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REMARKS ON CONVEXITY PROPERTIES OF NAKANO SPACES

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Dedicated to Professor Sompong Dhompongsa on the occasion of his 65th birthday

Abstract. We discuss the characterizations of all geometric properties of Nakano spaces $l^{\{p_j\}}$ appeared in [3] without assuming the boundedness of the sequence $\{p_j\}$.

1. Introduction

Geometric properties are important tools for studying the nonlinear functional analysis. For example, the problem of finding a nearest point in the best approximation context is solvable if a Banach space has a very nice geometric property. In the literature, many mathematicians have paid their attention on the classical sequence spaces ℓ^p and their generalizations. Recall that ℓ^p, where $1 \le p < \infty$, is the space of all real sequences $x := (x(j))$ such that

$$\sum_{j=1}^{\infty} |x(j)|^p < \infty.$$

The following two concepts are the natural generalizations of ℓ^p. To replace the function $t \mapsto |t|^p$ by the more general convex function $M : [0,\infty) \to [0,\infty]$, it leads to the concept of Orlicz sequence spaces. On the other hand, it is interesting to study the variable p, that is, the space of all real sequences $x := (x(j))$ such that

$$\sum_{j=1}^{\infty} |\lambda x(j)|^{p_j} < \infty$$

for some $\lambda > 0$. The latter space is known as the Nakano sequence space.

In [3], Dhompongsa investigated many geometric properties of the Nakano sequence space $l^{\{p_j\}}$. There are many interesting idea appeared there but, unfortunately, this paper has not been widely known as it should be. It was not even be reviewed in MathSciNet or Zentralblatt MATH. It is the author's purpose to draw the readers' attention to this paper.

It should be noted that the assumption that the sequence $\{p_j\}$ is bounded is imposed in the work of Dhompongsa. In this paper, we prove that this condition turns out to be a necessary condition of some of his results and some can be proved without this assumption.

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Recall that the Nakano sequence space $l^{\{p_j\}}$, where $1 \leq p_j < \infty$, is the space of all real sequences $x = (x(j))$ such that

$$\varrho(\lambda x) := \sum_{j=1}^{\infty} |\lambda x(j)|^{p_j} < \infty$$

for some $\lambda > 0$ equipped with the norm defined by

$$\|x\| = \inf\left\{\lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

In fact, Nakano defined the norm, for each $x = (x(j))$ such that $\sum\limits_{j=1}^{\infty} \frac{1}{p_j}\left|\frac{x(j)}{\lambda}\right|^{p_j} < \infty$ for some $\lambda > 0$, by

$$\|x\|' = \inf\left\{\lambda > 0 : \sum_{j=1}^{\infty} \frac{1}{p_j}\left|\frac{x(j)}{\lambda}\right|^{p_j} \leq 1\right\}.$$

However, both spaces are isometrically equal (see [4]).

2. Results

Let e_i stand for the standard basis for $\mathbb{R}^\infty$. That is, $e_j = (\delta_{jk})_k$ for all j.

Lemma 2.1. *The following statements are equivalent:*

(1) *The sequence $\{p_j\}$ is unbounded;*
(2) *There exists a norm-one element x such that*

$$\varrho(\lambda x) = \infty$$

for all $\lambda > 1$.

Proof. (1)⇒(2) Without loss of generality, we may assume that $p_j > j^2$ for all $j \in \mathbb{N}$. Let $x := (x(j))$ where $x(j) = \left(\frac{1}{2}\right)^{1/j}$ for all $j \in \mathbb{N}$. It is easy to see that (2) holds.

(2)⇒(1) We assume that $\{p_j\}$ is bounded and $p_j \leq p$ for some $p > 0$. Then, for each norm-one element $x \in \ell^{\{p_j\}}$, we have

$$\varrho(\lambda x) \leq \lambda^p \varrho(x) \leq \lambda^p < \infty$$

for all $\lambda > 1$. □

We recall the closed subspace

$$h^{\{p_j\}} := \{x \in \ell^{\{p_j\}} : \varrho(\lambda x) < \infty \text{ for all } \lambda > 0\}.$$

Note that $h^{\{p_j\}} = l^{\{p_j\}}$ if and only if the sequence $\{p_j\}$ is bounded. Thus, in [3], only geometric properties of $h^{\{p_j\}}$ are characterized. A careful reading allows us to prove only the following four geometric properties, namely, k-rotundity, reflexivity, property (H) and uniform λ-property. For each of the first three properties, we prove that the boundedness of the sequence $\{p_j\}$ is its necessary condition. While the last property, the characterization in [3] still holds even if we drop away the boundedness of $\{p_j\}$.

2.1. **k-Rotundity.** A Banach space X is said to be *k-rotund*, where $k \geq 1$, if for any norm-one elements $x_1, \ldots, x_{k+1} \in X$ with $\|x_1 + \cdots + x_{k+1}\| = k+1$ implies $x_1, \ldots, x_{k+1}$ are linearly dependent.

Theorem 2.2. *If $\ell^{\{p_j\}}$ is k-rotund, then $\{p_j\}$ is bounded.*

Proof. Suppose the assertion does not hold. Thus, by Lemma 2.1, there exists a norm-one element $x := (x(j))$ such that $\varrho(\lambda x) = \infty$ for all $\lambda > 1$. Without loss of generality, we may assume that $x(j) \neq 0$ for all $j = 1, \ldots, k+1$. For each $n = 1, \ldots, k+1$, we put

$$x_n := x(n)e_n + \sum_{j=k+2}^{\infty} x(j)e_j.$$

Then $\{x_1, \ldots, x_{k+1}\}$ is a linearly independent subset of norm-one elements of $\ell^{\{p_j\}}$. Furthermore,

$$1 = \left\| \sum_{j=k+2}^{\infty} x(j)e_j \right\| \leq \left\| \frac{x_1 + \cdots + x_{k+1}}{k+1} \right\| \leq 1.$$

This is a contradiction. □

2.2. **Reflexivity.** A Banach space X is said to be *reflexive* if the canonical map from X into its second dual X^{**} is surjective. Equivalently, every bounded sequence in X has a weakly convergent subsequence.

Lemma 2.3. *Let x be an element in Lemma 2.1. Then*

$$\inf\{\|x - y\| : y \in h^{\{p_j\}}\} = 1.$$

Proof. The inequality $\inf\{\|x - y\| : y \in h^{\{p_j\}}\} \leq 1$ is obvious. To see the reverse inequality, we first note that the subspace F of all real sequences with finitely many nonzero is dense in $h^{\{p_j\}}$. Now let $y := \sum_{j=1}^{n} y(j)e_j \in F$ where $n \in \mathbb{N}$. It follows then that

$$\|x - y\| \geq \left\| \sum_{j=n+1}^{\infty} x(j)e_j \right\| = 1.$$

This completes the proof. □

Theorem 2.4. *If $\ell^{\{p_j\}}$ is reflexive, then $\{p_j\}$ is bounded.*

Proof. Suppose not, by Lemma 2.1, there exists a norm-one element $x = (x(j))$ such that $\varrho(\lambda x) = \infty$ for all $\lambda > 1$. For each $n \in \mathbb{N}$, we put

$$x_n := \sum_{j=n}^{\infty} x(j)e_j.$$

It is obvious that $\{x_n\}$ is a sequence of norm-one elements in $\ell^{\{p_j\}}$. If the sequence $\{x_n\}$ has a weakly convergent subsequence, then the whole sequence is a weakly null sequence. By the Hahn-Banach Theorem, there exists a norm-one bounded linear functional f such that

$$f(x) = \inf\{\|x - y\| : y \in h^{\{p_j\}}\} = 1,$$

and $f(y) = 0$ for all $y \in h^{\{p_j\}}$.

On the other hand, we have $x_n - x \in h^{\{p_j\}}$ and hence

$$f(x_n) = f(x) + f(x_n - x) = 1$$

for all $n \in \mathbb{N}$ which is a contradiction. □

2.3. **Property (H).** A Banach space X is said to have *property (H)* if weak convergence and norm convergence of any sequence of norm-one elements coincide.

Theorem 2.5. *If $\ell^{\{p_j\}}$ has property (H), then $\{p_j\}$ is bounded.*

Proof. See [2, Theorem 2]. □

2.4. **Uniform λ-property.** A norm-one element e of a Banach space is said to be an *extreme point* if it cannot be a midpoint of any two distinct norm-one elements. For a norm-one element x, we define

$$\lambda(x) := \sup\{\lambda \in [0,1] : x = \lambda e + (1-\lambda)y, e \text{ is an extreme point}, \|y\| \le 1\}.$$

A Banach space X is said to have *uniform λ-property* if

$$\lambda(X) := \inf\{\lambda(x) : \|x\| = 1\} > 0.$$

It is easy to see that if x is an extreme point, then $\lambda(x) = 1$ but the converse does not hold. However, it is not difficult to prove that if $\lambda(x) = 1$, then x is a limit point of the set of extreme points.

Theorem 2.6.

$$\lambda(\ell^{\{p_j\}}) = \inf\{\lambda(x) : \varrho(x) = 1\}.$$

Proof. It suffices to prove that

$$\lambda(l^{\{p_i\}}) \ge \inf\{\lambda(x) : \varrho(x) = 1\} =: \lambda_0.$$

Let $x = (x(j)) \in \ell^{\{p_i\}}$ be a norm-one element such that $\varrho(x) < 1$. Then, for any $\alpha \in (0,1)$,

$$\varrho\Big(\frac{x}{1-\alpha}\Big) = \infty.$$

For every $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that

$$\sum_{j=1}^{k_n} \Big|\frac{x(j)}{1-\frac{1}{n}}\Big|^{p_j} > 1.$$

We can choose $\alpha_n \in (0, \frac{1}{n})$ so that

$$\sum_{j=1}^{k_n} \Big|\frac{x(j)}{1-\alpha_n}\Big|^{p_j} + \sum_{j=k_n+1}^{\infty} |x(j)|^{p_j} = 1.$$

Define

$$y := \sum_{j=1}^{k_n} \frac{x(j)}{1-\alpha_n} e_j + \sum_{j=k_n+1}^{\infty} x(j) e_j$$

and

$$z = \sum_{j=k_n+1}^{\infty} x(j)e_j.$$

Then $\varrho(y) = 1$ and $\|z\| \leq 1$. Moreover,

$$x = (1 - \alpha_n)y + \alpha_n z.$$

Hence, by Proposition 2.12 of [1],

$$\lambda(x) \geq (1 - \alpha_n)\lambda(y) \geq (1 - \alpha_n)\lambda_0.$$

Letting $n \to \infty$ yields $\lambda(x) \geq \lambda_0$ and then $\lambda(\ell^{\{p_j\}}) \geq \lambda_0$. This completes the proof. □

The following is also proved in [5] without assuming the boundedness of the sequence $\{p_j\}$.

Proposition 2.7. *A norm-one element $x = (x(j)) \in \ell^{\{p_j\}}$ is an extreme point if and only if*

(1) $\varrho(x) = 1$ *and*
(2) *the cardinality of* $\{j \in \mathbb{N} : x(j) \neq 0$ *and* $p_j = 1\} \leq 1$.

Supplement to the original proof of [3, Theorem 5], we have the following result.

Theorem 2.8. *The space $\ell^{\{p_j\}}$ has uniform λ-property if and only if $w := \#\{j \in \mathbb{N} : p_j = 1\} < \infty$. Furthermore,*

$$\lambda(\ell^{\{p_j\}}) = \begin{cases} 1/w & \text{if } w \geq 1, \\ 1 & \text{if } w = 0. \end{cases}$$

Proof. We need only prove the last assertion when $w = 0$ and 1. In these cases, by Proposition 2.7, $\{x \in \ell^{\{p_i\}} : \|x\| = 1$ and $\varrho(x) = 1\}$ is just the set of all extreme points. Therefore, by the observation before Theorem 2.6, $\lambda(x) = 1$ for all norm-one elements $x \in \ell^{\{p_i\}}$ with $\varrho(x) = 1$. □

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