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DIRECT SUMS OF BANACH SPACES WITH FPP WHICH FAIL TO BE UNIFORMLY NON-SQUARE

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Dedicated to Professor Sompong Dhompongsa on the occasion of his 65th birthday.

Abstract. This is a survey on some recent results on direct sums of Banach spaces, especially concerning uniform non-ℓ_1^n-ness and weak nearly uniform smoothness with application to the fixed point property for nonexpansive mappings.

1. Introduction and preliminaries

Recently the present authors have discussed uniform non-ℓ_1^n-ness and weak nearly uniform smoothness for ψ-direct sums of Banach spaces ([9, 11, 13, 14, 15, 16, 17, 18]). The starting point on these themes is the following: *A ψ-direct sum $X \oplus_\psi Y$ is uniformly non-square* (*UNSQ*) *if and only if X and Y are UNSQ and neither $\psi = \psi_1$ nor $\psi = \psi_\infty$, where ψ_1 and ψ_∞ are the corresponding convex functions to the ℓ_1- and ℓ_∞-norms, respectively* ([9]). Our first concern is to extend this result to the uniform non-ℓ_1^n-ness and also to investigate the extreme cases, ℓ_1- and ℓ_∞-sums ([12, 14, 15]). The next interest is to extend the above result to the N Banach spces case. In the course of trying this we treated the weak nearly uniform smoothness ([13, 16], cf. [17, 18]). In the 2-dimensional case we have the following: *A ψ-direct sum $X \oplus_\psi Y$ is weakly nearly uniformly smooth* (*WNUS*) *if and only if X and Y are WNUS and $\psi \neq \psi_1$* ([13]). This was extended to the N-dimensional case by introducing a class of convex functions $\Psi_N^{(1)}$ which yield partial ℓ_1-norms; we need to remove these functions more than the function ψ_1 ([16], see also [17, 18]).

The aim of this paper is to present a survey on these results in relation to the fixed point property for non-expansive mappings (FPP). In particular, keeping it in mind that *all uniformly non-square Banach spaces have FPP* (García-Falset, et al. [8]), we shall present a plenty of direct sums of Banch spaces with FPP which are not uniformly non-square.

A norm $\|\cdot\|$ on $\mathbb{C}^N$ is called *absolute* if $\|(z_1,\dots,z_N)\| = \|(|z_1|,\dots,|z_N|)\|$ for all $(z_1,\dots,z_N) \in \mathbb{C}^N$, and *normalized* if $\|(1,0,\dots,0)\| = \cdots = \|(0,\dots,0,1)\| = 1$. A norm $\|\cdot\|$ on $\mathbb{C}^N$ is called *monotone* if

$$|z_j| \le |w_j| \text{ for all } 1 \le j \le N \implies \|(z_1,\dots,z_N)\| \le \|(w_1,\dots,w_N)\|.$$

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We note that a norm $\|\cdot\|$ on $\mathbb{C}^N$ is absolute if and only if it is monotone (Bhatia [1], cf. [18]). For any absolute normalized norm $\|\cdot\|$ on $\mathbb{C}^N$ let

$$\psi(s) = \Big\|(1 - \sum_{i=1}^{N-1} s_i, s_1, \dots, s_{N-1})\Big\| \text{ for } s = (s_1, \dots, s_{N-1}) \in \Delta_N, \tag{1.1}$$

where

$$\Delta_N = \Big\{ s = (s_1, \dots, s_{N-1}) \in \mathbb{R}^{N-1} : \sum_{i=1}^{N-1} s_i \le 1,\ s_i \ge 0 \Big\}.$$

Then ψ is convex (continuous) on Δ_N and satisfies the following:

$$(A_0) \qquad \psi(0, \dots, 0) = \psi(1, 0, \dots, 0) = \cdots = \psi(0, \dots, 0, 1) = 1,$$

$$(A_1)\ \psi(s_1, \dots, s_{N-1}) \ge \Big(\sum_{i=1}^{N-1} s_i\Big) \psi\Big(\frac{s_1}{\sum_{i=1}^{N-1} s_i}, \dots, \frac{s_{N-1}}{\sum_{i=1}^{N-1} s_i}\Big) \quad \text{if } 0 < \sum_{i=1}^{N-1} s_i \le 1,$$

$$(A_2)\ \psi(s_1, \dots, s_{N-1}) \ge (1 - s_1) \psi\Big(0, \frac{s_2}{1 - s_1}, \dots, \frac{s_{N-1}}{1 - s_1}\Big) \quad \text{if } 0 \le s_1 < 1,$$

$$\cdots\cdots\cdots$$

$$(A_N)\ \psi(s_1, \dots, s_{N-1}) \ge (1 - s_{N-1}) \psi\Big(\frac{s_1}{1 - s_{N-1}}, \dots, \frac{s_{N-2}}{1 - s_{N-1}}, 0\Big) \text{ if } 0 \le s_{N-1} < 1.$$

In fact, the condition (A_0) means that the norm $\|\cdot\|$ is normalized. For the others, since $\|\cdot\|$ is monotone, we have

$$\begin{aligned}
(M_1)\ \Big\|(1 - \sum_{i=1}^{N-1} s_i, s_1, \dots, s_{N-1})\Big\| &\ge \|(0, s_1, \dots, s_{N-1})\|, \\
(M_2)\ \Big\|(1 - \sum_{i=1}^{N-1} s_i, s_1, \dots, s_{N-1})\Big\| &\ge \Big\|(1 - \sum_{i=1}^{N-1} s_i, 0, s_2, \dots, s_{N-1})\Big\|, \\
&\cdots\cdots\cdots \\
(M_N)\ \Big\|(1 - \sum_{i=1}^{N-1} s_i, s_1, \dots, s_{N-1})\Big\| &\ge \Big\|(1 - \sum_{i=1}^{N-1} s_i, s_1, \dots, s_{N-2}, 0)\Big\|.
\end{aligned}$$

By interpreting $(M_1) - (M_N)$ in words of ψ we obtain $(A_1) - (A_N)$.

Let Ψ_N denote the class of all convex functions ψ on Δ_N satisfying $(A_0) - (A_N)$. Then, conversely, for any $\psi \in \Psi_N$ let

$$\|(z_1, \dots, z_N)\|_\psi = \begin{cases} \Big(\sum_{j=1}^N z_j\Big) \psi\Big(\frac{|z_2|}{\sum_{j=1}^N z_j}, \dots, \frac{|z_N|}{\sum_{j=1}^N z_j}\Big) & \text{if } (z_1, \dots, z_N) \ne (0, \dots, 0), \\ 0 & \text{if } (z_1, \dots, z_N) = (0, \dots, 0). \end{cases} \tag{1.2}$$

Then $\|\cdot\|_\psi$ is an absolute normalized norm on $\mathbb{C}^N$ and satisfies (1.1) ([21]; see [2] for the case $N=2$). We refer to the norm $\|\cdot\|_\psi$ as ψ-norm. The ℓ_p-norms

$$\|(z_1,\dots,z_N)\|_p = \begin{cases} \{|z_1|^p+\cdots+|z_N|^p\}^{1/p} & \text{if } 1\le p<\infty, \\ \max\{|z_1|,\dots,|z_N|\} & \text{if } p=\infty \end{cases}$$

are basic examples and their corresponding convex functions ψ_p are given by

$$\psi_p(s_1,\dots,s_{N-1}) = \begin{cases} \left\{\left(1-\sum_{i=1}^{N-1} s_i\right)^p + s_1^p+\cdots+s_{N-1}^p\right\}^{1/p} & \text{if } 1\le p<\infty, \\ \max\{1-\sum_{i=1}^{N-1} s_i, s_1,\dots,s_{N-1}\} & \text{if } p=\infty. \end{cases}$$

In particular the function $\psi_1(t)=1$ corresponds to the ℓ_1-norm. For all $\psi\in\Psi_N$ we have $\|\cdot\|_\infty\le\|\cdot\|_\psi\le\|\cdot\|_1$ ([21]).

Let $X_1,\dots,X_N$ be Banach spaces and let $\psi\in\Psi_N$. The ψ-direct sum $(X_1\oplus\cdots\oplus X_N)_\psi$ is their direct sum $X_1\oplus\cdots\oplus X_N$ equipped with the norm

$$\|(x_1,\dots,x_N)\|_\psi := \|(\|x_1\|,\dots,\|x_N\|)\|_\psi \text{ for } (x_1,\dots,x_N)\in X_1\oplus\cdots\oplus X_N$$

([9, 22]). As usual S_X stands for the unit sphere of a Banach space X. X is called *uniformly non-square* provided there exists ε $(0<\varepsilon<1)$ such that

$$\min\{\|x+y\|,\|x-y\|\}\le 2(1-\varepsilon) \text{ for all } x,y\in S_X.$$

More generally, X is called *uniformly non-*ℓ_1^n provided there exists ε $(0<\varepsilon<1)$ such that for all $x_1,\dots,x_n\in S_X$ there exists $\theta=(\theta_j)$ (an n-tuple of signs) for which

$$\left\|\sum_{j=1}^n \theta_j x_j\right\| \le n(1-\varepsilon). \tag{1.3}$$

Here the unit sphere S_X can be replaced with the colsed unit ball of X (cf. [11]). If $n=2$, uniform non-ℓ_1^2-ness coincides with uniform non-squareness. If $n=3$, uniform non-ℓ_1^3 spaces are called *uniformly non-octahedral.* If $n=1$, the formal definition is possible, but no Banach space is uniformly non-ℓ_1^1. Every uniformly non-ℓ_1^n space is uniformly non-ℓ_1^{n+1}.

A Banach space X is said to have the *fixed point property* (resp. *weak fixed point property*) *for nonexpansive mappings* if every nonexpansive self-mapping T of any nonempty bounded closed (resp. weakly compact) convex subset C of X has a fixed point, where T is called *nonexpansive* if $\|Tx-Ty\|\le\|x-y\|$ for all $x,y\in C$. We say the former as *FPP* (resp. *WFPP*) in short.

2. Uniform non-ℓ_1^n-ness

In this section we shall discuss uniform non-ℓ_1^n-ness for direct sums of Banach spaces.

Theorem 2.1 (Kato-Saito-Tamura [10])**.** *The following are equivalent.*
(i) $X\oplus_\psi Y$ *is uniformly non-square.*
(ii) X *and* Y *are uniformly non-square and* $\psi\ne\psi_1,\psi_\infty$.

This is extended to the uniform non ℓ_1^n-ness.

Theorem 2.2 (Kato-Saito-Tamura [12]). *Assume that neither X nor Y is uniformly non-ℓ_1^{n-1}. Then the following are equivalent.*

(i) $X \oplus_\psi Y$ *is uniformly non-ℓ_1^n.*

(ii) X *and* Y *are uniformly non-ℓ_1^n and $\psi \neq \psi_1, \psi_\infty$.*

Remark 2.3. (i) Theorem 2.2 includes Theorem 2.1 as the case $n = 2$, since no Banach space is uniformly non-ℓ_1^1.

(ii) We cannot remove the condition that neither X nor Y is uniformly non-ℓ_1^{n-1}.

Theorem 2.1 asserts that $X \oplus_1 Y$ and $X \oplus_\infty Y$ cannot be uniformly non-square for all X and Y. This is also readily seen by the fact that ℓ_1^2 and ℓ_∞^2 are not uniformly non-square since these spaces are regarded as subspaces of $X \oplus_1 Y$ and $X \oplus_\infty Y$, respectively. On the other hand, Theorem 2.2 indicates that if X and Y are uniformly non-ℓ_1^{n-1} (or if one of them is so for $X \oplus_\infty Y$), $X \oplus_1 Y$ and $X \oplus_\infty Y$ can be uniformly non-ℓ_1^n ($n \geq 3$). Thus we shall confine ourselves to these extreme cases.

Theorem 2.4 (Kato-Tamura [14]). *The following are equivalent.*

(i) $X \oplus_1 Y$ *is uniformly non-ℓ_1^n, $n \geq 3$.*

(ii) *There exist $n_1, n_2 \in \mathbb{N}$ with $n_1 + n_2 = n - 1$ such that X is uniformly non-$\ell_1^{n_1+1}$ and Y is uniformly non-$\ell_1^{n_2+1}$.*

As the case $N = 3$ we have the following.

Corollary 2.5 (Kato-Saito-Tamura [12]). *The following are equivalent.*

(i) $X \oplus_1 Y$ *is uniformly non-ℓ_1^3.*

(ii) X *and* Y *are uniformly non-square.*

For the ℓ_∞-sum we have the following ([12]): *Let X and Y be uniformly non-square. Then $X \oplus_\infty Y$ is uniformly non-ℓ_1^3. The converse is not true* (see Example 1 below). For three Banach spaces we have the following.

Theorem 2.6 (Kato-Saito-Tamura [12]). *The following are equivalent.*

(i) $(X \oplus Y \oplus Z)_\infty$ *is uniformly non-ℓ_1^3.*

(ii) X*,* Y *and* Z *are uniformly non-square.*

Example 2.7. Let X, Y and Z be uniformly non-square and let $W = Y \oplus_\infty Z$. Then $X \oplus_\infty W = (X \oplus Y \oplus Z)_\infty$ is uniformly non-ℓ_1^3 by Theorem 2.6, while $W = Y \oplus_\infty Z$ is not uniformly non-square.

Theorem 2.4 is extended as follows.

Theorem 2.8 (Kato-Tamura [14]). *The following are equivalent.*

(i) $(X_1 \oplus \cdots \oplus X_N)_1$ *is uniformly non-ℓ_1^n.*

(ii) *There exist N positive integers $n_1, \ldots, n_N$ with $n_1 + n_2 + \cdots + n_N = n - 1$ such that X_j is uniformly non-$\ell_1^{n_j+1}$ for all $1 \leq j \leq N$.*

The space $(X_1 \oplus \cdots \oplus X_n)_1$ cannot be uniformly non-ℓ_1^n. To the contrary, by Theorem 2.8 we have the next result which extends Corollary 2.5.

Theorem 2.9 (Kato-Tamura [14]). *The following are equivalent.*

(i) $(X_1 \oplus \cdots \oplus X_n)_1$ *is uniformly non-ℓ_1^{n+1}.*

(ii) $X_1, \ldots, X_n$ *are uniformly non-square.*

Concerning the ℓ_∞-sum we have the following result which extends Theorem 2.6.

Theorem 2.10 (Kato-Tamura [15]). *Let $n \geq 2$. The following are equivalent.*
(i) $(X_1 \oplus \cdots \oplus X_{2^n-1})_\infty$ *is uniformly non-*ℓ_1^{n+1}.
(ii) $X_1, \dots, X_{2^n-1}$ *are uniformly non-square.*

3. Weak nearly uniform smootheness

First we shall discuss partial ℓ_1-norms, which are recently introduced by the present authors.

Definition 3.1 (cf. [16]). An absolute normalized norm $\|\cdot\|$ on $\mathbb{C}^N$ is called *partial ℓ_1-norm* if there esists $\boldsymbol{a} = (a_1, \dots, a_N) \in \mathbb{R}_+^N$ such that with some nonempty proper subset T of $\{1, \dots, N\}$

$$\|(a_1, \dots, a_N)\| = \|(\chi_T(1)a_1, \dots, \chi_T(N)a_N)\| + \|(\chi_{T^c}(1)a_1, \dots, \chi_{T^c}(N)a_N)\|,$$

where $(\chi_T(1)a_1, \dots, \chi_T(N)a_N)$ and $(\chi_{T^c}(1)a_1, \dots, \chi_{T^c}(N)a_N)$ are nonzero. Let $\Psi_N^{(1)}$ denote the class of convex functions $\psi \in \Psi_N$ for which $\|\cdot\|_\psi$ is a partial ℓ_1-norm.

Theorem 3.2 ([16]). *Let $\psi \in \Psi_N$. The following are equivalent.*
(i) $\psi \in \Psi_N^{(1)}$.
(ii) *There exists $\boldsymbol{a} = (a_1, \dots, a_N) \in \mathbb{R}_+^N$ such that with some nonempty proper subset T of $\{1, \dots, N\}$*

$$\|(a_1, \dots, a_N)\|_\psi = \|(\chi_T(1)a_1, \dots, \chi_T(N)a_N)\|_\psi + \|(\chi_{T^c}(1)a_1, \dots, \chi_{T^c}(N)a_N)\|_\psi,$$

where $\|(\chi_T(1)a_1, \dots, \chi_T(N)a_N)\|_\psi = \|(\chi_{T^c}(1)a_1, \dots, \chi_{T^c}(N)a_N)\|_\psi = 1$.
(iii) *There exists $(s_1, \dots, s_{N-1}) \in \Delta_N$ with $0 < M := \sum_{i=1}^{N-1} \chi_S(i)s_i < 1$ for some nonempty subset S of $\{1, \dots, N-1\}$ such that*

$$\begin{aligned}\psi(s_1, \dots, s_{N-1}) &= M\psi\left(\frac{\chi_S(1)s_1}{M}, \dots, \frac{\chi_S(N-1)s_{N-1}}{M}\right)\\ &\quad +(1-M)\psi\left(\frac{\chi_{S^c}(1)s_1}{1-M}, \dots, \frac{\chi_{S^c}(N-1)s_{N-1}}{1-M}\right),\end{aligned}$$

where χ_S denotes the characteristic function of the set S.

We note that the implication (i) $\Rightarrow$ (ii) is obtained owing to the sharp triangle inequality ([11]). The equivalence of (i) and (iii) is merely reformulation of Definition 3.1 by means of the convex function ψ.

Example 3.3 (cf. [18]). Let $N \geq 3$. We consider the absolute normalized norm

$$\|(a_1, \dots, a_N)\| = \max\left\{|a_1|, \dots, |a_N|, \frac{1}{2}\sum_{j=1}^N |a_j|\right\}.$$

The corresponding convex function $\psi \in \Psi_N$ is given by

$$\psi(s_1, \dots, s_{N-1}) = \max\left\{1 - \sum_{i=1}^{N-1} s_i,\ s_1,\ \dots,\ s_{N-1},\ \frac{1}{2}\right\}.$$

Since

$$\|(1,\dots,1)\|_\psi = \|(1,1,0,\dots,0)\|_\psi + \|(0,0,1,\dots,1)\|_\psi,$$

this norm is a partial ℓ_1-norm and hence $\psi \in \Psi_N^{(1)}$.

Proposition 3.4 ([16])**.** *Let $\psi \in \Psi_N$ be strictly convex. Then $\psi \notin \Psi_N^{(1)}$.*

Proof. Assume that $\psi \in \Psi_N^{(1)}$. Then by Theorem 3.2 there exist $\boldsymbol{a} = (a_1,\dots,a_N) \in \mathbb{R}_+^N$ and a nonempty proper subset T of $\{1,\dots,N\}$ such that

$$\|(a_1,\dots,a_N)\|_\psi = \|(\chi_T(1)a_1,\dots,\chi_T(N)a_N)\|_\psi + \|(\chi_{T^c}(1)a_1,\dots,\chi_{T^c}(N)a_N)\|_\psi$$

and

$$\|(\chi_T(1)a_1,\dots,\chi_T(N)a_N)\|_\psi = \|(\chi_{T^c}(1)a_1,\dots,\chi_{T^c}(N)a_N)\|_\psi = 1.$$

Therefore the norm $\|\cdot\|_\psi$ is not strictly convex, which is a contradiction. (Note that $\|\cdot\|_\psi$ is strictly convex if and only if ψ is strictly convex.) □

Example 3.5 ([16])**.** Let $1 < p \le \infty$. Then $\psi_p \notin \Psi_N^{(1)}$.

Indeed if $1 < p < \infty$, the ℓ_p-norm, and hence ψ_p is strictly convex. Therefore we have $\psi_p \notin \Psi_N^{(1)}$ by Proposition 3.4. Let $p = \infty$. Suppose that $\psi_\infty \in \Psi_N^{(1)}$. Then there exist $(a_1,\dots,a_N) \in \mathbb{R}_+^N$ and a nonempty proper subset T of $\{1,\dots,N\}$ such that

$$\|(a_1,\dots,a_N)\|_\infty = \|(\chi_T(1)a_1,\dots,\chi_T(N)a_N)\|_\infty + \|(\chi_{T^c}(1)a_1,\dots,\chi_{T^c}(N)a_N)\|_\infty,$$

where $(\chi_T(1)a_1,\dots,\chi_T(N)a_N)$ and $(\chi_{T^c}(1)a_1,\dots,\chi_{T^c}(N)a_N)$ are nonzero. Since

$$\|(a_1,\dots,a_N)\|_\infty = \|(\chi_T(1)a_1,\dots,\chi_T(N)a_N)\|_\infty$$

or

$$\|(a_1,\dots,a_N)\|_\infty = \|(\chi_{T^c}(1)a_1,\dots,\chi_{T^c}(N)a_N)\|_\infty,$$

we have a contradiction. Consequently, $\psi_\infty \notin \Psi_N^{(1)}$.

Now, a Banach space X is called *weakly nearly uniformly smooth* (WNUS) ([19], cf. [20]) if there exist $\varepsilon < 1$ and $\nu > 0$ such that for any basic sequence $\{x_n\}$ in B_X and any $0 < t < \nu$ there is $k > 0$ so that $\|x_1 + tx_k\| \le 1 + t\varepsilon$. According to García-Falset [6] *X is WNUS if and only if X is reflexive and $R(X) < 2$.* Here the *García-Falset coefficient* $R(X)$ is defined by

$$R(X) = \sup\{\liminf_{n\to\infty} \|x_n + x\|\},$$

where the supremum is taken over all weakly null sequences $\{x_n\}$ in B_X and all $x \in B_X$. It is known that *uniformly convex, resp., uniformly smooth spaces are WNUS, and WNUS spaces have FPP* (García-Falset [7]).

Theorem 3.6 (Kato-Tamura [16])**.** *Let $X_1,\dots,X_N$ be infinite dimensional. Let $\psi \in \Psi_N$. Then the following are equivalent.*

(i) *$(X_1 \oplus \cdots \oplus X_N)_\psi$ is weakly nearly uniformly smooth.*

(ii) *All $X_1,\dots,X_N$ are weakly nearly uniformly smooth and $\psi \notin \Psi_N^{(1)}$.*

Remark 3.7. The implication (ii) ⇒ (i) holds without the assumption on dimension. We refer the reader to [16] for the other cases on dimension.

Since strictly convex functions in Ψ_N are not in $\Psi_N^{(1)}$, the following result by Dhompongsa et al. are obtained as a corollary.

Corollary 3.8 (Dhompongsa-Kaewcharoen-Kaewkhao [3]). *Let $\psi \in \Psi_N$ be strictly convex. Then the following are equivalent.*
(i) $(X_1 \oplus \cdots \oplus X_N)_\psi$ *is weakly nearly uniformly smooth.*
(ii) *All $X_1, \ldots, X_N$ are weakly nearly uniformly smooth.*

4. Applications to FPP

It is well known that *all uniformly non-square spaces have FPP* (Theorem A below). In this section we shall construct some direct sums of Banach spaces with FPP which are not uniformly non-square.

We shall first discuss FPP for uniformly non-octahedral spaces. We need some previous results. For $0 \le a \le 1$ let

$$R(a, X) = \sup\left\{\liminf_{n\to\infty} \|x_n + x\|\right\}, \tag{4.1}$$

where the supremum is taken over all $x \in X$ with $\|x\| \le a$ and all weakly null sequences $\{x_n\}$ in the unit ball of X such that $\lim_{n,m\to\infty; n\neq m} \|x_n - x_m\| \le 1$ (Domínguez Benavides [4]).

Theorem A (Domínguez Benavides [4]). *Let $R(1, X) < 2$. Then X has WFPP.*

In 2006 García-Falset et.al obtained the next result.

Theorem B (García-Falset, et al. [8]). *Let X be uniformly non-square. Then $R(1, X) < 2$, and hence X has FPP.*

Since all uniformly non-square spaces have FPP, it is natural to ask whether all uniformly non-octahedral (uniformly non ℓ_1^3) spaces have FPP. We have the following.

Theorem 4.1 (Kato-Tamura [15]). *Let X be uniformly non-octahedral. If X is isometric to an ℓ_∞-sum of 3 Banach spaces, then X has FPP, while X is not uniformly non-square.*

More generally we have

Theorem 4.2 (Kato-Tamura [15]). *Let X be uniformly non-ℓ_1^{n+1}. If X is isometric to an ℓ_∞-sum of $2^n - 1$ Banach spaces, then X has FPP, while X is not uniformly non-square.*

To present a proof of Theorem 4.2 we need the next result.

Lemma 4.3 ([15]). *For all Banach spaces $X_1, \ldots, X_N$*

$$R(1, (X_1 \oplus \cdots \oplus X_m)_\infty) = \max_{1\le j\le m} R(1, X_j).$$

Proof of Theorem 4.2. Assume that $X = (X_1 \oplus \cdots \oplus X_{2^n-1})_\infty$ is uniformly non-ℓ_1^{n+1}. Then, by Theorem 3.6 all $X_1, \ldots, X_{2^n-1}$ are uniformly non-square. Therefore, by Theorem B

$$R(1, X_j) < 2 \text{ for all } 1 \le j \le 2^n - 1.$$

Hence, by Lemma 4.3 we have

$$R(1, (X_1 \oplus \cdots \oplus X_m)_\infty) < 2,$$

which implies that $X = (X_1 \oplus \cdots \oplus X_m)_\infty$ has WFPP by Theorem A. Since X is reflexive, X has FPP. □

Example 4.4. Since L_p, $1 < p < \infty$, is uniformly convex, a fortiori, uniformly non-square, the space $X = (L_p \oplus L_p \oplus L_p)_\infty$ is uniformly non-octahedoral by Theorem 2.6. Therefore X has FPP by Theorem 4.1, while it is not uniformly non-square.

Next by using Theorem 3.6 we shall construct a plenty of Banach spaces with FPP failing to be UNSQ.

Proposition 4.5 (Kato-Tamura [16])**.** *Let $\varphi \in \Psi_2$, $\varphi \neq \psi_1$ and define $\psi \in \Psi_N$ by*

$$\psi(s_1, \ldots, s_{N-1}) = \max\left\{ \|(1 - \sum_{i=1}^{N-1} s_i, s_1)\|_\varphi,\ \|(s_1, s_2)\|_\varphi,\ \|(s_2, s_3)\|_\varphi, \ldots, \|(s_{N-2}, s_{N-1})\|_\varphi \right\}$$

for $(s_1, \ldots, s_{N-1}) \in \Delta_N$.

Then $\psi \notin \Psi_N^{(1)}$ and $\|\cdot\|_\psi$ is not uniformly non-square.

We note that the corresponding norm is

$$\|(a_1, a_2 \ldots, a_N)\|_\psi = \max\{\|(a_1, a_2)\|_\varphi, \|(a_2, a_3)\|_\varphi, \ldots, \|(a_{N-1}, a_N)\|_\varphi\} \quad \text{for } (a_1, \ldots, a_N) \in \mathbb{C}^N$$

In fact, we considered first this norm which is in AN_N, and the above ψ was derived by $\psi(s) = \|(1 - \sum_{i=1}^{N-1} s_i, s_1, \ldots, s_{N-1})\|$. Therefore $\psi \in \Psi_N$. Then we have $\psi \notin \Psi_N^{(1)}$ and the norm $\|\cdot\|_\psi$ is not UNSQ.

Theorem 4.6 (Kato-Tamura [16])**.** *Let $X_1, \ldots, X_N$ be weakly nearly uniformly smooth, $N \ge 3$. Let $\psi \in \Psi_N$ be as in Proposition 4.5. Then $(X_1 \oplus \cdots \oplus X_N)_\psi$ has FPP, whereas it is not uniformly non-square.*

Proof. By Theorem 3.6 and Remark 3.7, $(X_1 \oplus \cdots \oplus X_N)_\psi$ is WNUS, and hence has FPP. On the other hand, it is not UNSQ since $(\mathbb{C}^N, \|\cdot\|_\psi)$ is not so by Proposition 4.5. □

By Theorem 3.6 and Remark 3.7 we also obtain

Corollary 4.7. *Let $X_1, \ldots, X_N$ be weakly nearly uniformly smooth, $N \ge 3$. Then $(X_1 \oplus \cdots \oplus X_N)_\infty$ has FPP, whereas it is not uniformly non-square.*

Example 4.8. Let

$$\psi(s_1, \ldots, s_{N-1})$$

$$= \max\left\{\|(1-\sum_{i=1}^{N-1} s_i, s_1)\|_2,\ \|(s_1,s_2)\|_2,\ \|(s_2,s_3)\|_2,\dots,\|(s_{N-2},s_{N-1})\|_2\right\}$$
$$\text{for } (s_1,\dots,s_{N-1})\in\Delta_N.$$

The corresponding norm

$$\|(a_1,a_2\dots,a_N)\|_\psi = \max\{\|(a_1,a_2)\|_2, \|(a_2,a_3)\|_2,\dots,\|(a_{N-1},a_N)\|_2\}$$

is not partial ℓ_1 by Proposition 4.5. Since L_{p_j}, $1 < p_j < \infty$ $(1 \le j \le N)$, are uniformly convex and hence WNUS, the space $X = (L_{p_1} \oplus \cdots \oplus L_{p_N})_\psi$ has FPP, while it is not uniformly non-square by Theorem 4.6. Also, the ℓ_∞-sum $X = (L_{p_1} \oplus \cdots \oplus L_{p_N})_\infty$, which is not uniformly non-square, has FPP by Theorem 3.6.

5. Concluding remarks

As another notion of direct sum of Banach spaces the Z-direct sum is discussed (cf. [5]). Let Z be a finite dimensional normed space $(\mathbb{R}^N, \|\cdot\|_Z)$, whose norm is monotone in $\mathbb{R}^N_+$, that is, $\|(a_1,\dots,a_N)\|_Z \le \|(b_1,\dots,b_N)\|_Z$ if $0 \le a_j \le b_j$ for all $1 \le j \le N$. The Z-direct sum $(X_1 \oplus \cdots \oplus X_N)_Z$ of $X_1,\dots,X_N$ is their direct sum equipped with the norm

$$\|(x_1,\dots,x_N)\|_Z := \|(\|x_1\|,\dots,\|x_N\|)\|_Z \text{ for } (x_1,\dots,x_N) \in X_1 \oplus \cdots \oplus X_N,$$

where the norm $\|\cdot\|_Z$ on $\mathbb{R}^N$ is assumed to be absolute without loss of generality. Clearly, the Z-direct sum is more general than the ψ-direct sum. On the other hand, as is mensioned in [5], any Z-direct sum is isometrically isomorphic to a ψ-direct sum. Thus, these notions are equivalent. This is true for a more general direct sum ([18]).

According to Dowling and Saejung [5], a norm $\|\cdot\|$ on $\mathbb{C}^N$ is said to have *property* T_1^N if for all $\boldsymbol{a} = (a_1,\dots,a_N), \boldsymbol{b} = (b_1,\dots,b_N) \in \mathbb{C}^N$ with $\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = \frac{1}{2}\|\boldsymbol{a}+\boldsymbol{b}\| = 1$ it follows that supp $\boldsymbol{a} \cap$ supp $\boldsymbol{b} \ne \emptyset$, where supp $\boldsymbol{a} = \{j : a_j \ne 0\}$. Also $\|\cdot\|$ is said to have *property* T_∞^N if for all $\boldsymbol{a} = (a_1,\dots,a_N), \boldsymbol{b} = (b_1,\dots,b_N) \in \mathbb{C}^N$ with $\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = \|\boldsymbol{a}+\boldsymbol{b}\| = 1$ it follows that supp $\boldsymbol{a} \cap$ supp $\boldsymbol{b} \ne \emptyset$. Using these notions, they characterized the uniform non-squareness of a Z-direct sum, or equivalently, a ψ-direct sum under the condition that the norm $\|\cdot\|$ (or $\|\cdot\|_\psi$) is strictly monotone (for $N = 3$ without this condition). These properties for an absolute norm are interpreted in words of partial ℓ_1-norms or the class $\Psi_N^{(1)}$ as follows.

Proposition 5.1 (Kato-Tamura [18]). *Let $\psi \in \Psi_N$.*

(i) *The ψ-norm $\|\cdot\|_\psi$ has property T_1^N if and only if $\psi \notin \Psi_N^{(1)}$.*

(ii) *The ψ-norm $\|\cdot\|_\psi$ has property T_∞^N if and only if $\psi^* \notin \Psi_N^{(1)}$, where ψ^* is the dual function of ψ which corresponds to the dual norm of $\|\cdot\|_\psi$.*

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