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MIXED DUALITY FOR A CLASS OF NONDIFFERENTIABLE MULTIOBJECTIVE PROGRAMMING PROBLEMS

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Dedicated to Professor Sompong Dhompongsa on the occasion of his 65th birthday

Abstract. In this paper, we introduce a class of nondifferentiable multiobjective programs with inequality and equality constraints, in which every component of the objective function contains a term involving the support function of a compact convex set. A mixed type dual for the primal problem is formulated. We establish weak and strong duality theorems for efficient solutions under (V, ρ)-invexity assumptions. Some special cases of our duality results are presented.

1. Introduction and preliminaries

The concept of efficiency was handled in game theory, optimal decision problems and optimization problems. In 1968, Geoffrion [6] introduced a slightly restricted definition of efficiency called proper efficiency. In virtue of proper efficiency, Weir [16] established some duality results between primal problem and Wolfe type dual problem and extended the duality results of Wolfe [17] for scalar convex programming problems, then duality results for scalar nonconvex programming problems to vector valued programs were established.

A new model for studying duality in nonlinear programming was given by Mond and Weir [13]. Based on the results in [14, 15], Egudo [5] formulated Wolfe type and Mond-Weir type dual problems and established duality theorems under generalized convexity assumptions.

Later, Xu [18] introduced a mixed type dual problem for differentiable multiobjective programs, in which Wolfe type and Mond-Weir type duals were special cases. More duality results were presented under generalized (F, ρ)-convexity assumptions. Subsequently, Bector *et al.* [3] devoted to the study of mixed duality for (generalized) fractional programming problems. Ahmad [1] introduced mixed duality for nondifferentiable programming with a square root term. Duality theorems for nondifferentiable static multiobjective programming problem with a square root term were obtained by Lal *et al.* [10].

On the other hand, Mond and Schechter [12] introduced firstly symmetric duality and optimality conditions for nondifferentiable multiobjective programming problems involving a support function. Yang *et al.* [19] formulated a mixed dual

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problem for a nondifferentiable multiobjective programming problem involving support function of a compact convex set. They established only weak duality theorems for efficient solutions by using the generalized (F,ρ)-convexity.

Recently, Kim and Bae [7] introduced nondifferentiable multiobjective programs involving support function of a compact convex set and linear function. They gave a mixed type dual problem and established weak and strong duality theorems under generalized (F,α,ρ,d)-convexity assumptions. Subsequently, Bae *et al.* [2] formulated Mond-Weir type and Wolfe type dual models and presented weak and strong duality theorems for efficient solutions by using generalized convexity conditions. Very recently, Kim *et al.* [8] introduced a G-mixed dual problem for a class of nondifferentiable multiobjective programs with inequality and equality constraints in which each component of the objective function contains a term involving the support function of a compact convex set. Weak, strong and converse duality theorems were proved by them.

In this paper, we introduce a mixed type dual problem. Our mixed dual is unifying the Wolfe and Mond-Weir type duals which was considered in Bae *et al.* [2]. Mixed duality relations are established by using more generalized convexity.

We consider the following nondifferentiable multiobjective programming problem involving the support function of a compact convex set.

$$
\begin{aligned}
(\mathbf{MP})_{\mathbf{E}} \quad \text{Minimize} \quad & (f_1(x)+s(x|D_1),\ldots,f_p(x)+s(x|D_p)) \\
\text{subject to} \quad & g_j(x) \leqq 0, j\in M=\{1,2,\ldots,m\}, \\
& h_l(x)=0, l\in Q=\{1,2,\ldots,q\},\ x\in X,
\end{aligned}
$$

where $f_i, g_j, h_l : \mathbb{R}^n \to \mathbb{R}$ are differentiable functions, $i\in P=\{1,2,\ldots,p\}$, $X=\{x\in\mathbb{R}^n | g(x)\leqq 0, h(x)=0\}$, D_i is a compact convex subset of $\mathbb{R}^n$.

Definition 1.1. A point $x^0\in X$ is said to be an efficient solution of $(\mathbf{MP})_{\mathbf{E}}$ if there exists no other $x\in X$ such that

$$f_{i_0}(x)+s(x|D_{i_0}) < f_{i_0}(x^0)+s(x^0|D_{i_0}),\ \text{for some } i_0\in P,$$

and

$$f_i(x)+s(x|D_i) \leqq f_i(x^0)+s(x^0|D_i),\ \text{for all } i\in P.$$

Definition 1.2. Let D be a compact convex set in $\mathbb{R}^n$. The support function $s(\cdot|D)$ is defined by

$$s(x|D) := max\{x^T y : y\in D\}.$$

The support function $s(\cdot|D)$ has a subdifferential. The subdifferential of $s(\cdot|D)$ at x is given by

$$\partial s(x|D) := \{z\in D : z^T x = s(x|D)\}.$$

The support function $s(\cdot|D)$ is convex and everywhere finite, that is, there exists $z\in D$ such that

$$s(y|D) \geq s(x|D) + z^T(y-x) \text{ for all } y\in D.$$

Equivalently,

$$z^T x = s(x|D).$$

Now, we define a differentiable (V,ρ)-invex function due to [9].

Definition 1.3. A vector function $f : \mathbb{R}^n \to \mathbb{R}^p$ is said to be (V,ρ)-invex at $u \in \mathbb{R}^n$ with respect to the function η and $\theta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if there exist $\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}$ and $\rho_i \in \mathbb{R},\ i \in P$, such that for any $x \in \mathbb{R}^n$

$$\alpha_i(x,u)[f_i(x) - f_i(u)] \geqq \nabla f_i(u)\eta(x,u) + \rho_i \|\theta(x,u)\|^2.$$

If this inequality is replaced by strict inequality, then f is called strictly (V,ρ)-invex.

For each $k \in P$, we consider the following scalarizing problem $\mathbf{P_k(x^0)}$ of $\mathbf{(MP)_E}$ due to the one in [4].

$$\begin{aligned} \text{Minimize} \quad & f_k(x) + s(x|D_k) \\ \text{subject to} \quad & f_i(x) + s(x|D_i) \leqq f_i(x^0) + s(x^0|D_i), i \neq k \in P, \\ & g_j(x) \leqq 0,\ j \in M, \\ & h_l(x) = 0,\ l \in Q. \end{aligned}$$

In order to establish strong duality results, we need the following theorem between $\mathbf{(MP)_E}$ and $\mathbf{P_k(x^0)}$.

Theorem 1.4. *x^0 is an efficient solution of $\mathbf{(MP)_E}$ if and only if x^0 solves $\mathbf{P_k(x^0)}$ for every $k = 1, 2, \ldots, p$.*

Proof. Assume that x^0 is not a solution of $\mathbf{P_k(x^0)}$. Then there exists $x \in X$ such that

$$f_k(x^0) + s(x^0|D_k) > f_k(x) + s(x|D_k), k \in P, \tag{1.1}$$

$$f_i(x^0) + s(x^0|D_i) \geqq f_i(x) + s(x|D_i), i \neq k. \tag{1.2}$$

From (1.1) and (1.2), we conclude that x^0 is not efficient for $\mathbf{(MP)_E}$.

Conversely, let x^0 solve $\mathbf{P_k(x^0)}$ for every $k \in P$, then for all $x \in X$ with $f_i(x^0) + s(x^0|D_i) \geqq f_i(x) + s(x|D_i), i \neq k$, we have $f_k(x^0) + s(x^0|D_k) \leqq f_k(x) + s(x|D_k)$. Then, there exists no other $x \in X$ such that $f_i(x) + s(x|D_i) \leqq f_i(x^0) + s(x^0|D_i)$, $i \in P$ with strict inequality holding for at least one i. This implies that x^0 is efficient for $\mathbf{(MP)_E}$. □

2. Mixed type duality

We propose the following mixed dual problem $\mathbf{(MD)_E}$ to $\mathbf{(MP)_E}$:

$$\begin{aligned} \mathbf{(MD)_E} \quad \text{Maximize} \quad & \Big(f_1(u) + u^T w_1 + \sum_{j \in J_0} \mu_j g_j(u) + \sum_{l \in K_0} \nu_l h_l(u), \ldots, \\ & f_p(u) + u^T w_p + \sum_{j \in J_0} \mu_j g_j(u) + \sum_{l \in K_0} \nu_l h_l(u)\Big) \\ \text{subject to} \quad & \sum_{i=1}^{p} \lambda_i (\nabla f_i(u) + w_i) + \sum_{j=1}^{m} \mu_j \nabla g_j(u) \\ & + \sum_{l=1}^{q} \nu_l \nabla h_l(u) = 0, \end{aligned} \tag{2.1}$$

$$\sum_{j\in J_\alpha}\mu_j g_j(u)+\sum_{l\in K_\alpha}\nu_l h_l(u)\geqq 0, \alpha=1,\ldots,r, \tag{2.2}$$

$$\lambda_i\geqq 0,\ w_i\in D_i,\ i\in P,\ \sum_{i=1}^{p}\lambda_i=1 \tag{2.3}$$

$$\mu_j\geqq 0, j\in M,\ \nu_l\in \mathbb{R}, l\in Q, \tag{2.4}$$

where $J_\alpha\subset M, \alpha=0,1,\ldots,r$ with $\cup_{\alpha=0}^{r}J_\alpha=M$ and $J_\alpha\cap J_\beta=\emptyset$ if $\alpha\neq\beta, K_\alpha\subset Q, \alpha=0,1,\ldots,r$ with $\cup_{\alpha=0}^{r}K_\alpha=Q$ and $K_\alpha\cap K_\beta=\emptyset$ if $\alpha\neq\beta$.

Now, we establish weak and strong duality relations between $(\mathbf{MP})_\mathbf{E}$ and $(\mathbf{MD})_\mathbf{E}$. First, we give a weak duality result by using suitable (V,ρ)-invexity.

Theorem 2.1 (Weak Duality)**.** *Let x and (u,w,λ,μ,ν) be feasible solutions of $(\mathbf{MP})_\mathbf{E}$ and $(\mathbf{MD})_\mathbf{E}$, respectively. If $\sum_{j\in J_\alpha}\mu_j g_j(\cdot)+\sum_{l\in K_\alpha}\nu_l h_l(\cdot)$, $\alpha=1,\ldots,r$ is strictly (V,σ)-invex at u with respect to η, $f_i(\cdot)+(\cdot)^T w_i$, $i\in P$ are (V,ρ)-invex at u with respect to η and $\sum_{j\in J_0}\mu_j g_j(\cdot)+\sum_{l\in K_0}\nu_l h_l(\cdot)$ is (V,ρ)-invex at u with respect to η with $\sigma\geqq 0$, $\sum_{i=1}^{p}\lambda_i\rho_i\geqq 0$, then the following cannot hold:*

$$f_{i_0}(x)+s(x|D_{i_0})<f_{i_0}(u)+u^T w_{i_0}+\sum_{j\in J_0}\mu_j g_j(u)+\sum_{l\in K_0}\nu_l h_l(u), \tag{2.5}$$

for some $i_0\in P$,

$$f_i(x)+s(x|D_i)\leqq f_i(u)+u^T w_i+\sum_{j\in J_0}\mu_j g_j(u)+\sum_{l\in K_0}\nu_l h_l(u),\ \textit{for all } i\in P. \tag{2.6}$$

Proof. Let x and (u,w,λ,μ,ν) be feasible solutions of $(\mathbf{MP})_\mathbf{E}$ and $(\mathbf{MD})_\mathbf{E}$, respectively. Since $\beta(x,u)>0$, we have

$$\beta(x,u)\Big(\sum_{j\in J_\alpha}\mu_j g_j(x)+\sum_{l\in K_\alpha}\nu_l h_l(x)\Big)\leqq \beta(x,u)\Big(\sum_{j\in J_\alpha}\mu_j g_j(u)+\sum_{l\in K_\alpha}\nu_l h_l(u)\Big).$$

By the strictly (V,σ)-invexity of $\sum_{j\in J_\alpha}\mu_j g_j(u)+\sum_{l\in K_\alpha}\nu_l h_l(u)$, for $\alpha=1,\ldots,r$, we have

$$\Big[\sum_{j\in J_\alpha}\mu_j\nabla g_j(u)+\sum_{l\in K_\alpha}\nu_l\nabla h_l(u)\Big]\eta(x,u)+\sigma\|\theta(x,u)\|^2<0. \tag{2.7}$$

Suppose contrary to the results that (2.5) and (2.6) hold. Since $x^T w_i\leqq s(x|D_i)$, $i\in P$, $\sum_{j\in J_0}\mu_j g_j(x)\leqq 0$ and $\sum_{l\in K_0}\nu_l h_l(u)=0$, we have

$$\begin{aligned}
&f_{i_0}(x)+x^T w_{i_0}+\sum_{j\in J_0}\mu_j g_j(x)+\sum_{l\in K_0}\nu_l h_l(x)\\
&\qquad<f_{i_0}(u)+u^T w_{i_0}+\sum_{j\in J_0}\mu_j g_j(u)+\sum_{l\in K_0}\nu_l h_l(u),\ \text{for some } i_0\in P,\\
&f_i(x)+x^T w_i+\sum_{j\in J_0}\mu_j g_j(x)+\sum_{l\in K_0}\nu_l h_l(x)\\
&\qquad\leqq f_i(u)+u^T w_i+\sum_{j\in J_0}\mu_j g_j(u)+\sum_{l\in K_0}\nu_l h_l(u),\ \text{for all } i\in P.
\end{aligned}$$

Since $f_i(\cdot)+(\cdot)^T w_i,\ i \in P$ are (V,ρ)-invex at u with respect to η and $\sum_{j\in J_0}\mu_j g_j(\cdot)+\sum_{l\in K_0}\nu_l h_l(\cdot)$ is (V,ρ)-invex at u with respect to η, we obtain

$$\begin{aligned}\alpha_{i_0}(x,u)\Big[f_{i_0}(x)+x^T w_{i_0}&+\sum_{j\in J_0}\mu_j g_j(x)+\sum_{l\in K_0}\nu_l h_l(x)\\ &-f_{i_0}(u)-u^T w_{i_0}-\sum_{j\in J_0}\mu_j g_j(u)-\sum_{l\in K_0}\nu_l h_l(u)\Big]\\ &>\Big[\nabla f_{i_0}(u)+w_{i_0}+\sum_{j\in J_0}\mu_j\nabla g_j(u)+\sum_{l\in K_0}\nu_l\nabla h_l(u)\Big]\eta(x,u)\\ &+\rho_{i_0}\|\theta_{i_0}(x,u)\|^2,\ \text{for some } i_0\in P,\end{aligned}$$

$$\begin{aligned}\alpha_i(x,u)\Big[f_i(x)+x^T w_i&+\sum_{j\in J_0}\mu_j g_j(x)+\sum_{l\in K_0}\nu_l h_l(x)\\ &-f_i(u)-u^T w_i-\sum_{j\in J_0}\mu_j g_j(u)-\sum_{l\in K_0}\nu_l h_l(u)\Big]\\ &\geqq\Big[\nabla f_i(u)+w_i+\sum_{j\in J_0}\mu_j\nabla g_j(u)+\sum_{l\in K_0}\nu_l\nabla h_l(u)\Big]\eta(x,u)\\ &+\rho_i\|\theta_i(x,u)\|^2,\ \text{for all } i\in P.\end{aligned}$$

Since $\lambda_i \geqq 0,\ i\in P$ and $\sum_{i=1}^p \lambda_i = 1$, we have

$$\Big[\sum_{i=1}^p\lambda_i(\nabla f_i(u)+w_i)+\sum_{j\in J_0}\mu_j\nabla g_j(u)+\sum_{l\in K_0}\nu_l\nabla h_l(u)\Big]\eta(x,u) + \sum_{i=1}^p\lambda_i\rho_i\|\theta(x,u)\|^2 \leqq 0. \tag{2.8}$$

By (2.7) and (2.8), we obtain

$$\begin{aligned}\Big[\sum_{i=1}^p\lambda_i(\nabla f_i(u)+w_i)&+\sum_{j=1}^m\mu_j\nabla g_j(u)+\sum_{l=1}^q\nu_l\nabla h_l(u)\Big]\eta(x,u)\\ &<-\sum_{i=1}^p\lambda_i\rho_i\|\theta(x,u)\|^2-\sigma\|\theta(x,u)\|^2\leqq 0.\end{aligned}$$

This inequality contradicts (2.1). □

Remark 2.2. If the strictly (V,σ)-invexity assumption of $\sum_{j\in J_\alpha}\mu_j g_j(\cdot)+\sum_{l\in K_\alpha}\nu_l h_l(\cdot)$ is replaced by (V,σ)-invexity, then our Theorem 2.1 also holds between $(\mathbf{MP})_{\mathbf{E}}$ and $(\mathbf{MD})_{\mathbf{E}}$ with $\lambda_i>0, \sum_{i=1}^p\lambda_i=1$.

Theorem 2.3. *Suppose that the assumptions of Theorem 2.1 are held. If* $(u^0,w^0,\lambda^0,\mu^0,\nu^0)$ *is a feasible solution of* $(\mathbf{MD})_{\mathbf{E}}$ *such that* u^0 *is a feasible solution of* $(\mathbf{MP})_{\mathbf{E}}$ *and* $(u^0)^T w_i^0 = s(u^0|D_i),\ i\in P$, *then* u^0 *is an efficient solution of* $(\mathbf{MP})_{\mathbf{E}}$ *and* $(u^0,w^0,\lambda^0,\mu^0,\nu^0)$ *is an efficient solution of* $(\mathbf{MD})_{\mathbf{E}}$.

Proof. It follows on the lines of Egudo (Ref.[5], Corollary 1) along with Theorem 2.1. □

Theorem 2.4 (Strong Duality). *If x^0 is an efficient solution of $(\mathbf{MP})_{\mathbf{E}}$ and satisfy a constraint qualification [11] for $\mathbf{P_k(x^0)}$ for at least one $k \in P$ then there exist $\lambda^0 \in \mathbb{R}^p, \mu^0 \in \mathbb{R}^m, \nu^0 \in \mathbb{R}^q$ and $w_i^0 \in D_i,\ i \in P$ such that $(x^0, w^0, \lambda^0, \mu^0, \nu^0)$ is a feasible solution of $(\mathbf{MD})_{\mathbf{E}}$ and $(x^0)^T w_i^0 = s(x^0|D_i),\ i \in P$. Moreover, if the assumptions of Theorem 2.1 are satisfied then $(x^0, w^0, \lambda^0, \mu^0, \nu^0)$ is an efficient solution of $(\mathbf{MD})_{\mathbf{E}}$.*

Proof. Since x^0 is an efficient solution of $(\mathbf{MP})_{\mathbf{E}}$, then from Theorem 1.4, x^0 solves $\mathbf{P_k(x^0)}$ for all $k \in P$. By the assumption, there exists at least one $k \in P$ for which x^0 satisfies a constraint qualification [11] for $\mathbf{P_k(x^0)}$. Now from the Kuhn-Tucker necessary conditions [11], there exist $\lambda_i^0 \geqq 0,\ i \neq k,\ 0 \leqq \mu^0 \in \mathbb{R}^m, \nu^0 \in \mathbb{R}^q$ and $w_i^0 \in D_i,\ i \in P$ such that

$$\lambda_k^0(\nabla f_k(x^0) + w_k^0) + \sum_{i \neq k} \lambda_i^0(\nabla f_i(x^0) + w_i^0) + \sum_{j=1}^{m} \mu_j^0 \nabla g_j(x^0) + \sum_{l=1}^{q} \nu_l^0 \nabla h_l(x^0) = 0, \tag{2.9}$$

$$(w_i^0)^T x^0 = s(x^0|D_i),\ i \in P,$$

$$\sum_{j=1}^{m} \mu_j^0 g_j(x^0) = 0. \tag{2.10}$$

Now dividing (2.9) and (2.10) by $1 + \sum_{i \neq k} \lambda_i$ and letting

$$\lambda_k^0 = \frac{1}{1 + \sum_{i \neq k} \lambda_i} > 0, \quad \lambda_i^0 = \frac{\lambda_i}{1 + \sum_{i \neq k} \lambda_i} \geqq 0, \text{ for all } i \neq k$$

and

$$\mu^0 = \frac{\mu}{1 + \sum_{i \neq k} \lambda_i} \geqq 0, \quad \nu^0 = \frac{\nu}{1 + \sum_{i \neq k} \lambda_i}.$$

Then, we have $\sum_{i=1}^{p} \lambda_i^0(\nabla f_i(x^0) + w_i^0) + \sum_{j=1}^{m} \mu_j^0 \nabla g_j(x^0) + \sum_{l=1}^{q} \nu_l^0 \nabla h_l(x^0) = 0$, and $\sum_{j \in J_\alpha} \mu_j^0 g_j(x^0) + \sum_{l \in K_\alpha} \nu_l^0 h_l(x^0) \geqq 0, \alpha = 1, \ldots, r$. So $(x^0, w^0, \lambda^0, \mu^0, \nu^0)$ satisfy (2.1), (2.2), (2.3) and (2.4). Thus, $(x^0, w^0, \lambda^0, \mu^0, \nu^0)$ is feasible for $(\mathbf{MD})_{\mathbf{E}}$.

Hence, $(x^0, w^0, \lambda^0, \mu^0, \nu^0)$ is efficient for $(\mathbf{MD})_{\mathbf{E}}$ from Remark 2.2. □

3. Special cases

Case 1. If $D_i = \emptyset, i \in P, Q = \emptyset, J_0 = M$, then our mixed dual problem $(\mathbf{MD})_{\mathbf{E}}$ reduces to Wolfe Vector Dual (**WVD**) in Egudo [5].

$$(\mathbf{WVD}) \quad \text{Maximize} \quad (f_1(u) + \mu^T g(u), \ldots, f_p(u) + \mu^T g(u))$$

$$\text{subject to} \quad \sum_{i=1}^{p} \lambda_i \nabla f_i(u) + \nabla \mu^T g(u) = 0,$$

$$\lambda_i \geqq 0, \sum_{i=1}^{p} \lambda_i = 1, \mu \geqq 0.$$

If $D_i = \emptyset, i \in P, Q = \emptyset, J_\alpha = M$, then our mixed dual problem $(\mathbf{MD})_\mathbf{E}$ reduces to Mond-Weir Vector Dual (**DVOP**) in Egudo [5].

$$\begin{aligned} (\mathbf{DVOP}) \quad & \text{Maximize} \quad (f_1(u), \ldots, f_p(u)) \\ & \text{subject to} \quad \sum_{i=1}^{p} \lambda_i \nabla f_i(u) + \nabla \mu^T g(u) = 0, \\ & \qquad \mu^T g(u) \geqq 0, \\ & \qquad \lambda_i \geqq 0, \sum_{i=1}^{p} \lambda_i = 1, \mu \geqq 0, \end{aligned}$$

Case 2. If $P = \{1\}, s(x|D) = (x^T B x)^{1/2}$ and $Q = \emptyset$, then our mixed dual problem $(\mathbf{MD})_\mathbf{E}$ reduces to the one in Ahmad [1].

Case 3. If $s(x|D_i) = (x^T B_i x)^{1/2}, i \in P$, $Q = \emptyset, J_0 = M$ and $X = \{x| - g(x) \leqq 0\}$, then our mixed dual problem $(\mathbf{MD})_\mathbf{E}$ reduces to $(\mathbf{VOP})_\mathbf{1}$ in Lal *et al.* [10].

$$\begin{aligned} (\mathbf{VDP})_\mathbf{1} \quad & \text{Maximize} \quad (f_1(u) + u^T B_1 w - \mu^T g(u), \ldots, f_p(u) + u^T B_p w - \mu^T g(u)) \\ & \text{subject to} \quad \mu^T g_x(u) = \sum_{i=1}^{p} \lambda_i f_{i_x}(u) + \sum_{i=1}^{p} \lambda_i B_i w, \\ & \qquad w^T B_i w \leqq 1, \lambda_i \geqq 0, \sum_{i=1}^{p} \lambda_i = 1, \mu \geqq 0. \end{aligned}$$

If $s(x|D_i) = (x^T B_i x)^{1/2}, i \in P, Q = \emptyset, J_\alpha = M$, then our mixed dual problem $(\mathbf{MD})_\mathbf{E}$ reduces to $(\mathbf{VDP})_\mathbf{2}$ in Lal *et al.* [10].

$$\begin{aligned} (\mathbf{VDP})_\mathbf{2} \quad & \text{Maximize} \quad (f_1(u) + u^T B_1 w, \ldots, f_p(u) + u^T B_p w) \\ & \text{subject to} \quad \mu^T g_x(u) = \sum_{i=1}^{p} \lambda_i f_{i_x}(u) + \sum_{i=1}^{p} \lambda_i B_i w, \\ & \qquad \mu^T g(u) \leqq 0, w^T B_i w \leqq 1, \\ & \qquad \lambda_i \geqq 0, \sum_{i=1}^{p} \lambda_i = 1, \mu \geqq 0. \end{aligned}$$

Case 4. $D_i = \emptyset, i \in P, Q = \emptyset$, then our mixed dual problem $(\mathbf{MD})_\mathbf{E}$ reduces to the mixed dual (**MDP**) in Xu [18].

$$\begin{aligned} (\mathbf{MDP}) \quad & \text{Maximize} \quad f(u) + \mu_{J_1}^T g_{J_1}(u) e \\ & \text{subject to} \quad \nabla f(u)^T \lambda + \nabla g(u)^T \mu = 0, \\ & \qquad \mu_{J_2}^T g_{J_2}(u) \geqq 0, \lambda_i \geqq 0, \sum_{i=1}^{p} \lambda_i = 1, \mu \geqq 0. \end{aligned}$$

where J_1 is a subset of M and $J_2 = M/J_1$.

Case 5. If $J_\alpha = \emptyset$ and $K_\alpha = \emptyset$, then our mixed dual problem $(\mathbf{MD})_\mathbf{E}$ reduces to (**WVODE**) in Bae *et al.* [2].

$$
\begin{aligned}
(\mathbf{WVODE}) \quad \text{Maximize} \quad & (f_1(u) + u^T w_1 + \mu^T g(u) + \nu^T h(u), \ldots, \\
& \qquad f_p(u) + u^T w_p + \mu^T g(u) + \nu^T h(u)) \\
\text{subject to} \quad & \sum_{i=1}^p \lambda_i (\nabla f_i(u) + w_i) + \nabla \mu^T g(u) + \nabla \nu^T h(u) = 0, \\
& w_i \in D_i, \lambda_i \geqq 0, \sum_{i=1}^p \lambda_i = 1, \mu_j \geqq 0, j \in M.
\end{aligned}
$$

If $J_0 = \emptyset$ and $K_0 = \emptyset$, then our mixed dual problem $(\mathbf{MD})_\mathbf{E}$ reduces to (**MVODE**) in Bae *et al.* [2].

$$
\begin{aligned}
(\mathbf{MVODE}) \quad \text{Maximize} \quad & (f_1(u) + u^T w_1, \ldots, f_p(u) + u^T w_p) \\
\text{subject to} \quad & \sum_{i=1}^p \lambda_i (\nabla f_i(u) + w_i) + \nabla \mu^T g(u) + \nabla \nu^T h(u) = 0, \\
& \mu^T g(u) + \nu^T h(u) \geqq 0, \\
& w_i \in D_i, \lambda_i \geqq 0, \sum_{i=1}^p \lambda_i = 1, \mu_j \geqq 0, j \in M.
\end{aligned}
$$

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