, then J is one-to-one and strictly monotone; for instance, see [30].

3. SEMITOPOLOGICAL SEMIGROUPS AND INVARIANT MEANS

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from S to S are continuous. In the case when S is commutative, we denote st by $s + t$. A commutative semigroup S with identity is a directed system when the binary relation is defined by $s \preceq t$ if and only if $\{t\} \cup (S + t) \subset \{s\} \cup (S + s)$. Let $B(S)$ be the Banach space of all bounded real valued functions on S with supremum norm and let $C(S)$ be the subspace of $B(S)$ of all bounded real valued continuous functions on S . Let μ be an element of $C(S)^*$ (the dual space of $C(S)$). We denote by $\mu(f)$ the value of μ at $f \in C(S)$. Sometimes, we denote by $\mu_t(f(t))$ or $\mu_t f(t)$ the value $\mu(f)$. For each $s \in S$ and $f \in C(S)$, we define two functions $l_s f$ and $r_s f$ as follows:

$$(l_s f)(t) = f(st) \quad \text{and} \quad (r_s f)(t) = f(ts)$$

for all $t \in S$. An element μ of $C(S)^*$ is called a *mean* on $C(S)$ if $\mu(e) = \|\mu\| = 1$, where $e(s) = 1$ for all $s \in S$. We know that $\mu \in C(S)^*$ is a mean on $C(S)$ if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s), \quad \forall f \in C(S).$$

A mean μ on $C(S)$ is called *left invariant* if $\mu(l_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. Similarly, a mean μ on $C(S)$ is called *right invariant* if $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. A left and right invariant mean on $C(S)$ is called an *invariant* mean on $C(S)$. The following theorem is in [30, Theorem 1.4.5].

Theorem 3.1 ([30]). *Let S be a commutative semitopological semigroup. Then there exists an invariant mean on $C(S)$, i.e., there exists an element $\mu \in C(S)^*$ such that $\mu(e) = \|\mu\| = 1$ and $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$.*

Theorem 3.2 ([30]). *Let S be a semitopological semigroup. Let μ be a right invariant mean on $C(S)$. Then*

$$\sup_s \inf_t f(ts) \leq \mu(f) \leq \inf_s \sup_t f(ts), \quad \forall f \in C(S).$$

Similarly, let μ be a left invariant mean on $C(S)$. Then

$$\sup_s \inf_t f(st) \leq \mu(f) \leq \inf_s \sup_t f(st), \quad \forall f \in C(S).$$

Let S be a semitopological semigroup. For any $f \in C(S)$ and $c \in \mathbb{R}$, we write

$$f(s) \rightarrow c, \quad \text{as } s \rightarrow \infty_R$$

if for each $\varepsilon > 0$ there exists an $\omega \in S$ such that

$$|f(t\omega) - c| < \varepsilon, \quad \forall t \in S.$$

We denote $f(s) \rightarrow c$, as $s \rightarrow \infty_R$ by

$$\lim_{s \rightarrow \infty_R} f(s) = c.$$

When S is commutative, we also denote $s \rightarrow \infty_R$ by $s \rightarrow \infty$.

Theorem 3.3 ([30]). *Let $f \in C(S)$ and $c \in \mathbb{R}$. If*

$$f(s) \rightarrow c, \quad \text{as } s \rightarrow \infty_R,$$

then $\mu(f) = c$ for all right invariant mean μ on $C(S)$.

Theorem 3.4 ([30]). *If $f \in C(S)$ fulfills*

$$f(ts) \leq f(s), \quad \forall t, s \in S,$$

then

$$f(t) \rightarrow \inf_{w \in S} f(w), \quad \text{as } t \rightarrow \infty_R.$$

Theorem 3.5 ([30]). *Let S be a commutative semitopological semigroup and let $f \in C(S)$. Then the following are equivalent:*

- (i) $f(s) \rightarrow c$, as $s \rightarrow \infty$;
- (ii) $\sup_w \inf_t f(t+w) = \inf_w \sup_t f(t+w) = c$.

Let E be a Banach space and let C be a nonempty, closed and convex subset of E . Let S be a semitopological semigroup and let $\mathcal{S} = \{T_s : s \in S\}$ be a family of nonexpansive mappings of C into itself. Then $\mathcal{S} = \{T_s : s \in S\}$ is called a *continuous representation* of S as nonexpansive mappings on C if $T_{st} = T_s T_t$ for all $s, t \in S$ and $s \mapsto T_s x$ is continuous for each $x \in C$. The following definition [28] is crucial in the nonlinear ergodic theory of abstract semigroups. Let S be a topological space and let $C(S)$ be the Banach space of all bounded real valued continuous functions on S with supremum norm. Let E be a reflexive Banach space. Let $u : S \rightarrow E$ be a continuous function such that $\{u(s) : s \in S\}$ is bounded and let μ be a mean on $C(S)$. Then there exists a unique element z_0 of E such that

$$\mu_s \langle u(s), x^* \rangle = \langle z_0, x^* \rangle, \quad \forall x^* \in E^*.$$

We call such z_0 the *mean vector* of u for μ and denote by $\tau(\mu)u$, i.e., $\tau(\mu)u = z_0$. In particular, if $\mathcal{S} = \{T_s : s \in S\}$ is a continuous representation of S as nonexpansive mappings on C and $u(s) = T_s x$ for all $s \in S$, then there exists $z_0 \in C$ such that

$$\mu_s \langle T_s x, x^* \rangle = \langle z_0, x^* \rangle, \quad \forall x^* \in E^*.$$

We denote such z_0 by $T_\mu x$.

4. STRONG CONVERGENCE THEOREMS

Let Y be a nonempty subset of a Banach space E and let Y^* be a nonempty subset of the dual space E^* . Then, we can define the *annihilator* Y_\perp^* of Y^* and the *annihilator* Y^\perp of Y as follows:

$$Y_\perp^* = \{x \in E : f(x) = 0 \text{ for all } f \in Y^*\}$$

and

$$Y^\perp = \{f \in E^* : f(x) = 0 \text{ for all } x \in Y\}.$$

We know the following result from Megginson [25].

Lemma 4.1 ([25]). *Let A be a nonempty subset of E . Then*

$$(A^\perp)_\perp = \overline{\text{span}}A,$$

where $\overline{\text{span}}A$ is the smallest closed linear subspace of E containing A .

Let $T : E \rightarrow E$ be a bounded linear operator. Then, the *adjoint* mapping $T^* : E^* \rightarrow E^*$ is defined as follows:

$$\langle x, T^* x^* \rangle = \langle Tx, x^* \rangle$$

for any $x \in E$ and $x^* \in E^*$. We know that T^* is also a bounded linear operator and $\|T\| = \|T^*\|$. If S and T are bounded linear operators from E into itself and $\alpha \in \mathbb{R}$, then $(S + T)^* = S^* + T^*$ and $(\alpha S)^* = \alpha(S)^*$. Let I be the identity operator on E . Then, I^* is the identity operator on E^* . Let $T^{**} : E^{**} \rightarrow E^{**}$ be the adjoint of T^* . Then we have $T^{**}(\pi(E)) \subset \pi(E)$ and $\pi^{-1}T^{**}\pi = T$, where π is the natural embedding from E into its second dual space E^{**} ; see [25].

Lemma 4.2. *Let S be a commutative semitopological semigroup with identity. Let E be a strictly convex, smooth and reflexive Banach space, let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as linear contractive operators of E into itself and let $F(\mathcal{S})$ be the set of common fixed points of T_s , $s \in S$. Then $JF(\mathcal{S})$ is a closed linear subspace of E^* and $JF(\mathcal{S}) = \bigcap_{s \in S} F(T_s^*) = \{z - T_s z : z \in E, s \in S\}^\perp$, where $J : E \rightarrow E^*$ is the duality mapping and T_s^* is the adjoint operator of T_s .*

Proof. From Corollary 2.3, we have

$$\langle x - T_s x, Ju \rangle = 0$$

for all $x \in E$, $s \in S$ and $u \in F(\mathcal{S})$. We also have that

$$\begin{aligned} \langle x - T_s x, Ju \rangle = 0 &\Leftrightarrow \langle x, Ju \rangle = \langle T_s x, Ju \rangle \\ &\Leftrightarrow \langle x, Ju \rangle = \langle x, T_s^* Ju \rangle \\ &\Leftrightarrow \langle x, (I^* - T_s^*) Ju \rangle = 0, \end{aligned}$$

where I^* is the identity operator in E^* . Since this equation holds for all $x \in E$, we have $(I^* - T_s^*)Ju = 0$ and hence $T_s^*Ju = Ju$. Then $JF(\mathcal{S}) \subset F(T_s^*)$ for all $s \in S$. This implies that $JF(\mathcal{S}) \subset \bigcap_{s \in S} F(T_s^*)$. Since $\|T_s^*\| = \|T_s\| \leq 1$, we can get the same fact about T_s^* . So, we obtain that

$$J_* \bigcap_{s \in S} F(T_s^*) \subset \bigcap_{s \in S} F(T_s^{**}),$$

where $J_* : E^* \rightarrow E^{**}$ is the duality mapping of E^* . Under assumptions on E , we know that $J_* = J^{-1}$ and $T_s^{**} = T_s$. Then, we have

$$\bigcap_{s \in S} F(T_s^*) \subset J \bigcap_{s \in S} F(T_s^{**}) = J \bigcap_{s \in S} F(T_s) = JF(\mathcal{S}).$$

So, we obtain that $\bigcap_{s \in S} F(T_s^*) = JF(\mathcal{S})$ and hence $JF(\mathcal{S})$ is a closed linear subspace of E^* . Finally, we show that $\bigcap_{s \in S} F(T_s^*) = \{z - T_s z : z \in E, s \in S\}^\perp$. Let $S_s = I - T_s$ for all $s \in S$, where $I : E \rightarrow E$ is the identity operator on E . If $x^* \in \{z \in E^* : S_s^* z = 0 : s \in S\}$, then we have

$$\langle S_s y, x^* \rangle = \langle y, S_s^* x^* \rangle = 0$$

for all $y \in E$ and $s \in S$. This implies $x^* \in \{z - T_s z : z \in E, s \in S\}^\perp$. We know that $S_s^* = I^* - T_s^*$ and $\{z \in E^* : S_s^* z = 0 : s \in S\} = \bigcap_{s \in S} F(T_s^*)$. So, we have $\bigcap_{s \in S} F(T_s^*) \subset \{z - T_s z : z \in E, s \in S\}^\perp$. On the other hand, if $x^* \in \{z - T_s z : z \in E, s \in S\}^\perp$, then we have $\langle S_s y, x^* \rangle = 0$ for all $y \in E$ and $s \in S$. Since

$$\langle y, S_s^* x^* \rangle = \langle S_s y, x^* \rangle = 0$$

for all $y \in E$ and $s \in S$, we have $S_s^* x^* = 0$ and hence $x^* \in F(T_s^*)$ for all $s \in S$. This implies $\{z - T_s z : z \in E, s \in S\}^\perp \subset \bigcap_{s \in S} F(T_s^*)$. Then, we have

$$\bigcap_{s \in S} F(T_s^*) = \{z - T_s z : z \in E, s \in S\}^\perp.$$

This completes the proof. □

Theorem 4.3. *Let S be a commutative semitopological semigroup with identity. Let E be a strictly convex, smooth and reflexive Banach space, let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as linear contractive operators of E into itself and let $\{S_\alpha : \alpha \in I\}$ be a net of contractive linear operators of E into itself such that $F(\mathcal{S}) \subset F(S_\alpha)$ for all $\alpha \in I$. Suppose $T_s \circ S_\alpha = S_\alpha \circ T_s$ for all $\alpha \in I$ and $s \in S$. Then, the following are equivalent:*

- (1) $\{S_\alpha x\}$ converges to an element of $F(\mathcal{S})$ for all $x \in E$;
- (2) $\{S_\alpha x\}$ converges to 0 for all $x \in (JF(\mathcal{S}))_\perp$;
- (3) $\{S_\alpha x - T_s \circ S_\alpha x\}$ converges to 0 for all $x \in E$ and $s \in S$.

Furthermore, if (1) holds, then $\{S_\alpha x\}$ converges to $R_{F(\mathcal{S})}x \in F(\mathcal{S})$, where $R_{F(\mathcal{S})} = J^{-1}\Pi_{JF(\mathcal{S})}J$ and $\Pi_{JF(\mathcal{S})}$ is the generalized projection of E^* onto $JF(\mathcal{S})$.

Proof. Suppose (1). Then, for any $x \in E$, $S_\alpha x \in R(x; F(S_\alpha)) \subset R(x; F(\mathcal{S}))$ for all $\alpha \in I$. We know from Lemma 2.5 that $R(x; F(\mathcal{S})) \cap F(\mathcal{S})$ consists of at most one point. Since $R(x; F(\mathcal{S}))$ is closed and $\{S_\alpha x\}$ converges strongly to an element z of $F(\mathcal{S})$, we have $R(x; F(\mathcal{S})) \cap F(\mathcal{S}) = \{z\}$. Let Rx be the unique element z of $R(x; F(\mathcal{S})) \cap F(\mathcal{S})$. Then, a mapping $R : E \rightarrow F(\mathcal{S})$ defined by $z = Rx$ is a retraction of E onto $F(\mathcal{S})$. Furthermore, we know from Corollary 2.3 that $\langle x - S_\alpha x, Ju \rangle = 0$ for all $u \in F(S_\alpha)$ and $\alpha \in I$. So, we have

$$(4.1) \quad \langle x - Rx, Ju \rangle = 0, \quad \forall u \in F(\mathcal{S})$$

From $Rx \in F(\mathcal{S})$, we also have $\langle x - Rx, JRx \rangle = 0$ and thus

$$(4.2) \quad \langle x - Rx, JRx - Ju \rangle = 0$$

for all $u \in F(\mathcal{S})$. So, from Lemmas 2.7 and 2.8, R is the unique sunny generalized nonexpansive retraction of E onto $F(\mathcal{S})$. Therefore, from Theorem 2.10, we have

$$R = R_{F(\mathcal{S})} = J^{-1}\Pi_{JF(\mathcal{S})}J,$$

where $\Pi_{JF(\mathcal{S})}$ is the generalized projection of E^* onto $JF(\mathcal{S})$. If $x \in (JF(\mathcal{S}))_{\perp}$, then we have $\langle x, Ju \rangle = 0$ for all $u \in F(\mathcal{S})$. From (4.1), we also have $\langle x - Rx, Ju \rangle = 0$ for all $u \in F(\mathcal{S})$. So, we get $\langle Rx, Ju \rangle = 0$ for all $u \in F(\mathcal{S})$. This implies $Rx \in (JF(\mathcal{S}))_{\perp}$. From $Rx \in F(\mathcal{S}) \cap (JF(\mathcal{S}))_{\perp}$ and $F(\mathcal{S}) \cap (JF(\mathcal{S}))_{\perp} = \{0\}$, we have that $S_{\alpha}x \rightarrow R_{F(\mathcal{S})}x = 0$. Then, we obtain (2).

Suppose (2). From Lemma 4.2, $JF(\mathcal{S})$ is a closed linear subspace of E^* . Then, we have from [2, 3, 10, 9] that for any $x \in E$,

$$x = R_{F(\mathcal{S})}x + P_{(JF(\mathcal{S}))_{\perp}}x,$$

where $P_{(JF(\mathcal{S}))_{\perp}}$ is the metric projection of E onto $(JF(\mathcal{S}))_{\perp}$. So, we have from (2) that

$$\begin{aligned} S_{\alpha}x &= S_{\alpha}(R_{F(\mathcal{S})}x + P_{(JF(\mathcal{S}))_{\perp}}x) \\ &= S_{\alpha}R_{F(\mathcal{S})}x + S_{\alpha}P_{(JF(\mathcal{S}))_{\perp}}x \\ &= R_{F(\mathcal{S})}x + S_{\alpha}P_{(JF(\mathcal{S}))_{\perp}}x \\ &\rightarrow R_{F(\mathcal{S})}x \in F(\mathcal{S}). \end{aligned}$$

Then, we obtain (1). Furthermore, we know from Corollary 2.3 that $x - T_sx \in (JF(\mathcal{S}))_{\perp}$ for all $x \in E$ and $s \in S$. So, we have from (2) that $S_{\alpha}(x - T_sx) \rightarrow 0$. Thus we have from $T_s \circ S_{\alpha} = S_{\alpha} \circ T_s$ that for any $x \in E$ and $s \in S$,

$$\begin{aligned} S_{\alpha}x - T_s \circ S_{\alpha}x &= S_{\alpha}x - S_{\alpha} \circ T_sx \\ &= S_{\alpha}(x - T_sx) \rightarrow 0. \end{aligned}$$

Then, we obtain (3).

Suppose (3). We have from (3) and $T_s \circ S_{\alpha} = S_{\alpha} \circ T_s$ that for any $x \in E$ and $s \in S$,

$$\begin{aligned} S_{\alpha}(x - T_sx) &= S_{\alpha}x - S_{\alpha}(T_sx) \\ &= S_{\alpha}x - S_{\alpha} \circ T_s(x) \\ &= S_{\alpha}x - T_s \circ S_{\alpha}(x) \\ &\rightarrow 0. \end{aligned}$$

So, we have that $\{S_{\alpha}y\}$ converges to 0 for all $y \in \{x - T_sx : x \in E, s \in S\}$. From Lemmas 4.2 and 4.1, we have

$$(JF(\mathcal{S}))_{\perp} = (\{z - T_s z : z \in E, s \in S\}^{\perp})_{\perp} = \overline{\text{span}}\{x - T_sx : x \in E, s \in S\}.$$

Take $x \in (JF(\mathcal{S}))_{\perp}$. Then, for any $\epsilon > 0$, we have that there exist $\{a_i\}_{i=1}^n \subset \mathbb{R}$ and $\{y_i\}_{i=1}^n \subset \{x - T_sx : x \in E, s \in S\}$ such that $\|x - \sum_{i=1}^n a_i y_i\| < \epsilon$. Thus we have

$$\|S_{\alpha}x\| = \left\| S_{\alpha} \sum_{i=1}^n a_i y_i + \left(S_{\alpha}x - S_{\alpha} \sum_{i=1}^n a_i y_i \right) \right\|$$

$$\begin{aligned} &\leq \left\| S_\alpha \sum_{i=1}^n a_i y_i \right\| + \left\| S_\alpha x - S_\alpha \sum_{i=1}^n a_i y_i \right\| \\ &\leq \left\| S_\alpha \sum_{i=1}^n a_i y_i \right\| + \left\| x - \sum_{i=1}^n a_i y_i \right\| \\ &\leq \sum_{i=1}^n |a_i| \|S_\alpha y_i\| + \epsilon \end{aligned}$$

and hence

$$\limsup_\alpha \|S_\alpha x\| \leq \limsup_\alpha \left(\sum_{i=1}^n |a_i| \|S_\alpha y_i\| + \epsilon \right) = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have that for any $x \in (JF(\mathcal{S}))_\perp$, $S_\alpha x$ converges to 0. Then, we obtain (2).

Furthermore, if (1) holds, then we have from the proof of (1) that for any $x \in E$, $\{S_\alpha x\}$ converges strongly to $R_{F(\mathcal{S})}x \in F(\mathcal{S})$. \square

Using Theorem 4.3, we have the following useful result.

Theorem 4.4. *Let S be a commutative semitopological semigroup with identity. Let E be a strictly convex, smooth and reflexive Banach space, let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as linear contractive operators of E into itself and let $\{T_n : n \in \mathbb{N}\}$ be a sequence of linear contractive operators of E into itself such that $F(\mathcal{S}) \subset F(T_n)$ for all $n \in \mathbb{N}$. Let $S_n = T_n \circ T_{n-1} \circ \dots \circ T_1$ for all $n \in \mathbb{N}$ and suppose that $T_s \circ S_n = S_n \circ T_s$ for all $n \in \mathbb{N}$ and $s \in S$. Then, the following are equivalent:*

- (1) $\{S_n x\}$ converges to an element of $F(\mathcal{S})$ for all $x \in E$;
- (2) $\{S_n x\}$ converges to 0 for all $x \in (JF(\mathcal{S}))_\perp$;
- (3) $S_n x - T_s \circ S_n x \rightarrow 0$ for all $x \in E$ and $s \in S$.

Furthermore, if (1) holds, then $\{S_n x\}$ converges to $R_{F(\mathcal{S})}x \in F(\mathcal{S})$, where $R_{F(\mathcal{S})} = J^{-1}\Pi_{JF(\mathcal{S})}J$ and $\Pi_{JF(\mathcal{S})}$ is the generalized projection of E^* onto $JF(\mathcal{S})$.

Proof. For any $n \in \mathbb{N}$, $S_n = T_n \circ T_{n-1} \circ \dots \circ T_1$ is a linear contractive operator on E and $F(\mathcal{S}) \subset F(S_n)$. Furthermore, from the assumption, $T_s \circ S_n = S_n \circ T_s$ for all $n \in \mathbb{N}$ and $s \in S$. So, we have the desired result from Theorem 4.3 \square

5. APPLICATIONS

In this section, using Theorems 4.3 and 4.4, we obtain some strong convergence theorems for commutative families of linear contractive mappings in Banach spaces.

In 2003, Bauschk, Deutsch, Hundal and Park showed the following theorem [4].

Theorem 5.1. *Let T be a contractive linear operator on a Hilbert space H ; i.e. $\|T\| \leq 1$, and let M be a closed linear subspace of H . Consider the following statements;*

- (1) $\lim_{n \rightarrow \infty} \|T^n x - P_M x\| = 0$ for all $x \in H$;
- (2) $M = F(T)$ and $T^n x$ converges to 0 for all $x \in M^\perp$;
- (3) $M = F(T)$ and $T^n x - T^{n+1}x \rightarrow 0$ for all $x \in E$.

Then, all statements are equivalent.

Using Theorem 4.3, we can obtain an extension of the above theorem to commutative families of linear contractive mappings in Banach spaces.

Theorem 5.2. *Let E be a strictly convex, smooth and reflexive Banach space and let M be a closed linear subspace of E such that there exists a sunny generalized nonexpansive retraction R of E onto M . Let S be a commutative semitopological semigroup with identity and let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as linear contractive operators of E into itself. Then the following are equivalent:*

- (1) $\{T_s x\}$ converges to the element Rx of M for all $x \in E$;
- (2) $M = F(\mathcal{S})$ and $\{T_s x\}$ converges to 0 for all $x \in (JM)_\perp$;
- (3) $M = F(\mathcal{S})$ and $T_s x - T_{s+t} x \rightarrow 0$ for all $x \in E$ and $t \in S$.

Furthermore, if (1) holds, then $R = R_{F(\mathcal{S})} = J^{-1}\Pi_{JF(\mathcal{S})}J$, where $\Pi_{JF(\mathcal{S})}$ is the generalized projection of E^* onto $JF(\mathcal{S})$.

Proof. If (1) holds, then it is obvious that $F(\mathcal{S}) \subset M$. In fact, let $z \in F(\mathcal{S})$ and $z \notin M$. Since $T_s z = z$, $s \in S$ and $\{T_s z\}$ converges to the element Rz , we have $Mz = z$ and hence $z \in M$. This is a contradiction. Conversely, take $z \in M$. Then we have $Rz = z$. Since $\{T_s z\}$ converges to the element $Rz = z$, we have that for any $t \in S$, $\{T_{s+t} z\}$ converges to the element $T_t z$ because T_t is continuous. On the other hand, $\{T_{s+t} z\}$ converges to the element z . So, we have $T_t z = z$. This implies $M \subset F(\mathcal{S})$. Then we get $M = F(\mathcal{S})$. Define $S_s = T_s$ for all $s \in S$. Then, we have $F(\mathcal{S}) \subset F(S_s)$ and $T_t \circ S_s = S_s \circ T_t$ for all $s, t \in S$. So, we have the desired result from Theorem 4.3. □

Remark 5.3. If M is a closed linear subspace of a Hilbert space H , then there exists the metric projection P of H onto M . In a Hilbert space, the metric projection P of H onto M is coincident with the sunny generalized nonexpansive retraction R_M of H onto M .

Applying Theorem 4.4, we obtain a strong convergence theorem of Mann’s type for commutative semigroups of linear contractive operators in a Banach space. Before obtaining this result, we need the following lemma.

Lemma 5.4 (Eshita and Takahashi [8]). *Let $\{\alpha_n\}$ be a sequence of $[0, 1]$ such that $\sum_{n=1}^\infty (1 - \alpha_n) = \infty$. Let $\{b_n\}$ and $\{\varepsilon_n\}$ be sequences of $[0, \infty)$ such that*

$$b_{n+1} \leq \alpha_n b_n + (1 - \alpha_n)\varepsilon_n, \quad \forall n \in \mathbb{N}$$

and $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Then $\lim_{n \rightarrow \infty} b_n = 0$.

Theorem 5.5. *Let S be a commutative semitopological semigroup with identity. Let E be a strictly convex, smooth and reflexive Banach space and let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as linear contractive operators of E into itself. Let $\{\mu_n\}$ be a sequence of means on $C(S)$ which is strongly asymptotically invariant, i.e., for each $s \in S$, $\|l_s^* \mu_n - \mu_n\| \rightarrow 0$, where l_s^* is the adjoint operator of l_s . Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\sum_{n=1}^\infty (1 - \alpha_n) = \infty$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in E$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_{\mu_n} x_n, \quad n \in \mathbb{N}$$

converges strongly to the element Rx of $F(\mathcal{S})$, where $R = R_{F(\mathcal{S})} = J^{-1}\Pi_{JF(\mathcal{S})}J$ and $\Pi_{JF(\mathcal{S})}$ is the generalized projection of E^* onto $JF(\mathcal{S})$.

Proof. Let $T_n = \alpha_n I + (1 - \alpha_n)T_{\mu_n}$ for all $n \in \mathbb{N}$, where I is the identity operator on E and let $S_n = T_n \circ T_{n-1} \circ \cdots \circ T_1$ for all $n \in \mathbb{N}$. Then, we have that $x_{n+1} = S_n x$. For any $n \in \mathbb{N}$, we have $\|T_n\| \leq 1$ and $F(\mathcal{S}) \subset F(T_n)$. Indeed, we have

$$\|T_n\| = \|\alpha_n I + (1 - \alpha_n)T_{\mu_n}\| \leq \alpha_n \|I\| + (1 - \alpha_n)\|T_{\mu_n}\| \leq 1.$$

We show $F(\mathcal{S}) \subset F(T_n)$. If $x \in F(\mathcal{S})$, then $T_n x = \alpha_n Ix + (1 - \alpha_n)T_{\mu_n}x = x$ and hence $F(\mathcal{S}) \subset F(T_n)$. We also have that $T_s \circ T_{\mu_n} = T_{\mu_n} \circ T_s$ for all $n \in \mathbb{N}$ and $s \in S$. In fact, we have that for any $x \in E$, $s \in S$, $y^* \in E^*$ and $n \in \mathbb{N}$,

$$\begin{aligned} \langle T_s T_{\mu_n} x, y^* \rangle &= \langle T_{\mu_n} x, T_s^* y^* \rangle \\ &= (\mu_n)_t \langle T_t x, T_s^* y^* \rangle \\ &= (\mu_n)_t \langle T_s T_t x, y^* \rangle \\ &= (\mu_n)_t \langle T_t T_s x, y^* \rangle \\ &= \langle T_{\mu_n} T_s x, y^* \rangle. \end{aligned}$$

Then $T_s \circ T_{\mu_n} = T_{\mu_n} \circ T_s$.

Next, we show that $T_s \circ S_n = S_n \circ T_s$ for all $s \in S$. When $n = 1$, we have that for any $s \in S$, $x \in E$ and $y^* \in E^*$,

$$\begin{aligned} \langle T_s \circ S_1 x, y^* \rangle &= \langle T_s \circ T_1 x, y^* \rangle \\ &= \langle T_s (\alpha_1 I + (1 - \alpha_1)T_{\mu_1} x, y^* \rangle) \\ &= \langle (\alpha_1 I + (1 - \alpha_1)T_{\mu_1} x, T_s^* y^* \rangle) \\ &= \alpha_1 \langle x, T_s^* y^* \rangle + (1 - \alpha_1)(\mu_1)_t \langle T_t x, T_s^* y^* \rangle \\ &= \alpha_1 \langle T_s x, y^* \rangle + (1 - \alpha_1)(\mu_1)_t \langle T_s T_t x, y^* \rangle \\ &= \alpha_1 \langle T_s x, y^* \rangle + (1 - \alpha_1)(\mu_1)_t \langle T_t T_s x, y^* \rangle \\ &= \langle \alpha_1 T_s x + (1 - \alpha_1)T_{\mu_1} T_s x, y^* \rangle \\ &= \langle T_1 \circ T_s x, y^* \rangle \\ &= \langle S_1 \circ T_s x, y^* \rangle. \end{aligned}$$

Then $T_s \circ S_1 = S_1 \circ T_s$. Suppose that for some $k \in \mathbb{N}$, $T_s \circ S_k = S_k \circ T_s$ for all $s \in S$. Then, we have that for any $s \in S$,

$$\begin{aligned} T_s \circ S_{k+1} &= T_s \circ T_{k+1} \circ S_k \\ &= T_s (\alpha_k S_k + (1 - \alpha_k)T_{\mu_k} \circ S_k) \\ &= \alpha_k T_s \circ S_k + (1 - \alpha_k)T_s T_{\mu_k} \circ S_k \\ &= \alpha_k T_s \circ S_k + (1 - \alpha_k)T_{\mu_k} T_s \circ S_k \\ &= \alpha_k I \circ S_k T_s + (1 - \alpha_k)T_{\mu_k} \circ S_k T_s \\ &= T_{k+1} \circ S_k T_s \\ &= S_{k+1} \circ T_s. \end{aligned}$$

Then, by induction, we have that $T_s \circ S_n = S_n \circ T_s$ for all $n \in \mathbb{N}$ and $s \in S$. To complete the proof, it is sufficient by Theorem 4.4 to show that for any $s \in S$,

$$\|x_n - T_s x_n\| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

From $0 \in F(\mathcal{S})$, we have that

$$\begin{aligned} \|x_{n+1}\| &= \|\alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n\| \\ &\leq \alpha_n \|x_n\| + (1 - \alpha_n) \|T_{\mu_n} x_n\| \\ &\leq \alpha_n \|x_n\| + (1 - \alpha_n) \|x_n\| \\ &\leq \|x_n\| \end{aligned}$$

and hence $\|x_n\| \leq \|x\|$. Using this, we have that for any $s \in S$ and $y^* \in E^*$,

$$\begin{aligned} |\langle T_{\mu_n} x_n - T_s \circ T_{\mu_n} x_n, y^* \rangle| &= |(\mu_n)_t \langle T_t x_n, y^* \rangle - (\mu_n)_t \langle T_t x_n, T_s^* y^* \rangle| \\ &= |(\mu_n)_t \langle T_t x_n, y^* \rangle - (\mu_n)_t \langle T_{s+t} x_n, y^* \rangle| \\ &= |(\mu_n - l_s^* \mu_n)_t \langle T_t x_n, y^* \rangle| \\ &\leq \|\mu_n - l_s^* \mu_n\| \sup_{t \in S} |\langle T_t x_n, y^* \rangle| \\ &\leq \|\mu_n - l_s^* \mu_n\| \|x_n\| \|y^*\| \\ &\leq \|\mu_n - l_s^* \mu_n\| \|x\| \|y^*\| \end{aligned}$$

and hence

$$(5.1) \quad \|T_{\mu_n} x_n - T_s \circ T_{\mu_n} x_n\| \rightarrow 0.$$

We also have that for any $s \in S$,

$$\begin{aligned} \|x_{n+1} - T_s x_{n+1}\| &= \|\alpha_n (x_n - T_s x_n) + (1 - \alpha_n) (T_{\mu_n} x_n - T_s T_{\mu_n} x_n)\| \\ &\leq \alpha_n \|x_n - T_s x_n\| + (1 - \alpha_n) \|T_{\mu_n} x_n - T_s T_{\mu_n} x_n\|. \end{aligned}$$

We obtain from (5.1) and Lemma 5.4 that

$$\lim_{n \rightarrow \infty} \|T_s x_n - x_n\| = 0, \quad \forall s \in S.$$

By Theorem 4.4, $\{x_n\}$ converges strongly to the element Rx of $F(\mathcal{S})$, where $R = R_{F(\mathcal{S})} = J^{-1} \Pi_{JF(\mathcal{S})} J$ and $\Pi_{JF(\mathcal{S})}$ is the generalized projection of E^* onto $JF(\mathcal{S})$. This completes the proof. \square

From Theorem 4.3, we can show a mean strong convergence theorem for commutative semigroups of contractive linear operators in a Banach space; see Yosida [39].

Theorem 5.6. *Let S be a commutative semitopological semigroup with identity. Let E be a strictly convex, smooth and reflexive Banach space, let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as linear contractive operators of E into itself. Let $\{\mu_\alpha\}$ be a net of strongly asymptotically invariant means on $C(S)$, i.e., for each $s \in S$, $\|l_s^* \mu_\alpha - \mu_\alpha\| \rightarrow 0$, where l_s^* is the adjoint operator of l_s . Then, for each $x \in E$, $\{T_{\mu_\alpha} x\}$ converges strongly to the element Rx of $F(\mathcal{S})$, where $R = R_{F(\mathcal{S})} = J^{-1} \Pi_{JF(\mathcal{S})} J$ and $\Pi_{JF(\mathcal{S})}$ is the generalized projection of E^* onto $JF(\mathcal{S})$.*

Proof. For any α , the operator T_{μ_α} is a contractive linear operator. Furthermore, we have $F(\mathcal{S}) \subset F(T_{\mu_\alpha})$ and $T_s \circ T_{\mu_\alpha} = T_{\mu_\alpha} \circ T_s$ for any α and $s \in S$. In fact, if $u \in F(\mathcal{S})$, then we have that for any $y^* \in E^*$,

$$\langle T_{\mu_\alpha} u, y^* \rangle = (\mu_\alpha)_t \langle T_t u, y^* \rangle = (\mu_\alpha)_t \langle u, y^* \rangle = \langle u, y^* \rangle$$

and hence $T_{\mu_\alpha} u = u$. This implies $F(\mathcal{S}) \subset F(T_{\mu_\alpha})$. We also have that for any $x \in E$, $s \in S$, $y^* \in E^*$ and α ,

$$\begin{aligned} \langle T_s T_{\mu_\alpha} x, y^* \rangle &= \langle T_{\mu_\alpha} x, T_s^* y^* \rangle \\ &= (\mu_\alpha)_t \langle T_t x, T_s^* y^* \rangle \\ &= (\mu_\alpha)_t \langle T_s T_t x, y^* \rangle \\ &= (\mu_\alpha)_t \langle T_t T_s x, y^* \rangle \\ &= \langle T_{\mu_\alpha} T_s x, y^* \rangle. \end{aligned}$$

Then $T_s \circ T_{\mu_\alpha} = T_{\mu_\alpha} \circ T_s$. To complete the proof, it is sufficient to show that $T_{\mu_\alpha} x - T_s T_{\mu_\alpha} x \rightarrow 0$ for all $x \in E$ and $s \in S$. In fact, we have

$$\begin{aligned} |\langle T_{\mu_\alpha} x - T_s T_{\mu_\alpha} x, y^* \rangle| &= |(\mu_\alpha)_t \langle T_t x, y^* \rangle - (\mu_\alpha)_t \langle T_t x, T_s y^* \rangle| \\ &= |(\mu_\alpha)_t \langle T_t x, y^* \rangle - (\mu_\alpha)_t \langle T_{s+t} x, y^* \rangle| \\ &\leq \|\mu_\alpha - l_s^* \mu_\alpha\| \sup_{t \in S} |\langle T_t x, y^* \rangle| \\ &\leq \|\mu_\alpha - l_s^* \mu_\alpha\| \|x\| \|y^*\| \end{aligned}$$

and hence

$$\|T_{\mu_\alpha} x - T_s T_{\mu_\alpha} x\| \leq \|\mu_\alpha - l_s^* \mu_\alpha\| \|x\|.$$

So, we obtain that $T_{\mu_\alpha} x - T_s T_{\mu_\alpha} x \rightarrow 0$ for each $x \in E$ and $s \in S$. Using Theorem 4.3, $\{T_{\mu_\alpha} x\}$ converges strongly to the element Rx of $F(\mathcal{S})$, where $R = R_{JF(\mathcal{S})} = J^{-1} \Pi_{JF(\mathcal{S})} J$ and $\Pi_{JF(\mathcal{S})}$ is the generalized projection of E^* onto $JF(\mathcal{S})$. This completes the proof. \square

Remark 5.7. In Theorem 5.6, note that the point $z = \lim_\alpha T_{\mu_\alpha} x$ is characterized by the sunny generalized nonexpansive retraction $R = R_{F(\mathcal{S})} = J^{-1} \Pi_{JF(\mathcal{S})} J$ of E onto $F(\mathcal{S})$. Such a result is still new even if the operators T_s , $s \in S$ are linear.

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