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A NEWTON-LIKE METHOD FOR SOLVING GENERALIZED OPERATOR EQUATIONS AND VARIATIONAL INEQUALITIES

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Abstract. In this paper, we present a semilocal convergence analysis of a Newton-like method for solving the generalized operator equations in Hilbert spaces and also discuss the convergence analysis of the proposed algorithm under weak conditions. We establish sharp generalizations of Kantorovich theory for operator equations when the derivative is not necessarily invertible. As a simple consequence of our result, we discuss the existence and uniqueness of solutions of mixed variational inequality problems. Finally, we give numerical examples for the equations involving single valued as well as multi-valued mappings.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, respectively. Throughout the paper, we denote the set D a closed convex subset of $\mathcal{H}$, D_0 the interior of D, $B_r[x]$ the set

$$\{y \in \mathcal{H} : \|y - x\| \leq r\},$$

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and Φ the collection of all continuous, nondecreasing functions $\phi : [0,\infty) \to [0,\infty)$.

Let us recall some basic definitions.

Definition 1.1. Let $\mathcal{H}$ be a Hilbert space. An operator $T : \mathcal{H} \to \mathcal{H}$ is said to be:

(1) *monotone* if

$$\langle Tx - Ty, x - y\rangle \geq 0$$

for all $x, y \in \mathcal{H}$;

(2) *strongly monotone* if there exists $k_1 > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq k_1\|x - y\|^2$$

for all $x, y \in \mathcal{H}$;

(3) *Lipschitz continuous* if there exists $k_2 \geq 0$ such that

$$\|Tx - Ty\| \leq k_2\|x - y\|$$

for all $x, y \in \mathcal{H}$.

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Definition 1.2. A multi-valued operator $G : \mathcal{H} \to 2^{\mathcal{H}}$ is said to be:

(1) *monotone* if

$$\langle y_2 - y_1, x_2 - x_1\rangle \geq 0$$

for all $x_1, x_2 \in \mathcal{H}$, $y_1 \in Gx_1$ and $y_2 \in Gx_2$;

(2) *maximal monotone* if it is monotone and there is no other monotone operator whose graph contains strictly the graph $\mathcal{G}(G)$ of G, where graph of G is defined by

$$\mathcal{G}(G) = \{(x, y) \in \mathcal{H} \times \mathcal{H} : x \in D(G), y \in Gx\};$$

(3) *strongly monotone* if there exists $\alpha > 0$ such that

$$\langle y_2 - y_1, x_2 - x_1\rangle \geq \alpha\|x_1 - x_2\|^2 \tag{1.1}$$

for all $x_1, x_2 \in \mathcal{H}$, $y_1 \in Gx_1$ and $y_2 \in Gx_2$.

A well-known example (see [7, 8]) of a maximal monotone operator is the subgradient

$$\partial\phi(x) = \{z \in \mathcal{H} : \phi(x) - \phi(y) \leq \langle z, x - y\rangle, \forall y \in \mathcal{H}\}$$

of a proper lower semi-continuous convex function $\phi : \mathcal{H} \to (-\infty, \infty]$.

In the sequel, we regard the statements $[x, y] \in G$, $G(x) \ni y$, $-y + G(x) \ni 0$ and $y \in G(x)$ as synonymous. In [7], it is shown that, if G is maximal monotone, then G is closed in the sense that

$$[x_m, y_m] \in G,\ \lim_{m\to\infty} x_m = x,\ \lim_{m\to\infty} y_m = y \implies [x, y] \in G.$$

In this paper, we consider the following problem: Let $F : D \to \mathcal{H}$ be an operator which is Fréchet differentiable at each point of D_0 and $G : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator. Find $x \in \mathcal{H}$ such that

$$Fx + Gx \ni 0. \tag{1.2}$$

Examples of the variational inclusion (1.2) are as follows:

(1) If $G = \partial\phi$, then the problem (1.2) reduces to the following problem:

Find $x \in \mathcal{H}$ such that

$$\langle Fx, y - x\rangle \geq \phi(x) - \phi(y) \tag{1.3}$$

for all $y \in \mathcal{H}$, which is called the *mixed variational inequality* and has been studied by many authors (see, for example, [4, 5, 9]).

(2) Let $G = \partial\delta_K$, where $\partial\delta_K$ is the indicator function of a nonempty closed and convex subset K of $\mathcal{H}$ defined by

$$\partial\delta_K(x) = \begin{cases} 0, & x \in K; \\ \infty, & \text{otherwise.} \end{cases}$$

In this case, the problem (1.2) reduces to the following problem:

Find $x \in K$ such that

$$\langle Fx, y - x\rangle \geq 0$$

for all $y \in K$, which is the classical variational inequality (see [6, 12]).

For solving the operator equation (1.2), the generalized Newton method is given by

$$F'_{x_n}x_{n+1} + Gx_{n+1} \ni F'_{x_n}x_n - Fx_n \tag{1.4}$$

for all $n \geq 0$, where F'_x denotes the Fréchet derivative of F at the point $x \in D_0$. The convergence of the generalized Newton method (1.4) can be found in [1, 2, 10, 11, 13, 14, 15, 16, 17].

In [13, Theorem 2.10], the existence and uniqueness of solutions of the problem (1.2) was discussed and the following semi-local convergence analysis of (1.4) was given.

Theorem 1.3. *Let F be an operator defined on a closed convex set D of a Hilbert space $\mathcal{H}$ and has the Fréchet derivative at each point of D_0. Assume that $G : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator satisfying* (1.1). *For some $x_0 \in D_0$, assume that the operators F and G satisfy the following conditions:*

(C1) *there exists $y_0 \in \mathcal{H}$ such that $y_0 \in G(x_0)$ and $\|F(x_0) + y_0\| \leq \beta$ for some $\beta > 0$;*

(C2) *$\langle F'_{x_0}x, x\rangle \geq c_0\|x\|^2$ for all $x \in \mathcal{H}$ and for some real number c_0;*

(C3) *$\|F'_x - F'_y\| \leq K\|x - y\|$ for all $x, y \in D_0$ and for some $K \geq 0$.*

Let $c_0 + \alpha > 0$ and denote $d = \frac{\beta}{c_0+\alpha}$ and $h = \frac{Kd}{c_0+\alpha}$. Assume that $h < \frac{1}{2}$ and $B_r[x_0] \subseteq D_0$, where $r = \frac{2d}{1+\sqrt{1-2h}}$. Then we have the following:

(1) *The operator equation* (1.2) *has a unique solution x^* in $B_{r_1}[x_0] \cap D$, where $r_1 = \frac{2d}{1-\sqrt{1-2h}}$.*

(2) *The sequence $\{x_n\}$ generated by* (1.4) *remains in $B_r[x_0]$ and converges to x^*.*

(3) *The following error estimate holds:*

$$\|x_{n+1} - x^*\| \leq \frac{d}{h}\gamma^{n+1} \tag{1.5}$$

for all $n \geq 1$, where $\gamma = 1 - \sqrt{1-2h}$.

In [2, 17], the authors discussed the semilocal convergence analysis of (1.4) using all the conditions of Theorem 1.3 and the center Lipschitz condition

$$\|F'_x - F'_{x_0}\| \leq K_0\|x - x_0\| \tag{1.6}$$

for all $x, y \in D_0$ and for some $K_0 \geq 0$. In [18, 19, 20], the following general condition is considered for the convergence of Newton-like methods

$$\|F'_x - F'_y\| \leq \omega(\|x - y\|) \tag{1.7}$$

for all $x, y \in D_0$, where $\omega \in \Phi$. It is interesting to consider the following condition:

$$\|F'_x - F'_{x_0}\| \leq \omega_0(\|x - x_0\|) \tag{1.8}$$

for all $x \in D_0$, where $\omega_0 \in \Phi$. It is easy to see that (1.8) is a weaker assumption than (1.7).

Recently, Argyros and Hilout [3] have studied the convergence analysis of

$$x_{n+1} = x_n - A(x_n)^{-1}(F(x_n) + G(x_n)) \tag{1.9}$$

using the conditions (1.7) and (1.8) for solving the operator equation

$$F(x) + G(x) = 0, \tag{1.10}$$

where F is Fréchet differentiable, G is continuous operator defined on Banach spaces and $A(x)$ is an approximation of F'_x.

In the present paper, we introduce the following Newton-like method for finding the solution of the operator equation (1.2) in Hilbert spaces.

Algorithm 1.1. Let $A(x)$ be an approximation of F'_x for each $x \in D_0$. Starting with $x_0 \in D_0$ and, after $x_n \in D_0$ is defined, we define the next iterate x_{n+1} as follows:

$$A(x_n)x_{n+1} + Gx_{n+1} \ni A(x_n)x_n - Fx_n \tag{1.11}$$

for all $n \geq 0$.

Motivated by Argyros and Hilout [3], the purpose of this paper is to prove the semi-local convergence analysis of Algorithm 1.1 under both the conditions (1.7) and (1.8). The results presented in this paper improve and extend the corresponding results announced in [3, 13, 15]. As applications of our results, we discuss the solution of the nonlinear variational inequality. By numerical example, we show the applicability of our results.

2. Convergence analysis

Before giving our main convergence result, we establish two technical lemmas which are useful in the sequel.

Lemma 2.1. *Let c_0, l_0, H, l_1, α be some nonnegative real numbers and $\omega, \omega_0, \omega_1 \in \Phi$. Let $\eta > 0$. Assume that the scalar equation*

$$(c_0 + \alpha - l_0 - \omega_0(r))(\eta - r) + (H\omega(\eta) + \omega_1(r) + l_1)r = 0 \tag{2.1}$$

has a minimum positive zero r^* such that

$$H\omega(\eta) + \omega_1(r^*) + \omega_0(r^*) < c_0 + \alpha - l_0 - l_1. \tag{2.2}$$

Then the sequence $\{t_n\}$ defined by

$$\begin{cases} t_0 = 0, \ t_1 = \eta, \\ t_{n+1} = t_n + \frac{(H\omega(t_n - t_{n-1}) + \omega_1(t_n) + l_1)}{(c_0 + \alpha - l_0 - \omega_0(t_n))}(t_n - t_{n-1}) \end{cases} \tag{2.3}$$

is nondecreasing in $[\eta, r^*]$ with

$$t_n - t_{n-1} \leq d^{n-1}\eta \tag{2.4}$$

for all $n \geq 1$, where $d = \frac{(H\omega(\eta) + \omega_1(r^*) + l_1)}{(c_0 + \alpha - l_0 - \omega_0(r^*))}$ and it converges to its least upper bound t^*.

Proof. By the principle of induction, we show that the sequence $\{t_n\}$ remains in $[\eta, r^*]$ and holds (2.4). Since r^* is the minimum positive root of (2.1), we have $t_1 = \eta \leq r^*$. Hence our assertion holds for $n = 1$. Assume that our assertion holds for some positive integer $n = k$. Using (2.3), we have

$$t_{k+1} = t_k + \frac{(H\omega(t_k - t_{k-1}) + \omega_1(t_k) + l_1)}{(c_0 + \alpha - l_0 - \omega_0(t_k))}(t_k - t_{k-1})$$

$$
\begin{aligned}
&\leq t_k + d(t_k - t_{k-1}) \\
&\leq t_{k-1} + d(t_{k-1} - t_{k-2}) + d(t_k - t_{k-1}) \\
&\leq t_{k-1} + d^{k-1}\eta + d^k\eta \\
&\leq (1 + d + \cdots + d^k)\eta \\
&= \frac{1-d^k}{1-d}\eta \\
&< \frac{\eta}{1-d} = r^*.
\end{aligned}
$$

Hence t_{k+1} is in $[\eta, r^*]$. Using (2.3), we have

$$
\begin{aligned}
t_{k+1} - t_k &= \frac{(H\omega(t_k - t_{k-1}) + \omega_1(t_k) + l_1)}{(c_0 + \alpha - l_0 - \omega_0(t_k))}(t_k - t_{k-1}) \\
&\leq \frac{(H\omega(\eta) + \omega_1(r^*) + l_1)}{(c_0 + \alpha - l_0 - \omega_0(r^*))}(t_k - t_{k-1}) \\
&\leq d^k\eta.
\end{aligned}
$$

Thus (2.4) holds for $n = k + 1$. Therefore, $\{t_n\}$ defined by (2.3) is in $[\eta, r^*]$ and holds the estimate (2.4). By the definitions of ω, ω_0, ω_1 and (2.3), it follows that $t_n \leq t_{n+1}$ for all $n \geq 0$. Hence the sequence $\{t_n\}$ is nondecreasing, bounded above and as such it converges to its unique least upper bound t^* for some $t^* \in [\eta, r^*]$. This completes the proof. □

Lemma 2.2 ([7, 8, 13, 17])**.** *Let S be a bounded linear operator from $\mathcal{H}$ into $\mathcal{H}$ and G be a maximal monotone operator from $\mathcal{H}$ into $2^{\mathcal{H}}$ satisfying* (1.1). *Assume that*

(i) $\langle S(x), x\rangle \geq c_0\|x\|^2$ *for all $x \in \mathcal{H}$ and for some real number c_0;*

(ii) $c_0 + \alpha > 0$.

Then, for any $b \in \mathcal{H}$, there exists a unique $z \in \mathcal{H}$ satisfying the generalized equation

$$Sz + Gz \ni b.$$

Now, we ready to present the semilocal convergence analysis of (1.11).

Theorem 2.3. *Let F be an operator defined on a closed convex subset D of a Hilbert space $\mathcal{H}$ with values in $\mathcal{H}$ such that F is continuously Fréchet differentiable at each point of D_0 and G be a maximal monotone operator from $\mathcal{H}$ into $2^{\mathcal{H}}$ satisfying* (1.1). *Let $A(x)$ be an approximation of $F'_x, x \in D_0$. For some $x_0 \in D_0$, assume that the operators F, G, F'_x and $A(x)$ satisfy* (C1) *and* (1.7) *with $\omega \in \Phi$ and the following conditions:*

(C4) $\langle A(x_0)(x), x\rangle \geq c_0\|x\|^2$ *for all $x \in \mathcal{H}$ and for some real number c_0;*

(C5) $\|A(x) - A(x_0)\| \leq \omega_0(\|x - x_0\|) + l_0$ *for all $x \in D_0$, for some $l_0 \geq 0$ and for some $\omega_0 \in \Phi$;*

(C6) $\|F'_x - A(x)\| \leq \omega_1(\|x - x_0\|) + l_1$ *for all $x \in D_0$, for some $l_1 \geq 0$ and for some $\omega_1 \in \Phi$;*

(C7) *the function ω satisfies $\omega(ts) \leq h(t)\omega(s)$ for all $t \in [0, 1]$ and $s \in [0, \infty)$, where h is a continuous, positive and nondecreasing function defined on* $[0, 1]$.

Let $H = \int_0^1 h(t)dt$ *and* $\eta = \frac{\beta}{c_0+\alpha}$. *Assume that the scalar equation defined by* (2.1) *has a minimum positive zero* r^* *such that* (2.2) *is satisfied. Let* $B_{r^*}[x_0] \subseteq D_0$. *Then we have the following:*

(1) *The sequence* $\{x_n\}$ *generated by* (1.11) *is well defined, remains in* $B_{r^*}[x_0]$ *and converges to a solution* $x^* \in B_{r^*}[x_0]$ *of* (1.2). *Moreover, the following error estimates hold:*

$$\|x_n - x_{n-1}\| \leq t_n - t_{n-1}, \tag{2.5}$$

$$\|x_n - x_0\| \leq t_n \tag{2.6}$$

and

$$\|x_n - x^*\| \leq t^* - t_n, \tag{2.7}$$

where $\{t_n\}$ *is a sequence generated by* (2.3) *and* t^* *is the limit of sequence* $\{t_n\}$.

(2) *Further, if*

$$H\omega(r^*) + \omega_1(r^*) + \omega_0(r^*) < c_0 + \alpha - l_0 - l_1, \tag{2.8}$$

then the solution of (1.2) *is unique in* $B_{r^*}[x_0]$.

Proof. (1) It follows from Lemma 2.1 that the sequence $\{t_n\}$ defined by (2.3) is non-decreasing in $[\eta, r^*]$ and converges to some $t^* \in [\eta, r^*]$. By the principle of induction, we show that the sequence $\{x_n\}$ is well defined in $B_{r^*}[x_0]$ and the conditions (2.5)–(2.6) hold for all $n \geq 1$. It follows from the condition (C4) and Lemma 2.2 that the first iterate x_1 in (1.11) is defined uniquely. Using (1.1) and (1.11), we have

$$\begin{aligned} \alpha\|x_1 - x_0\|^2 &\leq \langle -A(x_0)(x_1 - x_0) - F(x_0) - y_0, x_1 - x_0\rangle \\ &\leq -\langle A(x_0)(x_1 - x_0), x_1 - x_0\rangle + \langle -F(x_0) - y_0, x_1 - x_0\rangle. \end{aligned}$$

Using (C1) and (C4), we have

$$\|x_1 - x_0\| \leq \frac{\beta}{c_0 + \alpha} = \eta = t_1 - t_0 < r^*.$$

Hence $x_1 \in B_{r^*}[x_0]$ and (2.5)–(2.6) hold for $n = 1$. Assume that vector x_k given by (1.11) is well defined in $B_{r^*}[x_0]$ and (2.5)-(2.6) hold for some positive integer $n = k$. Note that

$$\|A(x_k) - A(x_0)\| \leq \omega_0(\|x_k - x_0\|) + l_0 \leq \omega_0(t_k) + l_0.$$

Therefore, we have

$$\langle A(x_0)x - A(x_k)x, x\rangle \leq \|A(x_k) - A(x_0)\|\|x\|^2 \leq (\omega_0(t_k) + l_0)\|x\|^2$$

for all $x \in \mathcal{H}$, which implies that

$$\langle A(x_k)x, x\rangle \geq (c_0 - l_0 - \omega_0(t_k))\|x\|^2$$

for all $x \in \mathcal{H}$. Hence it follows from Lemma 2.2 that the vector x_{k+1} given by (1.11) is well defined in $\mathcal{H}$. Using (1.1), (1.7), (1.11) and (C6)–(C7), we have

$$\begin{aligned} &\alpha\|x_{k+1} - x_k\|^2 \\ \leq\ & \langle -A(x_k)(x_{k+1} - x_k) - F(x_k) + A(x_{k-1})(x_k - x_{k-1}) + F(x_{k-1}), x_{k+1} - x_k\rangle \\ =\ & \langle -A(x_k)(x_{k+1} - x_k), x_{k+1} - x_k\rangle \\ & + \langle -F(x_k) + F'_{x_{k-1}}(x_k - x_{k-1}) + F(x_{k-1}), x_{k+1} - x_k\rangle \end{aligned}$$

$$
\begin{aligned}
& +\langle -F'_{x_{k-1}}(x_k - x_{k-1}) + A(x_{k-1})(x_k - x_{k-1}), x_{k+1} - x_k\rangle \\
\leq\ & -(c_0 - l_0 - \omega_0(t_k))\|x_{k+1} - x_k\|^2 \\
& +\|F(x_k) - F(x_{k-1}) - F'_{x_{k-1}}(x_k - x_{k-1})\|\|x_{k+1} - x_k\| \\
& +\|F'_{x_{k-1}} - A(x_{k-1})\|\|x_k - x_{k-1}\|\|x_{k+1} - x_k\| \\
\leq\ & -(c_0 - l_0 - \omega_0(t_k))\|x_{k+1} - x_k\|^2 \\
& +\int_0^1 \|F'_{x_{k-1}+t(x_k - x_{k-1})} - F'_{x_{k-1}}\|\|x_k - x_{k-1}\|\|x_{k+1} - x_k\|dt \\
& +\|F'_{x_{k-1}} - A(x_{k-1})\|\|x_k - x_{k-1}\|\|x_{k+1} - x_k\| \\
\leq\ & -(c_0 - l_0 - \omega_0(t_k))\|x_{k+1} - x_k\|^2 \\
& +\int_0^1 h(t)\omega(\|x_k - x_{k-1}\|)\|x_k - x_{k-1}\|\|x_{k+1} - x_k\|dt \\
& +(\omega_1(\|x_k - x_0\|) + l_1)\|x_k - x_{k-1}\|\|x_{k+1} - x_k\|,
\end{aligned}
$$

which gives that

$$
\begin{aligned}
& \|x_{k+1} - x_k\| \\
\leq\ & \frac{1}{(c_0 + \alpha - l_0 - \omega_0(t_k))}\left(H\omega(\|x_k - x_{k-1}\|) + \omega_1(\|x_k - x_0\|) + l_1\right) \\
& \times\|x_k - x_{k-1}\| \\
\leq\ & \frac{1}{(c_0 + \alpha - l_0 - \omega_0(t_k))}\left(H\omega(t_k - t_{k-1}) + \omega_1(t_k) + l_1\right)(t_k - t_{k-1}) \\
\leq\ & t_{k+1} - t_k.
\end{aligned}
$$

Note

$$\|x_{k+1} - x_0\| \leq \|x_{k+1} - x_k\| + \cdots + \|x_1 - x_0\| \leq t_{k+1} < r^*.$$

Thus $x_{k+1} \in B_{r^*}[x_0]$, (2.5) and (2.6) hold for all $n = k+1$. By induction principle, the sequence $\{x_n\}$ remains in $B_{r^*}[x_0]$ and (2.5)–(2.6) hold. Note that the sequence $\{t_n\}$ majorizes the sequence $\{x_n\}$. Hence the sequence $\{x_n\}$ is a Cauchy sequence and hence converges to some $x^* \in B_{r^*}[x_0]$. Further, we observe that

$$
\begin{aligned}
\|x_{m+n} - x_n\| &\leq \|x_{m+n} - x_{m+n-1}\| + \cdots + \|x_{n+1} - x_n\| \\
&\leq t_{m+n} - t_{m+n-1} + \cdots + t_{n+1} - t_n \\
&= t_{m+n} - t_n.
\end{aligned} \tag{2.9}
$$

Letting limit as $m \to \infty$ in (2.9), we get (2.7).

Now, we show that x^* is a solution of (1.2). Write

$$y_n = -A(x_n)(x_{n+1} - x_n) - F(x_n).$$

Then, by (1.11), we have $y_n \in G(x_{n+1})$. Now, we can write

$$
\begin{aligned}
& \|y_n + F(x^*)\| \\
=\ & \|A(x_n)(x_{n+1} - x_n) + F(x_n) - F(x^*)\| \\
\leq\ & \|A(x_n) - A(x_0)\|\|x_{n+1} - x_n\| + \|A(x_0)\|\|x_{n+1} - x_n\|
\end{aligned}
$$

$$\begin{aligned}
&\quad + \int_0^1 \|F'_{x^*+t(x_n-x^*)} - F'_{x_0}\|\|x_n - x^*\|dt + \|F'_{x_0}\|\|x_n - x^*\| \\
&\leq (\omega_0(r^*) + \|A(x_0)\| + l_0)\|x_{n+1} - x_n\| + (\omega(r^*) + \|F'_{x_0}\|)\|x_n - x^*\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Hence $y_n \to -F(x^*)$ as $n \to \infty$. By the definition of G, we have $-F(x^*) \in G(x^*)$. Therefore, x^* is a solution of (1.2).

(2) Suppose that (2.8) holds. To prove the uniqueness of x^*, let y^* be another solution of (1.2) in $B_{r^*}[x_0]$. It is easy to see that

$$\langle A(x_n)x, x\rangle \geq (c_0 - \omega_0(t_n) - l_0)\|x\|^2$$

for all $n \geq 1$ and $x \in \mathcal{H}$. Since $-F(y^*) \in G(y^*)$, it follows from (1.1), (1.7), (1.11) and (C6)–(C7) that

$$\begin{aligned}
&\quad \alpha\|y^* - x_{n+1}\|^2 \\
&\leq -\langle -F(y^*) + A(x_n)(x_{n+1} - x_n) + F(x_n), y^* - x_{n+1}\rangle \\
&\leq \langle -A(x_n)(y^* - x_{n+1}) + A(x_n)(y^* - x_n) + F(x_n) - F(y^*), y^* - x_{n+1}\rangle \\
&= \langle -A(x_n)(y^* - x_{n+1}), y^* - x_{n+1}\rangle + \langle A(x_n)(y^* - x_n) + F(x_n) \\
&\quad -F(y^*), y^* - x_{n+1}\rangle \\
&= -\langle A(x_n)(y^* - x_{n+1}), y^* - x_{n+1}\rangle + \langle A(x_n)(y^* - x_n) \\
&\quad -F'_{x_n}(y^* - x_n), y^* - x_{n+1}\rangle \\
&\quad +\Big\langle \int_0^1 (F'_{x_n} - F'_{x_n+t(y^*-x_n)})(y^* - x_n)dt, y^* - x_{n+1}\Big\rangle \\
&\leq -(c_0 - l_0 - \omega_0(t_n))\|y^* - x_{n+1}\|^2 + (\omega_1(\|x_n - x_0\|) + l_1) \\
&\quad \times\|y^* - x_n\|\|y^* - x_{n+1}\| + H\omega(\|y^* - x_n\|)\|y^* - x_n\|\|y^* - x_{n+1}\|,
\end{aligned}$$

which gives

$$\begin{aligned}
&\quad \|y^* - x_{n+1}\| \\
&\leq \frac{1}{(c_0 + \alpha - l_0 - \omega_0(t_n))}(H\omega(\|y^* - x_n\|) + \omega_1(\|x_n - x_0\|) + l_1) \\
&\quad \times\|y^* - x_n\|.
\end{aligned} \tag{2.10}$$

Now, we prove, by induction, that

$$\|y^* - x_{n+1}\| \leq \frac{1}{(c_0 + \alpha - l_0 - \omega_0(r^*))}(H\omega(r^*) + \omega_1(r^*) + l_1))\|y^* - x_n\| \tag{2.11}$$

holds for all $n \geq 0$. Using (2.8), we see that (2.11) holds for $n = 0$. Assume that (2.11) holds for some positive integer $n = k$. Using (2.8), we note that

$$\|y^* - x_{k+1}\| \leq \|y^* - x_k\| \leq \cdots \leq \|y^* - x_0\| \leq r^*.$$

It follows from (2.10) that

$$\|y^* - x_{k+2}\| \leq \frac{1}{(c_0 + \alpha - \omega_0(t_1) - l_0)}(H\omega(r^*) + \omega_1(r^*) + l_1)\|y^* - x_{k+1}\|.$$

Hence (2.11) holds for $n = k + 1$. Thus (2.11) holds for all $n \geq 0$. It follows from (2.11) that $\lim_{n\to\infty} x_n = y^*$. Therefore, since we already proved that $\lim_{n\to\infty} x_n = x^*$, we have $x^* = y^*$. This completes the proof. □

Remark 2.4. Theorem 2.3 is an improvement of Theorem 1.3 in the following sense:

(1) In Theorem 2.3, the ω-type condition (1.7) is used, which is a generalization of the Lipschitz condition (C3) of Theorem 1.3.

(2) In Algorithm 1.1, F'_x is not involved.

(3) The ω-type center condition (C5) is used, which is a generalization of the center Lipsschitz condition (1.6).

(4) For the convergence of sequence $\{x_n\}$, majorant theory is adopted. This provides the domain for existence and uniqueness of solution of the operator equation (1.2).

For $A(x) = F'_x$ in Theorem 2.3, we have the following result.

Theorem 2.5. *Let F be an operator defined on a closed convex subset D of a Hilbert space $\mathcal{H}$ with values in $\mathcal{H}$ such that F is continuously Fréchet differentiable at each point of D_0 and G be a maximal monotone operator from $\mathcal{H}$ into $\mathcal{H}$ satisfying* (1.1). *For any $x_0 \in D$, assume that the operators F, G and F'_x satisfy* (C1)–(C2), (C7), (1.7) *and* (1.8) *with $\omega, \omega_0 \in \Phi$. Let $\eta = \frac{\beta}{c_0+\alpha}$ and assume that the scalar equation defined by*

$$(c_0 + \alpha - l_0 - \omega_0(r))(\eta - r) + Hr\omega(\eta) = 0$$

See Theorem 2.3 has a minimum positive zero r^* such that

$$H\omega(\eta) + \omega_0(r^*) < c_0 + \alpha - l_0$$

is satisfied. Suppose that $B_{r^*}[x_0] \subseteq D$. Then we have the following:

(1) *The sequence $\{x_n\}$ generated by* (1.4) *is well defined, remains in $B_{r^*}[x_0]$ and converges to a solution $x^* \in B_{r^*}[x_0]$ of* (1.2). *The error estimates* (2.5)–(2.7) *hold, where $\{t_n\}$ is a sequence generated by*

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+1} \quad = \quad t_n + \frac{H\omega(t_n - t_{n-1})(t_n - t_{n-1})}{(c_0 + \alpha - l_0 - \omega_0(t_n))}$$

and t^* is the limit of the sequence $\{t_n\}$.

(2) *Further, if $H\omega(r^*) + \omega_0(r^*) < c_0 + \alpha - l_0$, then the solution of (1.2) is unique in $B_{r^*}[x_0]$.*

Proof. For $A(x) = F'_x$ in Theorem 2.3, we can take $\omega_1(t) = 0$ for all $t \in [0, \infty)$ and $l_1 = 0$. Hence the condition (C6) satisfies trivially. Note that the conditions (C5) and (C6) reduce to (C2) and (1.8), respectively. Hence all the conditions of Theorem 2.3 are satisfied. Thus the remaning portion follows from Theorem 2.3. □

In the special case, when $A(x) = F'_x$, $\omega(t) = Kt$, $\omega_0(t) = K_0 t$, $\omega_1(t) = 0$ and $l_0 = l_1 = 0$, we have the following result which follows from Theorem 2.5.

Corollary 2.6. *Let F be an operator defined on a closed convex subset D of a Hilbert space $\mathcal{H}$ with values in $\mathcal{H}$ such that F is continuously Fréchet differentiable at each point of D_0 and G be a maximal monotone operator from $\mathcal{H}$ into $\mathcal{H}$ satisfying*

(1.1). *For any* $x_0 \in D$, *assume that the operators* F, G *and* F'_x *satisfy* (C1)–(C3) *and* (1.6). *Let* $\eta = \frac{\beta}{c_0+\alpha}$ *and assume that the scalar equation defined by*

$$(c_0 + \alpha - K_0 r)(\eta - r) + \frac{1}{2} K r \eta = 0$$

has a minimum positive zero r^* *such that*

$$\frac{1}{2} K\eta + K_0 r^* < c_0 + \alpha$$

is satisfied. Suppose that $B_{r^*}[x_0] \subseteq D$. *Then we have the following:*

(1) *The sequence* $\{x_n\}$ *generated by* (1.4) *is well defined, remains in* $B_{r^*}[x_0]$ *and converges to a solution* $x^* \in B_{r^*}[x_0]$ *of* (1.2). *The error estimates* (2.5)–(2.7) *hold, where* $\{t_n\}$ *is a sequence generated by*

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+1} = t_n + \frac{K(t_n - t_{n-1})^2}{2(c_0 + \alpha - K_0 t_n)} \tag{2.12}$$

and t^* *is the limit of the sequence* $\{t_n\}$.

(2) *Further, if* $\left(\frac{1}{2}K + K_0\right) r^* < c_0 + \alpha$, *then the solution of*(1.2) *is unique in* $B_{r^*}[x_0]$.

Now, we discuss the existence and uniqueness of solutions of the mixed variational inequality problem given by (1.3). The following result follows from Corollary 2.6.

Theorem 2.7. *Let* F *be an operator defined on a closed convex subset* D *of a Hilbert space* $\mathcal{H}$ *with values in* $\mathcal{H}$ *such that* F *is continuously Fréchet differentiable at each point of* D_0 *and* $\phi : \mathcal{H} \to (-\infty, \infty]$ *be a proper lower semi-continuous convex function. Let* $G = \partial\phi$ *be the subgradient of* ϕ. *For any* $x_0 \in D$, *assume that the operators* F, G *and* F'_x *satisfy* (C1)–(C3), (1.1) *and* (1.6). *Let* $\eta = \frac{\beta}{c_0+\alpha}$ *and assume that the scalar equation defined by*

$$(c_0 + \alpha - K_0 r)(\eta - r) + \frac{1}{2} K r \eta = 0$$

has a minimum positive zero r^* *such that*

$$\frac{1}{2} K\eta + K_0 r^* < c_0 + \alpha$$

is satisfied. Suppose that $B_{r^*}[x_0] \subseteq D$. *Then we have the following:*

(1) *The sequence* $\{x_n\}$ *generated by*

$$F'_{x_n}(x_{n+1}) + \partial\phi(x_{n+1}) \ni F'_{x_n}(x_n) - F(x_n) \tag{2.13}$$

is well defined, remains in $B_{r^*}[x_0]$ *and converges to a solution* $x^* \in B_{r^*}[x_0]$ *of the variational inequality given by* (1.3). *The error estimates* (2.5)–(2.7) *hold, where* $\{t_n\}$ *is a sequence generated by* (2.12) *and* t^* *is the limit of the sequence* $\{t_n\}$ *defined by* (2.12).

(2) *Further, if* $\left(\frac{1}{2}K + K_0\right) r^* < c_0 + \alpha$, *then the solution of* (1.3) *is unique in* $B_{r^*}[x_0]$.

3. Numerical examples

In this section, we provide some numerical examples.

Example 3.1. Let $\mathcal{H} = D = \mathbb{R}$ and consider the problem of finding the zero of

$$x^2 + 3x + 1 = 0. \tag{3.1}$$

For $F(x) = x^2+x+1$, $G(x) = 2x$ and $A(x) = F'_x = 2x+1$, the equation (3.5) can be modeled as the problem (1.2). Since G is single-valued, we express the generalized Newton iterates (1.11) in the form

$$A(x_n)(x_{n+1}) + G(x_{n+1}) = A(x_n)(x_n) - F(x_n),$$

which can we written as in the following form:

$$(3 + 2x_n)x_{n+1} = {x_n}^2 - 1. \tag{3.2}$$

For $x_0 = 0$, we get

$$\beta = c_0 = 1,\ \alpha = 2,\ \omega(t) = \omega_0(t) = 2t,\ h(t) = t,\ \omega_1(t) = 0,\ l_0 = l_1 = 0$$

for all $t \geq 0$, which gives that $\eta = \frac{1}{3}$ and $H = \frac{1}{2}$. In this case, the scalar equation (2.1) reduces to

$$6r^2 - 10r + 3 = 0. \tag{3.3}$$

The minimum positive root r^* of the scalar equation (3.3) is given by $r^* = \frac{5-\sqrt{7}}{6}$. Since $\frac{6-\sqrt{7}}{3} < 3$, the condition (2.2) is satisfied. Thus all the conditions of Theorem 2.3 are satisfied. Hence Theorem 2.3 guarantees that the sequence $\{x_n\}$ generated by (3.2) converges to a unique solution $x^* \in B_{r^*}[x_0]$ of (3.5). The error estimates are given by (2.5)–(2.7), where the scalar sequence $\{t_n\}$ is given by

$$t_0 = 0, \quad t_1 = \frac{1}{3}, \quad t_{n+1} = t_n + \frac{(t_n - t_{n-1})^2}{3 - 2t_n}. \tag{3.4}$$

The convergence analysis of (3.2) and the difference rate of the sequence $\{t_n\}$ is given in Table 1.

n	x_n	$t_{n+1} - t_n$
0	0	0.333333333333333
1	-0.333333333333333	0.047619047619048
2	-0.380952380952381	0.001013171225937
3	-0.381965552178318	0.000000459071693
4	-0.381966011250011	0.000000000000094
5	-0.381966011250105	0
6	-0.381966011250105	0
7	$\cdots$	$\cdots$

Table 1. The convergence of (3.2)

Example 3.2. Let $\mathcal{H} = D = \mathbb{R}$ and consider the problem of finding the zero of

$$F(x) + G(x) \ni 0, \tag{3.5}$$

where $F(x) = x^2 + 3x$, $G(x) = \partial g(x)$, $g(x) = |x|$ and $A(x) = F'_x = 2x + 3$ for all $x \in \mathbb{R}$. Recall that $\partial g(x)$ is the set defined by

$$\partial g(x) = \{z \in \mathbb{R} : g(x) - g(y) \geq z(x - y), \forall y \in \mathbb{R}\}.$$

A straightforward computation gives

$$G(x) = \partial g(x) = \begin{cases} -1, & x < 0; \\ [-1, 1], & x = 0; \\ 1, & x > 0. \end{cases}$$

In this case, the generalized Newton iterates (2.13) can be written in the following form:

$$2x_n x_{n+1} + G(x_{n+1}) \ni x_n^2. \tag{3.6}$$

For $x_0 = 0$, choose $y_0 = 0$ and

$$\beta = 1,\ c_0 = 3,\ \alpha = 0,\ \omega(t) = \omega_0(t) = 2t,$$

$$\omega_1(t) = 0,\ h(t) = t,\ l_0 = l_1 = 0,$$

which gives that $\eta = \frac{1}{3}$ and $H = \frac{1}{2}$. In this case, the scalar equation (2.1) reduces to the scalar equation given in (3.3). As in Example 3.1, the minimum positive root r^* of the scalar equation (3.3) is given by $r^* = \frac{5-\sqrt{7}}{6}$. Since $\frac{6-\sqrt{7}}{3} < 3$, the condition (2.2) is satisfied. Thus all the conditions of Corollary 2.6 are satisfied. Hence Corollary 2.6 guarantees that the sequence $\{x_n\}$ generated by (3.6) converges to a unique solution $x^* \in B_{r^*}[x_0]$ of (3.5). The error estimates are given by (2.5)–(2.7), where the scalar sequence $\{t_n\}$ is given by (3.4). Using the definition of G and (3.6), we can also verify that $\{x_n\}$ converges to 0. Indeed, $x_n = 0$ for all $n \geq 1$.

References

[1] I. K. Argyros, Y. J. Cho and S. Hilout, *Numerical Methods for Equations and its Applications*, CRC Press, Taylor and Francis, New York, 2012.

[2] I. K. Argyros and S. Hilout, *On the solution of generalized equations and variational inequalities*, Cubo **13** (2011), 45–60.

[3] I. K. Argyros and S. Hilout, *Improved generalized differentiability conditions for Newton-like methods*, J. Complexity **26** (2010), 316–333.

[4] C. Baiocchi and A. Capelo, *Variational and Quasi-variational inequalities, Applications to Free Boundary Problems*, Wiley, New York/London, 1984.

[5] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer-Verlag, Berlin, 1984.

[6] P. T. Harker and J. S. Pang, *Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications*, Math. Prog. **48** (1990), 161–220.

[7] G. J. Minty, *Monotone (nonlinear) operators in Hilbert space*, Duke Math. J. **29** (1973), 341–346.

[8] G. J. Minty, *On the monotonicity of the gradient of a convex function*, Pacific J. Math. **14** (1964), 243–247.

[9] M. A. Noor, *Some recent advances in variational inequalities I*, New Zeal. J. Math. **26** (1997), 53–80; II, 229–255.

[10] S. M. Robinson, *Generalized equations and their solutions, Part 1: Basic theory*, Math. Program. Study **10** (1979), 128–141.

[11] S. M. Robinson, *Generalized Equations*, in Math. Prog., A. Bachem, M. Gretschel and B. Korle, eds, the State of the Art, Springer, Berlin, 1982, pp. 346–367.

[12] G. Stampacchia, *Formes bilineaires sur les ensemble convexes*, C. R. Acad. Sci. Paris **258** (1964), 4413–4416.

[13] L. U. Uko, *Generalized equations and the generalized Newton method*, Math. Prog. **73** (1996), 251–268.

[14] L. U. Uko, *The generalized Newton's method*, J. Nig. Math. Soci. **10** (1991), 55–64.

[15] L. U. Uko, *Remarks on the generalized Newton method*, Math. Prog. **59** (1993), 405–412.

[16] L. U. Uko and J. O. Adeyeye, *Newton-type iterative schemes for generalized equations*, Nonlinear Anal. **63** (2005), 2289–2299.

[17] L. U. Uko and I. K. Argyros, *Generalized equations, variational inequalities and a weak Kantorovich theorem*, Numer. Algor. **52** (2009), 321–333.

[18] L. Kohaupt, *A Newton-like method for the numerical solution of nonlinear Fredholm-type operator equations*, Appl. Math. Comput. **218** (2012), 10129–10148.

[19] P. D. Proinov, *New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems*, J. Complexity **26** (2010), 3–42.

[20] J. A. Ezquerro and M. A. Hernández, *Generalized differentiability conditions for Newton's method*, IMA J. Numer. Anal. **22** (2002), 187–205.

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