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AN ENDPOINT THEOREM IN GENERALIZED L-SPACES WITH APPLICATIONS

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Dedicated to Sompomg Dhompongsa on the occasion of his 65th anniversary

Abstract. In this paper, we will introduce first the notions of measure of non-singletonsness (denoted by δ), δ-Cauchy sequence and δ-completeness in the setting of generalized L-spaces. Main result of the paper is a new endpoint theorem in generalized L-spaces from which we will derive an order-theoretic Cantor theorem in such spaces. Some examples are also given to support our main result. Our results generalize some recent results in the literature.

1. Introduction and preliminaries

Let X be a nonempty set, 2^X be the set of all subsets of X and let $T : X \to 2^X$ be a set-valued mapping on X. By definition, an element $x \in X$ is said to be an endpoint (also called strict fixed point or stationary point) of T if $Tx = \{x\}$.

The existence of endpoints of set-valued mappings has significant applications in the Optimization Theory, Operatorial Inclusions, Mathematical Economics and Variational Analysis; for more details see [1, 12, 14, 19, 20, 23, 24, 28]. Most of the existence results of endpoints and of asymptotic stationary points are proved in metric spaces and uniform spaces (see [1–4, 13, 14, 19–21, 25, 26, 28] and references therein). Recently, Jachymski [20] proved an endpoint theorem for a set-valued map on a metric space from which he derived the famous Ekeland variational principle and the order-theoretic Cantor theorem.

The first aim of this paper is to give a generalization of the above mentioned theorem of Jachymski in the setting of generalized L-spaces. Then, we derive an order-theoretic Cantor type theorem in such spaces.

2. Main results

We first introduce the notion of a measure of non-singletonsness in a very general setting.

Definition 2.1. Let X be a nonempty set. Then the generalized functional $\delta : 2^X \to [0, \infty]$ is called a measure of non-singletonsness if the following axioms are satisfied:

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(i) for each $A, B \in 2^X$, $A \subseteq B \Rightarrow \delta(A) \leq \delta(B)$.
(ii) If $A \neq \emptyset$, then $\delta(A) = 0 \Leftrightarrow A$ is a singleton.

Following M. Fréchet [16], we present now the concept of generalized L-space.

Definition 2.2. Let X be a nonempty set. Let

$$s(X) := \{(x_n)_{n\in\mathbb{N}} \mid x_n \in X,\ n \in \mathbb{N}\}.$$

Let $c(X)$ be a subset of $s(X)$ and $Lim : c(X) \to 2^X \setminus \emptyset$ be a set-valued operator. By definition the triple $(X, c(X), Lim)$ is called a generalized L-space (denoted by $(X, \to)$) if the following conditions are satisfied:

(i) if $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = \{x\}$.
(ii) if $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = \{x\}$, then for all subsequences $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and

$$Lim(x_{n_i})_{i\in\mathbb{N}} = \{x\}.$$

By definition, an element of $c(X)$ is said to be a convergent sequence and $Lim(x_n)_{n\in\mathbb{N}}$ is the set of all limits of this sequence. If $Lim(x_n)_{n\in\mathbb{N}} = \{x\}$, then we write

$$x_n \to x \text{ as } n \to \infty.$$

Remark 2.3. A generalized L-space is any set endowed with a structure implying a notion of convergence for sequences. For example, any topological space is a generalized L-space.

Definition 2.4. Let $(X, \to)$ be a generalized L-space. Then, a subset Y of X is called closed in $(X, \to)$ if and only if for each sequence $(x_n) \in c(Y)$ we have that $Lim(x_n)_{n\in\mathbb{N}} \subset Y$.

The concept of closedness introduced by Definition 2.4 does not coincide with the concept of closedness in a general topological space, but they coincide in first countable topological spaces.

Remark 2.5. Notice that, if in above definition $Lim : c(X) \to X$ (i.e., it is a single-valued operator), then we get the concept of L-space, which was also introduced by M. Fréchet.

Remark 2.6. An L-space is any generalized L-space endowed with a structure generating a notion of convergence for sequences with a unique limit. For example, Hausdorff topological spaces, metric spaces, different generalized metric spaces (in the sense that $d(x, y) \in \mathbb{R}^m_+$ or in in the sense that $d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$ or in the sense that $d(x, y) \in K$, where K is a cone in an ordered Banach space, etc.), 2-metric spaces, D-R-spaces, probabilistic metric spaces, syntopogenous spaces, are examples of L-spaces. For more details see Fréchet [16], Blumenthal [8] and I. A. Rus [22].

Definition 2.7. Let $(X, \to)$ be a generalized L-space and let δ be a measure of non-singletonsness on X. Let $\{x_n\}$ be a sequence in X. We say that:

(i) $\{x_n\}$ is a δ-Cauchy sequence if

$$\lim_{n\to\infty} \delta(\{x_n, x_{n+1}, x_{n+2}, \dots\}) = 0,$$

(ii) $(X, \to)$ is called δ-complete if for every δ-Cauchy sequence $\{x_n\}$ in X we have that $\{x_n\} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = \{\bar{x}\}$.

Example 2.8. Let (X, d) be a complete metric space. Let us consider the generalized diameter functional on X, i.e. let $\delta : 2^X \to [0, \infty]$ be given by $\delta(A) = \text{diam}(A)$ (where $\text{diam}(A) := \sup\{d(a, b) : a, b \in A\}$). Then, we have:

i) δ is a measure of non-singletonsness on X;

ii) $(X, \xrightarrow{d})$ is δ-complete, where $\xrightarrow{d}$ denotes the convergence generated by d.

Example 2.9. Let (X, d) be a complete b-metric space (see [5], [9], [10], ...) with constant $s > 1$. Let $\delta : 2^X \to [0, \infty]$ be given by $\delta(A) = \text{diam}(A)$. Then δ is a measure of non-singletonsness on X and $(X, \xrightarrow{d})$ is δ-complete.

Notice that, in the above cases, the notions of δ-Cauchy sequence and d-Cauchy sequence coincide.

To prove our main result we need the following intersection lemma.

Lemma 2.10. *Let $(X, \to)$ be a δ-complete generalized L-space. Let $\{A_n\}$ be a sequence of nonempty closed subsets of X such that*

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq A_{n+1} \supseteq \ldots \text{ and } \lim_{n\to\infty} \delta(A_n) = 0.$$

Then $\bigcap_{n=1}^{\infty} A_n = \{\overline{x}\}$.

Proof. Let $x_n \in A_n$. Since $\{A_n\}$ is a decreasing sequence, we get that $\{x_n, x_{n+1}, \ldots\} \subseteq A_n$ and, thus,

$$\delta(\{x_n, x_{n+1}, \ldots\}) \le \delta(A_n), \text{ for each } n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} \delta(A_n) = 0$, we get

$$\lim_{n\to\infty} \delta(\{x_n, x_{n+1}, \ldots\}) = 0,$$

and so $\{x_n\}$ is a δ-Cauchy sequence. Since $(X, \to)$ is δ-complete, we deduce that there exists an element $\overline{x} \in X$ such that $Lim(x_n)_{n\in\mathbb{N}} = \{\bar{x}\}$. Since $\{x_n, x_{n+1}, \ldots\} \subseteq A_n$ and A_n is closed, we have that $\overline{x} \in A_n$, for each $n \in \mathbb{N}$. Thus $\overline{x} \in \bigcap_{n=1}^{\infty} A_n$. Since, for each $n \in \mathbb{N}$, we have $\bigcap_{n=1}^{\infty} A_n \subseteq A_n$, we get that $\delta(\bigcap_{n=1}^{\infty} A_n) \le \lim_{n\to\infty} \delta(A_n) = 0$. Thus $\delta(\bigcap_{n=1}^{\infty} A_n) = 0$ and so $\bigcap_{n=1}^{\infty} A_n = \{\overline{x}\}$. □

Now we are ready to state our main result.

Theorem 2.11. *Let $(X, \to)$ be a δ-complete generalized L-space and $T : X \to 2^X$ be a set-valued map with nonempty closed values such that $Ty \subseteq Tx$ for each $y \in Tx$. Assume that, for any $x \in X$ and $\epsilon > 0$, there exists $y \in Tx$ such that $\delta(Ty) < \epsilon$. Then T has an endpoint.*

Proof. Let $x_0 \in X$. By our assumptions, there is $x_1 \in Tx_0$ such that $\delta(Tx_1) < 1$ and $Tx_1 \subseteq Tx_0$. By induction, we get a sequence $\{x_n\}_{n=1}^{\infty}$ such that

$$\delta(Tx_n) < \frac{1}{n} \text{ and } Tx_{n+1} \subseteq Tx_n, \text{ for each } n \in \mathbb{N}.$$

By Lemma 2.10, there exists $\overline{x} \in X$ such that $\bigcap_{n=1}^{\infty} Tx_n = \{\overline{x}\}$. Since $\overline{x} \in Tx_n$, by the assumption imposed, we get that $T\overline{x} \subseteq Tx_n$, for each $n \in \mathbb{N}$. Thus, $T\overline{x} \subseteq \bigcap_{n=1}^{\infty} Tx_n = \{\overline{x}\}$ and so $T\overline{x} = \{\overline{x}\}$. □

From Theorem 2.11, we obtain, as a consequence, the following result due to Jachymski [20].

Theorem 2.12. *Let (X, d) be a complete metric space and let $T : X \to 2^X$ be a set-valued map with nonempty closed values such that $Ty \subseteq Tx$ for each $y \in Tx$. Assume that for any $x \in X$ and $\epsilon > 0$, there exists $y \in Tx$ such that diam$(Ty) < \epsilon$. Then T has an endpoint.*

Another consequence of the main result can be obtained in the context of a b-metric space. We will first give the definition of a b-metric space.

Definition 2.13. (Bakhtin [5], Czerwik [11]) Let X be a set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}_+$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space with constant s.

Several examples of b-metric spaces are given by V. Berinde [7], S. Czerwik [11], [10], etc.

Remark 2.14. Notice that in a b-metric space (X, d) the following assertions hold:

(i) a convergent sequence has a unique limit;
(ii) $(X, \overset{d}{\to})$ is an L-space (see Fréchet [16], Blumenthal [8]);
(iii) in general, a b-metric is not continuous;
(iv) a continuous b-metric induce a topology on X (see Blumenthal [8]).

The following generic example was also given in [10].

Example 2.15. Let E be a Banach space, let P be a cone in E with $intP \neq \emptyset$ and let $\leq$ be a partial ordering with respect to P. A mapping $d : X \times X \to E$ is called a cone metric on the nonempty set X if the following axioms are satisfied:

1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2) $d(x, y) = d(y, x)$, for all $x, y \in X$
3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

The pair (X, d), where X is a nonempty set and d is a cone metric is called a cone metric space.

If the cone P is normal with constant K, then the cone metric $d : X \times X \to E$ is continuous.

Let E be a Banach space and P be a normal cone in E with the coefficient of normality denoted by K. Let $D : X \times X \to \mathbb{R}$ be defined by $D(x, y) = \|d(x, y)\|$, where $d : X \times X \to E$ is a cone metric. Then (X, D) is a b-metric space with constant $s := K \geq 1$.

Moreover, since the topology τ_d generated by the cone metric d coincides with the topology τ_D generated by the b-metric D, we have that the b-metric D is continuous too.

From Lemma 2.10, we obtain, as a consequence, the following result proved in [10]

Theorem 2.16. *Let (X, d) be a complete b-metric space. Then, for every descending sequence $\{A_n\}_{n\geq 1}$ of nonempty closed subsets of X such that diam$(A_n) \to 0$ as $n \to \infty$. Then the intersection $\bigcap_{n=1}^{\infty} A_n$ contains one and only one point.*

Thus, we also get the following extension of Jachymski's theorem from [20].

Theorem 2.17. *Let (X, d) be a complete b-metric space and let $T : X \to 2^X$ be a set-valued map with nonempty closed values such that $Ty \subseteq Tx$ for each $y \in Tx$. Assume that for any $x \in X$ and $\epsilon > 0$, there exists $y \in Tx$ such that diam$(Ty) < \epsilon$. Then T has an endpoint.*

Notice that, the above results in b-metric spaces generate a Cantor type intersection Lemma and an endpoints theorem in cone metric spaces, due to Example 2.15.

Now we illustrate our main result by the following examples.

Example 2.18. Let $X = \{0, 1, 2\}$ and consider on X a topology τ on X given by $\tau = \{\emptyset, X, \{1\}, \{2\}, \{0, 1\}, \{1, 2\}\}$. Let $\delta : 2^X \to [0, \infty]$ be given by $\delta(\emptyset) = \delta(\{0\}) = \delta(\{1\}) = \delta(\{2\}) = 0,\ \delta(\{0, 1\}) = \delta(\{0, 2\}) = \delta(\{1, 2\}) = 1, \delta(\{0, 1, 2\}) = 2$. Let $T : X \to 2^X$ be given by

$$T0 = \{0\},\ T1 = \{0, 1\} \text{ and } T2 = \{0, 1, 2\}.$$

Notice that $(X, \to)$ is a generalized L-space, where $\to$ is the convergence generated by τ. Then it is straightforward to show that all of the assumptions of Theorem 2.11 are satisfied and T has an endpoint $\bar{x} = 0$. Since the topological space (X, τ) is not metrizable (actually the topology τ is not Hausdorff) we can't invoke the above mentioned theorem of Jachymski to show the existence of an endpoint for T.

Example 2.19. Let $X = [0, 1]$ and let $\tau = \{\emptyset, X, [0, 1)\} \cup \{A : A \subseteq \mathbb{Q}^c \cap [0, 1]\}$ a topology on X. Let $\delta : 2^X \to [0, \infty]$ be defined as

$$\delta(A) = \begin{cases} 0, & A \text{ is either empty or a singleton} \\ 1. & \text{otherwise} \end{cases}$$

Let $T : X \to 2^X$ be given by

$$Tx = \begin{cases} \{1\}, & x \in \mathbb{Q} \\ \mathbb{Q} \cap [0, 1]. & \text{otherwise} \end{cases}$$

Let (x_n) be a δ-Cauchy sequence, i.e., $\lim_{n\to\infty} \delta(\{x_n, x_{n+1}, \dots\}) = 0$. Thus, there exists $k \in \mathbb{N}$ such that $\delta(\{x_n, x_{n+1}, \dots\}) < 1$ for $n \geq k$. Thus, by the definition of δ we get that $\delta(\{x_n, x_{n+1}, \dots\}) = 0$ and so the set $\{x_n, x_{n+1}, \dots\}$ is a singleton, for $n \geq k$. Hence $x_k = x_{k+1} = x_{k+2} = \dots$, that is, the sequence (x_n) is eventually constant. Hence, it is convergent with respect to the convergence generated by τ. Thus (X, τ) is δ-complete. Let $y = 1$. Then $y \in Tx$ for each $x \in X$ and $\delta(Ty) = 0 < \epsilon$ for each $\epsilon > 0$. Then from Theorem 2.11, we get T has an endpoint (notice again that (X, τ) isn't a Hausdorff space).

As an application of Theorem 2.11, we obtain the following generalization of the order-theoretic Cantor type theorem due to Granas and Horvath [17, 18].

Theorem 2.20. *Let $(X, \to)$ be a δ-complete generalized L-space endowed with a partial order $\preceq$. Assume that for any $x \in X$, the set $\{y \in X : x \preceq y\}$ is closed and given $\epsilon > 0$, there is $y \succeq x$ such that $\delta(\{z \in X : y \preceq z\}) < \epsilon$. Then $(X, \preceq)$ has a maximal element.*

Proof. For $x \in X$ let us define $Tx := \{y \in X : x \preceq y\}$. By hypothesis, T has closed values and by transitivity of $\preceq$, we have that $Ty \subseteq Tx$ for each $y \in Tx$. For each $x \in X$ there is $y \in Tx$ such that $\delta(Ty) < \epsilon$. Thus all of the assumptions of Theorem 2.11 are satisfied and so there is $\overline{x} \in X$ such that $T\overline{x} = \{\overline{x}\}$. Hence if $\overline{x} \preceq x$, i.e., $x \in T\overline{x}$ then $x = \overline{x}$, which means $\overline{x}$ is a maximal element. □

Remark 2.21. The above result takes place if we replace "partial order" with "preorder" (by "preorder" we mean a relation which is only reflexive and transitive).

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