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FIXED POINT THEOREMS FOR TWO HYBRID PAIRS OF NON-SELF MAPPINGS UNDER JOINT COMMON LIMIT RANGE PROPERTY IN METRIC SPACES

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Dedicated to Professor Sompong Dhompongsa on the occasion of his 65th birthday

Abstract. In this paper, we introduce the notion of joint common limit range property for two hybrid pairs ofnon-self mappings and utilize the same to obtain some coincidence and common fixed point theorems defined on an arbitrary set with values in metric spaces. Some illustrative examples are also given to highlight the realized improvements. Our results improve, generalize and extend some results of the existing literature especially the ones due to Liu et al. [Common fixed points of single-valued and multivalued maps, Internat. J. Math. Math. Sci. 19 (2005), 3045–3055].

1. Introduction and preliminaries

Nadler [25] proved the classical Banach fixed point theorem for set-valued mappings. The first ever use of a weak commutativity condition in a hybrid setting can be traced back to a Itoh and Takahashi paper of 1977 while the formal use of a weak commutativity condition essentially belongs to Sessa [34] which appeared in 1982. Kaneko and Sessa [23] weakened the notion of weak commutativity by extending the idea of compatibility (due to Jungck [18]) to a hybrid pair of mappings. Pathak [29] extended the concept of compatibility (due to Jungck [19]) by defining weak compatibility for hybrid pairs of mappings (including single valued case also) and utilized the same to prove results on existence of coincidence and common fixed points. Following this line of research, many authors have proved coincidence and common fixed point theorems in metric spaces satisfying hybrid-type contraction conditions (e.g. [4, 6, 7, 8, 29, 30, 37])

It is well known that strict contractive conditions do not ensure the existence of fixed points unless the underlying space is assumed to be compact or the contractive conditions are replaced by relatively stronger conditions. In 2004, Kamran [21] extended the idea of the property (E.A) (due to Aamri and Moutawakil [1]) to a hybrid pair of mappings and proved some fixed point results. Imdad and Ali [14] pointed out that the property (E.A) buys the suitable required containment between the range of one mapping into the range of other mapping of the pair. In 2005, Liu et al. [24] investigated a new property for two hybrid pairs of mappings and term

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the same as common property (E.A) which reduces to the property (E.A)whenever restricted to a single pair. By using this interesting property, they extended the results of Kamran [21]. Also, Ali and Imdad [3] studied the notion of non-compatible mappings (due to Pant [27]) in the hybrid setting.

In 2011, Samet and Vetro [33] pointed out an error in the proof of Theorem 1 of Rhoades et al. [32] and proved some results on coincidence points for a hybrid pair of mappings satisfying ϕ-contractive condition in the presence of the property (E.A). Damjanović et al. [5] obtained a coincidence point theorem for two hybrid pairs of mappings which improved the results of Gordji et al. [10]. Sintunavarat and Kumam [40] coined the idea of 'common limit range property' for single-valued mappings which never demands the completeness (or closedness) of the underlying subspaces. Most recently, Imdad et al. [15] defined the notion of common limit range property for a hybrid pair of mappings and proved some fixed point results in symmetric spaces. Motivated by the idea of Liu et al. [24], Imdad et al. [16] extended the notion of common limit range property to pair of self mappings and obtained some fixed point theorems in Menger and metric spaces. In the recent past, several authors have contributed to the vigorous development of metric fixed point theory for hybrid mappings (e.g. [2, 3, 9, 11, 12, 13, 14, 22, 26, 31, 35, 36, 38, 39, 41, 42]).

The aim of this paper is to define joint common limit range property for two hybrid pairs of non-self mappings and utilize the same to prove results on coincidence and common fixed points in metric spaces. We furnish some examples to support our main result besides deriving some related results. Our results improve and generalize a host of previously known results contained in [5, 24, 39] and the ones contained in cited references.

The following definitions and results will be needed in the sequel.

Definition 1.1. Let (X, d) be a metric space. A subset A of X is said to be

(1) closed if $A = \overline{A}$ where $\overline{A} = \{x \in X : d(x, A) = 0\}$,
(2) bounded if $\delta(A) < \infty$ where $\delta(A) = \sup\{d(a, b) : a, b \in A\}$.

Let (X, d) be a metric space. Then, on the lines of Nadler [25], we adopt

(1) $\mathcal{CL}(X) = \{A : A$ is a non-empty closed subset of $X\}$,
(2) $\mathcal{CB}(X) = \{A : A$ is a non-empty closed and bounded subset of $X\}$,
(3) For non-empty closed and bounded subsets A, B of X and $x \in X$,

$$d(x, A) = \inf\{d(x, a) : a \in A\}$$

and

$$H(A, B) = \max\{\{\sup d(a, B) : a \in A\}, \{\sup d(A, b) : b \in B\}\}.$$

It is easy to see that $(\mathcal{CB}(X), H)$ is a metric space wherein $\mathcal{CB}(X)$ is a metric space with the distance H which is known as the Hausdorff-Pompeiu metric on $CB(X)$ provided (X, d) is a metric space.

Definition 1.2 ([27])**.** Let (X, d) be a metric space with $F : X \to \mathcal{CB}(X)$ and $g : X \to X$. The pair of hybrid mappings (F, g) is said to be R-weakly commuting if, for given $x \in X$, $gFx \in CB(X)$, there exists some positive real number R such that $H(Fgx, gFx) \leq Rd(Fx, gx)$.

Definition 1.3 ([23])**.** Let (X, d) be a metric space with $F : X \to \mathcal{CB}(X)$ and $g : X \to X$. The pair of hybrid mappings (F, g) is said to be compatible if $gFx \in \mathcal{CB}(X)$ for all $x \in X$ and $\lim_{n\to\infty} H(Fgx_n, gFx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} gx_n = t \in A = \lim_{n\to\infty} Fx_n$.

Here it may be noted that compatible mappings need not be R-weakly commuting (see [27]). Also, on the points of coincidence R-weak commutativity is equivalent to commutativity and remains a necessary minimal condition for the existence of common fixed points for contractive type mappings.

Definition 1.4 ([3])**.** Let (X, d) be a metric space with $F : X \to \mathcal{CB}(X)$ and $g : X \to X$. The pair of hybrid mappings (F, g) is said to be non-compatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} gx_n = t \in A = \lim_{n\to\infty} Fx_n$ but $\lim_{n\to\infty} H(Fgx_n, gFx_n)$ is either non-zero or nonexistent.

Now we define the following definitions for non-self mappings:

Definition 1.5 ([20, 29])**.** Let (X, d) be a metric space whereas Y be an arbitrary non-empty set with $F : Y \to 2^X$ and $g : Y \to X$. The pair of hybrid mappings (F, g) is said to be weakly compatible if they commute at their coincidence points, that is, $gFx = Fgx$ whenever $gx \in Fx$.

Definition 1.6 ([12])**.** Let (X, d) be a metric space whereas Y be an arbitrary non-empty set with $F : Y \to 2^X$ and $g : Y \to X$. The pair of hybrid mappings (F, g) is said to be quasi-coincidentally commuting if $gx \in Fx$ (for $x \in X$ with $Fx, gx \in Y$) implies gFx is contained in Fgx.

Definition 1.7 ([12])**.** Let (X, d) be a metric space whereas Y be an arbitrary non-empty set with $F : Y \to 2^X$ and $g : Y \to X$. The mapping g is said to be coincidentally idempotent with respect to mapping F, if $gx \in Fx$ with $gx \in Y$ imply $ggx = gx$, that is, g is idempotent at coincidence points of the pair (F, g).

Definition 1.8 ([21])**.** Let (X, d) be a metric space whereas Y be an arbitrary non-empty set with $F : Y \to \mathcal{CB}(X)$ and $g : Y \to X$. Then the pair of hybrid mappings (F, g) is said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in Y, for some $t \in X$ and $A \in \mathcal{CB}(X)$ such that

$$\lim_{n\to\infty} gx_n = t \in A = \lim_{n\to\infty} Fx_n.$$

Definition 1.9 ([24])**.** Let (X, d) be a metric space whereas Y be an arbitrary non-empty set with $F, G : Y \to \mathcal{CB}(X)$ and $f, g : Y \to X$. Then the pairs of hybrid mappings (F, f) and (G, g) are said to satisfy the common property (E.A) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y, some $t \in Y$ and $A, B \in \mathcal{CB}(X)$ such that

$$\lim_{n\to\infty} Fx_n = A, \lim_{n\to\infty} Gy_n = B, \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = t \in A \cap B.$$

Definition 1.10. Let (X, d) be a metric space whereas Y be an arbitrary non-empty set with $F : Y \to \mathcal{CB}(X)$ and $g : Y \to X$. Then the pair of hybrid mappings (F, g) is said to satisfy the (CLRg) property if there exists a sequence $\{x_n\}$ in Y, for some $u \in Y$ and $A \in \mathcal{CB}(X)$ such that

$$\lim_{n\to\infty} gx_n = gu \in A = \lim_{n\to\infty} Fx_n.$$

2. Main result

In 2005, Liu et al. [24] proved some common point theorems for two hybrid pairs of mappings sharing common property (E.A) besides satisfying hybrid contractive conditions which generalizes certain results of Kamran [21]. The main result of Liu et al. [24] runs as follows:

Theorem 2.1 ([24, Theorem 2.3])**.** *Let f, g be two self mappings of the metric space (X, d) and let F, G be two mappings from X into $\mathcal{CB}(X)$ such that*

(1) *(f, F) and (g, G) satisfy the common property (E.A);*
(2) *for all $x \neq y$ in X,*

(2.1)
$$H(Fx, Gy) < \max\left\{d(fx, gy), \frac{d(fx, Fx) + d(gy, Gy)}{2}, \frac{d(fx, Gy) + d(gy, Fx)}{2}\right\}.$$

If $f(X)$ and $g(X)$ are closed subsets of X, then

(a) *f and F have a coincidence point;*
(b) *g and G have a coincidence point;*
(c) *f and F have a common fixed point provided that f is F-weakly commuting at v and $ffv = fv$ for $v \in C(f, F)$;*
(d) *g and G have a common fixed point provided that g is G-weakly commuting at v and $ggv = gv$ for $v \in C(g, G)$;*
(e) *f, g, F and G have a common fixed point provided that both (c) and (d) are true.*

One may notice that the notion of common property (E.A) requires the closedness of the underlying subspaces to ascertain the existence of coincidence points. Hence in order to remove this requirement, we introduced the notion of Joint Common Limit Range Property (in short (JCLR) property) for two hybrid pairs of non-self mappings as follows:

Definition 2.2. Let (X, d) be a metric space whereas Y be an arbitrary non-empty set with $F, G : Y \to \mathcal{CB}(X)$ and $f, g : Y \to X$. Then the pairs of hybrid mappings (F, f) and (G, g) are said to have the (JCLR) property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y and $A, B \in \mathcal{CB}(X)$ such that

$$\lim_{n\to\infty} Fx_n = A,$$
$$\lim_{n\to\infty} Gy_n = B,$$
$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = t \in A \cap B \cap f(Y) \cap g(Y)$$

i.e., there exist u and v in Y such that $t = fu = gv \in A \cap B$.

Now, we present some examples which demonstrate the utility of preceeding definition.

Example 2.3. Consider $Y = [0, 1] \subset [0, \infty) = X$ equipped with the usual metric. Define $F, G : Y \to \mathcal{CB}(X)$ and $f, g : Y \to X$ as follows:

$$fx = \begin{cases} 1 - x, & \text{if } 0 \le x \le \frac{1}{2}; \\ \frac{4}{5}, & \text{if } \frac{1}{2} < x \le 1. \end{cases} \qquad gx = \begin{cases} 1 - x^2, & \text{if } 0 \le x < \frac{1}{2}; \\ \frac{1}{2}, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

$$Fx = \begin{cases} \left[\frac{1}{2}, \frac{3}{4}\right], & \text{if } 0 \le x \le \frac{1}{2}; \\ \left[\frac{1}{4}, \frac{1}{2}\right), & \text{if } \frac{1}{2} < x \le 1. \end{cases} \quad Gx = \begin{cases} \left(\frac{1}{2}, \frac{3}{5}\right], & \text{if } 0 \le x < \frac{1}{2}; \\ \left[\frac{2}{5}, \frac{x+1}{2}\right], & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

If we choose the esteemed sequences $\{x_n\} = \left\{\frac{1}{2} - \frac{1}{n}\right\}_{n\in\mathbb{N}}$ and $\{y_n\} = \left\{\frac{1}{2} + \frac{1}{n}\right\}_{n\in\mathbb{N}}$ in Y, then one can verify that the pairs (F, f) and (G, g) share the (JCLR) property

$$\lim_{n\to\infty} Fx_n = \left[\frac{1}{2}, \frac{3}{4}\right], \lim_{n\to\infty} Gy_n = \left[\frac{2}{5}, 1\right], \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = \frac{1}{2},$$

where $f\left(\frac{1}{2}\right) = g\left(\frac{1}{2}\right) = \frac{1}{2} \in \left[\frac{1}{2}, \frac{3}{4}\right]$.

Notice that the (JCLR) property implies the common property (E.A) but the converse implication is not true in general. The following example substantiates this view point.

Example 2.4. In the setting of Example 2.3, replace the mappings f and g (besides retaining the rest):

$$fx = \begin{cases} 1 - x, & \text{if } 0 \le x < \frac{1}{2}; \\ \frac{4}{5}, & \text{if } \frac{1}{2} \le x \le 1. \end{cases} \quad gx = \begin{cases} 1 - x^2, & \text{if } 0 \le x \le \frac{1}{2}; \\ \frac{1}{2}, & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

If we consider the sequences as in Example 2.3, then one can verify that

$$\lim_{n\to\infty} Fx_n = \left[\frac{1}{2}, \frac{3}{4}\right], \lim_{n\to\infty} Gy_n = \left[\frac{2}{5}, 1\right], \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = \frac{1}{2}(= t),$$

where $\frac{1}{2} \in \left[\frac{1}{2}, \frac{3}{4}\right]$. Hence both the pairs (F, f) and (G, g) share the common property (E.A). However, there does not exists a point u in Y such that $t = fu$.

In an attempt to improve the main result of Liu et al. [24], we prove the following:

Theorem 2.5. *Let (X, d) be a metric space whereas Y be an arbitrary non-empty set with $F, G : Y \to \mathcal{CL}(X)$ and $f, g : Y \to X$. Suppose that*

(1) *the hybrid pairs (F, f) and (G, g) share the (JCLR) property,*
(2) *for all $x \ne y$ in Y and $0 < k < 2$*

$$H(Fx, Gy) < \max\Big\{d(fx, gy), \frac{k}{2}[d(fx, Fx) + d(gy, Gy)], \frac{k}{2}[d(fx, Gy) + d(gy, Fx)]\Big\}. \tag{2.2}$$

Then the pairs (F, f) and (G, g) have a coincidence point each.

In particular, if $Y \subset X$ and the pairs (F, f) and (G, g) are quasi-coincidentally commuting and coincidentally idempotent, then the pairs (F, f) and (G, g) have a common fixed point in X.

Proof. Since the pairs (F, f) and (G, g) share the (JCLR) property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y and $A, B \in \mathcal{CL}(X)$ such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Fx_n, \lim_{n\to\infty} gy_n = t \in B = \lim_{n\to\infty} Gy_n$$

implies that there exist u and v in X such that $t = fu = gv \in A \cap B$ for some $u, v \in Y$.

We assert that $fu \in Fu$. Suppose that $fu \notin Fu$, then using inequality (2.2), one gets

$$H(Fu, Gy_n) < \max\Big\{d(fu, gy_n), \frac{k}{2}[d(fu, Fu) + d(gy_n, Gy_n)], \frac{k}{2}[d(fu, Gy_n) + d(gy_n, Fu)]\Big\}.$$

Taking limit as $n \to \infty$, we have

$$\begin{aligned} H(Fu, B) &\leq \max\Big\{d(t,t), \frac{k}{2}[d(fu, Fu) + d(t, B)], \frac{k}{2}[d(t, B) + d(fu, Fu)]\Big\} \\ &= \frac{k}{2} d(fu, Fu) \\ &< d(fu, Fu) \end{aligned}$$

Since $t = fu = gv \in A \cap B$, it follows from the definition of Hausdorff metric that

$$d(fu, Fu) \leq H(B, Fu) < d(fu, Fu),$$

a contradiction. Hence $fu \in Fu$ which shows that the pair (F, f) has a coincidence point u in Y.

Now we show that $gv \in Gv$, if not, then using inequality (2.2), one obtains

$$\begin{aligned} d(gv, Gv) = d(fu, Gv) &\leq H(Fu, Gv) \\ &< \max\Big\{d(fu, gv), \frac{k}{2}[d(fu, Fu) + d(gv, Gv)], \\ &\quad \frac{k}{2}[d(fu, Gv) + d(gv, Fu)]\Big\} \\ &= \frac{k}{2} d(gv, Gv) \\ &< d(gv, Gv), \end{aligned}$$

a contradiction. Hence $gv \in Gv$ which shows that the pair (G, g) has a coincidence point v in Y.

Suppose that $Y \subset X$. Since u is a coincidence point of the pair (F, f), which is quasi-coincidentally commuting and coincidentally idempotent with respect to mapping F, we have $fu \in Fu$ and $ffu = fu$, therefore $fu = ffu \in f(Fu) \subset F(fu)$ which shows that fu is a common fixed point of the pair (F, f). Similarly, v is a coincidence point of the pair (G, g) which is quasi-coincidentally commuting and coincidentally idempotent with respect to mapping G, one can easily show that gv is a common fixed point of the pair (G, g). The analogous arguments work for the alternate statement as well. This completes the proof. □

Example 2.6. Let $Y = [0, 2] \subset [0, \infty) = X$ with the usual metric. Define $F, G : Y \to \mathcal{CL}(X)$ and $f, g : Y \to X$ as follows.

$$Fx = \begin{cases} \left[\frac{3}{5}, \frac{3}{2}\right], & \text{if } 0 \leq x \leq 1; \\ \left[\frac{1}{4}, \frac{1}{2}\right], & \text{if } 1 < x \leq 2. \end{cases} \quad Gx = \begin{cases} \left[\frac{3}{2}, 2\right], & \text{if } 0 \leq x < 1; \\ \left[\frac{1}{2}, \frac{2+x}{2}\right], & \text{if } 1 \leq x \leq 2. \end{cases}$$

$$fx = \begin{cases} 1, & \text{if } 0 \le x \le 1; \\ \frac{3x}{5}, & \text{if } 1 < x \le 2. \end{cases} \quad gx = \begin{cases} \frac{3x}{2}, & \text{if } 0 \le x < 1; \\ 1, & \text{if } 1 \le x \le 2. \end{cases}$$

Choosing two sequences $\{x_n\} = \left\{1 - \frac{1}{n}\right\}_{n\in\mathbb{N}}$ and $\{y_n\} = \left\{1 + \frac{1}{n}\right\}_{n\in\mathbb{N}}$ in Y, one can see that the pairs (F, f) and (G, g) enjoy the (JCLR) property, i.e.

$$\lim_{n\to\infty} f\left(1 - \frac{1}{n}\right) = 1 \in \left[\frac{3}{5}, \frac{3}{2}\right] = \lim_{n\to\infty} F\left(1 - \frac{1}{n}\right),$$

$$\lim_{n\to\infty} g\left(1 + \frac{1}{n}\right) = 1 \in \left[\frac{1}{2}, \frac{3}{2}\right] = \lim_{n\to\infty} G\left(1 + \frac{1}{n}\right),$$

where $1 = f(1) = g(1) \in \left[\frac{3}{5}, \frac{3}{2}\right] = \left[\frac{3}{5}, \frac{3}{2}\right] \cap \left[\frac{1}{2}, \frac{3}{2}\right]$. By a routine calculation one can show that the contractive condition (2.2) holds for every $x \neq y \in X$ and for some fixed $k \in (0, 2)$. Also $f(Y) = \left(\frac{3}{5}, \frac{6}{5}\right]$ and $g(Y) = \left[0, \frac{3}{2}\right)$. Hence $f(Y)$ and $g(Y)$ are not closed subsets of X. The pairs (F, f) and (G, g) are quasi-coincidentally commuting at $x = 1$, i.e. $f(1) \in F(1)$, $fF(1) = \left(\frac{3}{5}, \frac{9}{10}\right) \cup \{1\} \subset \left[\frac{3}{5}, \frac{3}{2}\right] = Ff(1)$ and $g(1) \in G(1)$, $gG(1) = \left[\frac{3}{4}, \frac{3}{2}\right) \cup \{1\} \subset \left[\frac{1}{2}, \frac{3}{2}\right] = Gg(1)$. Thus, all conditions of Theorem 2.5 are satisfied and $1 = f(1) = g(1) \in F(1) = G(1)$.

Theorem 2.7. *Let (X, d) be a metric space whereas Y be an arbitrary non-empty set with $F, G : Y \to \mathcal{CL}(X)$ and $f, g : Y \to X$. Suppose that the hybrid pairs (F, f) and (G, g) share the common property (E.A) and satisfy inequality (2.2). If $f(Y)$ and $g(Y)$ are closed subsets of X, then the pairs (F, f) and (G, g) have a point of coincidence.*

In particular, if $Y \subset X$ and the pairs (F, f) and (G, g) are quasi-coincidentally commuting and coincidentally idempotent, then the pairs (F, f) and (G, g) have a common fixed point in X.

Proof. If the pairs (F, f) and (G, g) share the common property (E.A), then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y and some $t \in X$, $A, B \in \mathcal{CL}(X)$ such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Fx_n, \ \lim_{n\to\infty} gy_n = t \in B = \lim_{n\to\infty} Gy_n.$$

As $f(Y)$ and $g(Y)$ are closed subsets of X, there exist u and v in X such that $t = fu = gv$ for some $u, v \in Y$. Hence the pairs (F, f) and (G, g) satisfy the (JCLR) property. The rest of the proof runs on the lines of the proof of Theorem 2.5. □

Remark 2.8. The conclusions of Theorem 2.5 and Theorem 2.7 remain true if inequality (2.2) is replaced by one of the following: For all $x \neq y$ in Y

(2.3)
$$H(Fx, Gy) < \max\{d(fx, gy), k[d(fx, Fx) + d(gy, Gy)], k[d(gy, Fx) + d(fx, Gy)]\},$$
where $0 \le k < 1$.

(2.4)
$$H(Fx, Gy) \le \lambda \max\left\{d(fx, gy), d(fx, Fx), d(gy, Gy), \frac{d(fx, Gy) + d(gy, Fx)}{2}\right\},$$
where $\lambda \in (0, 1)$.

(2.5)
$$H(Fx, Fy) \le \alpha d(fx, gy) + \beta \max\{d(fx, Fx), d(gy, Gy)\}$$

$$+\gamma \max \left\{ \begin{array}{c} d(fx, Gy) + d(gy, Fx), \\ d(fx, Fx) + d(gy, Gy) \end{array} \right\},$$

where $\alpha + \beta + 2\gamma < 1$.

By setting f, g, F and G suitably, one can deduce corollaries involving two as well as three non-self mappings. For the sake naturality, we only derive the following corollary involving a hybrid pair of non-self mappings:

Corollary 2.9. *Let (X, d) be a metric space whereas Y be an arbitrary non-empty set with $F : Y \to \mathcal{CL}(X)$ and $g : Y \to X$. Suppose that*

(1) *the hybrid pair (F, g) enjoys the (CLRg) property,*
(2) *for all $x \neq y$ in Y and $0 < k < 2$*

$$H(Fx, Fy) < \max \Big\{ d(fx, fy), \frac{k}{2}[d(fx, Fx) + d(fy, Fy)], \frac{k}{2}[d(fx, Fy) + d(fy, Fx)] \Big\}. \tag{2.6}$$

Then the pair (F, f) has a coincidence.

In particular, if $Y \subset X$ and the pair (F, f) is quasi-coincidentally commuting and coincidentally idempotent, then the pair (F, f) has a common fixed point in X.

Our next theorem involves a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies the following properties:

(1) ϕ is upper semi-continuous on $\mathbb{R}^+$,
(2) $0 < \phi(t) < t$ for each $t \in \mathbb{R}^+$.

Theorem 2.10. *Let (X, d) be a metric space whereas Y be an arbitrary non-empty set with $F, G : Y \to \mathcal{CL}(X)$ and $f, g : Y \to X$. Suppose that*

(1) *the hybrid pairs (F, f) and (G, g) share the (JCLR) property,*
(2) *for all $x \neq y \in Y$,*

$$H(Fx, Gy) \leq \phi(m(x, y)), \tag{2.7}$$

where

$$m(x, y) = \max \{d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)\}. \tag{2.8}$$

Then the pairs (F, f) and (G, g) have a coincidence point each.

In particular, if $Y \subset X$ and the pairs (F, f) and (G, g) are quasi-coincidentally commuting and coincidentally idempotent, then the pairs (F, f) and (G, g) have a common fixed point in X.

Proof. Suppose the pairs (F, f) and (G, g) share the (JCLR) property, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y and $A, B \in \mathcal{CL}(X)$ such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Fx_n, \lim_{n\to\infty} gy_n = t \in B = \lim_{n\to\infty} Gy_n$$

implies that there exist u and v in X such that $t = fu = gv \in A \cap B$ for some $u, v \in Y$. First we show that $fu \in Fu$. If not, then using inequality (2.7), one obtains

$$H(Fu, Gy_n) \leq \phi(m(u, y_n)), \tag{2.9}$$

where

$$m(u, y_n) = \max\{d(fu, gy_n), d(fu, Fu), d(gy_n, Gy_n), d(fu, Gy_n), d(gy_n, Fu)\}.$$

Taking limit as $n \to \infty$ in (2.9), we have

$$\begin{aligned} \lim_{n\to\infty} H(Fu, Gy_n) &\le \lim_{n\to\infty} \phi(m(u, y_n)) \\ H(Fu, B) &\le \phi\left(\lim_{n\to\infty} m(u, y_n)\right), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} \lim_{n\to\infty} m(u, y_n) &= \lim_{n\to\infty} \max\Big\{d(fu, gy_n), d(fu, Fu), d(gy_n, Gy_n), \\ &\quad d(fu, Gy_n), d(gy_n, Fu)\Big\} \\ &= d(fu, Fu). \end{aligned}$$

Hence (2.10) implies

$$\begin{aligned} H(B, Fu) &\le \phi(d(fu, Fu)) \\ &< d(fu, Fu). \end{aligned}$$

Since $t = fu = gv \in A \cap B$ and (owing to the definition of Hausdorff metric) it follows that

$$d(fu, Fu) \le H(fu, Fu) < d(fu, Fu),$$

which is a contradiction. Hence $fu \in Fu$ which shows that $u \in Y$ is a coincidence point of the pair (F, f).

Now we assert that $gv \in Gv$. On using inequality (2.7), one gets

$$\begin{aligned} d(gv, Gv) = d(fu, Gv) &\le H(Fu, Gv) \\ &\le \phi(m(u, v)), \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} m(u, v) &= \max\{d(fu, gv), d(fu, Fu), d(gv, Gv), d(fu, Gv), d(gv, Fu)\} \\ &= d(gv, Gv). \end{aligned}$$

Hence (2.11) implies

$$\begin{aligned} d(gv, Gv) \le H(Fu, Gv) &\le \phi(d(gv, Gv)) \\ &< d(gv, Gv), \end{aligned}$$

which is a contradiction. Then we have $gv \in Gv$ which shows that $v \in Y$ is a coincidence point of the pair (G, g).

The rest of the proof can be completed on the lines of the proof of Theorem 2.5. This completes the proof. □

Now we prove a more interesting result by introducing the notion of conditionally commuting (due to Pant and Pant [28]) for a hybrid pair of mappings which is the weakest form of the commutativity.

Definition 2.11 ([28])**.** Let (X, d) be a metric space whereas Y be an arbitrary non-empty set with $F : Y \to 2^X$ and $g : Y \to X$. The pair of hybrid mappings (F, g) is said to be conditionally commuting if they commute on a nonempty subset of the set of coincidence points whenever the set of their coincidences is nonempty.

Theorem 2.12. *Let (X, d) be a metric space whereas Y be an arbitrary non-empty set with $F, G : Y \to \mathcal{CL}(X)$ and $f, g : Y \to X$ satisfying inequality (2.2). Suppose that the hybrid pairs (F, f) and (G, g) enjoy the (JCLR) property. Then the pairs (F, f) and (G, g) have a point of coincidence.*

In particular, if $Y \subset X$ and $0 < k < 1$, then the pairs (F, f) and (G, g) have a common fixed point provided the pairs (F, f) and (G, g) are conditionally commuting.

Proof. In view of proof of Theorem 2.5, the pairs (F, f) and (G, g) have a coincidence point each u, v in Y. Suppose that $Y \subset X$. Since the pair (F, f) is conditionally commuting, two possible cases arise:

Case I: The pair (F, f) commutes at $u \in Y \subset X$, then $fu \in Fu$ so that $ffu \in f(Fu) \subset F(fu)$. Now we show that fu is a common fixed point of the pair (F, f). If it is not so, then using inequality (2.2), one gets

$$H(Ffu, Gy_n) < \max\Big\{d(ffu, gy_n), \frac{k}{2}[d(ffu, Ffu) + d(gy_n, Gy_n)], \frac{k}{2}[d(ffu, Gy_n) + d(gy_n, Ffu)]\Big\}.$$

Taking limit as $n \to \infty$, we have

$$\begin{aligned} H(Ffu, B) \leq& \max\Big\{d(ffu, fu), \frac{k}{2}[d(ffu, Ffu) + d(t, B)], \frac{k}{2}[d(ffu, B) + d(fu, Ffu)]\Big\} \\ =& \max\{d(ffu, fu), kd(ffu, fu)\} \\ =& d(fu, ffu). \end{aligned}$$

Since $t = fu = gv \in A \cap B$ and $ffu \in Ffu$, it follows (owing to the definition of Hausdorff metric) that

$$d(ffu, fu) \leq H(Ffu, B) < d(ffu, fu),$$

a contradiction. Hence $fu = ffu \in Ffu$ which shows that fu is a common fixed point of the pair (F, f).

Case II: If F and f do not commute at u, then by virtue of conditional commutativity of F and f, there exists a coincidence point of F and f at which F and f commute, i.e., there exists a point u' in Y such that $fu' \in Fu'$ and $ffu' \in f(Fu') \subset F(fu')$. Rest of the proof can be completed on the lines of the Case I when F and f commute at u.

Similarly, we can show that gv is a common fixed point of the pair (G, g). This completes the proof of the theorem. ☐

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