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# A NEW ACCURACY CRITERION FOR THE CONTRACTION PROXIMAL POINT ALGORITHM WITH TWO MONOTONE OPERATORS

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*This paper is dedicated to Professor Sompong Dhompongsa on the occasion of his 65th Birthday*

Abstract. An iterative algorithm is proposed to find a common zero of two maximal monotone operators $A$ and $B$ in a Hilbert space. This algorithm is a two-step procedure which alternates the operators $A$ and $B$. The features of this algorithm are its combination of regularization and contraction for the proximal point algorithm, and its strong convergence under different accuracy criteria on the errors.

## 1. Introduction

von Neumann (1933) initiated the study of finding a point in the intersection of two closed subspaces of a Hilbert space $H$ by the method of alternating projections. More precisely, let $K_1$ and $K_2$ be closed subspaces of $H$. Then von Neumann proved that, for each $x \in H$, the sequence $(x_n)$ defined by the alternating projections

$$x_0 := x,\ x_{2k-1} := P_{K_1} x_{2k-2},\ x_{2k} := P_{K_2} x_{2k-1},\ k = 1, 2, \dots \tag{1.1}$$

converges in norm to $P_{K_1 \cap K_2} x$. [Here $P_K$ denotes the orthogonal projection from $H$ onto a closed subspace $K$ of $H$.] A proof to this classic result of von Neumann can be found in [2].

Bregman [9] (1965) extended von Neumann's result to the case where $K_1$ and $K_2$ are two intersecting closed convex subsets by proving that the sequence $(x_n)$ generated by the same method (1.1) of alternating projections converges weakly to a point of the intersection $K_1 \cap K_2$, namely, a point $x^*$ that solves the convex feasibility problem

$$x^* \in K_1 \cap K_2. \tag{1.2}$$

[Note that in the case of Bregman's method, $P_{K_i}$ denotes the nearest point projection from $H$ onto the closed convex subset $K_i$ $(i = 1, 2)$.]

It had been an outstanding question whether or not the weak convergence of Bregman's alternating projection method above-mentioned can be in norm in the infinite-dimensional case. This question was eventually solved in the negative in 2004 by Hundal's counterexample [13]. This example is also useful in constructing other counterexamples (see [16, 23]).

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It is therefore an interesting topic of designing iterative methods that generate sequences converging in the norm topology to a point in the intersection $K_1 \cap K_2$. The study of this topic has connections to the work of [15, 17, 19, 22].

Another interesting extension of the convex feasibility problem (1.3) is the problem of finding a common zero of two maximal monotone operators, that is, the problem

$$x^* \in A^{-1}(0) \cap B^{-1}(0) \quad \text{or} \quad 0 \in Ax^* \cap Bx^*, \tag{1.3}$$

where $A$ and $B$ are two maximal monotone operators in $H$.

The most well-known method for finding a zero of a maximal monotone operator $A$ is Rochafellar's proximal point algorithm (PPA) [18] which has weak convergence, but no strong convergence [11], in general. Strongly convergent modifications of PPA can be found in [15, 17, 19, 22]. Note that the PPA essentially iterates the resolvent of the maximal monotone operator $A$.

Recently, in order to solve the problem (1.3) by strongly convergent resolvent methods, Boikanyo and Morosanu proposed several modifications of the PPA in their papers [4, 5, 7]. In particular, they considered in [7] the following alternating resolvent method that generates a sequence $(x_n)$ according to the rule

$$x_{2n+1} = J^A_{\beta_n}(\alpha_n u + (1-\alpha_n)x_{2n} + e_n), \quad n \geq 0, \tag{1.4}$$

$$x_{2n} = J^B_{\mu_n}(\lambda_n u + (1-\lambda_n)x_{2n-1} + e'_n), \quad n \geq 1, \tag{1.5}$$

where $u, x_0 \in H$, $A$ and $B$ are maximal monotone operators, $(e_n)$ and $(e'_n)$ are sequences of errors, and $\alpha_n, \lambda_n \in (0,1)$ and $\beta_n, \mu_n \in (0,\infty)$ are sequences of parameters. Here $J^A_\beta := (I+\beta A)^{-1}$ denotes the resolvent of $A$ of index $\beta > 0$. Boikanyo and Morosanu showed that the sequence $(x_n)$ generated by the algorithm (1.4)-(1.5) converges strongly to a solution of (1.3) which is nearest to the point $u$ from the common solution set $A^{-1}(0) \cap B^{-1}(0)$ provided the parameter sequences $(\alpha_n), (\lambda_n), (\beta_n)$ and $(\mu_n)$, and the error sequences $(e_n)$ and $(e'_n)$ satisfy certain appropriate conditions. The alternating resolvent method (1.4)-(1.5) extends the method given in [1] and also uses regularization of proximal point algorithm. In [21], Wang and Cui investigated the the contraction proximal point algorithm (CPPA) which generates a sequence $(x_n)$ by the recursion process (for a single maximal monotone operator $A$):

$$x_{n+1} = \alpha_n u + \delta_n x_n + \gamma_n J^A_{\beta_n} x_n + e_n, \quad n = 0, 1, \ldots, \tag{1.6}$$

where $\alpha_n, \delta_n, \gamma_n \in (0,1)$ with $\alpha_n + \delta_n + \gamma_n = 1$. They proved that the sequence $(x_n)$ thus generated converges strongly to a zero of $A$ (if any). Boikanyo and Morosanu [6] extended Wang and Cui's algorithm to the case of two maximal monotone operators. More precisely, they introduced the following algorithm

$$x_{2n+1} = \alpha_n u + \delta_n x_{2n} + \gamma_n J^A_{\beta_n} x_{2n} + e_n, \quad n = 0, 1, \ldots, \tag{1.7}$$

$$x_{2n} = \lambda_n u + \rho_n x_{2n-1} + \sigma_n J^B_{\mu_n} x_{2n-1} + e'_n, \quad n = 1, 2, \ldots, \tag{1.8}$$

where $\alpha_n, \delta_n, \gamma_n \in (0,1)$ are such that $\alpha_n + \delta_n + \gamma_n = 1$ and $\lambda_n, \rho_n, \sigma_n \in (0,1)$ such that $\lambda_n + \rho_n + \sigma_n = 1$. Note that this algorithm unifies the results in [21, 24, 3].

Recall in [18], to solve the inclusion problem

$$0 \in Ax \tag{1.9}$$

where $A$ is a maximal monotone operator in $H$ such that (1.9) is solvable, Rockafellar proposed the proximal point algorithm (PPA) that generates, with an initial guess $x_0 \in H$, the sequence $(x_n)$ via the resolvent iteration procedure

$$x_{n+1} = J^A_{\beta_n}(x_n + e_n), \tag{1.10}$$

where $J^A_{\beta_n}$ stands for the resolvent of $A$ of index $\beta_n$ and $(e_n)$ is the error sequence. In general, the following accuracy criterion on the error sequence $(e_n)$ applies:

$$\|e_n\| \leq \varepsilon_n \quad \text{with} \quad \sum_{n=0}^{\infty} \varepsilon_n < \infty \tag{I}$$

in order to ensure the convergence of PPA. In [18], Rockafellar also presented another accuracy criterion on the error sequence,

$$\|e_n\| \leq \eta_n \|\tilde{x}_n - x_n\| \quad \text{with} \quad \sum_{n=0}^{\infty} \eta_n < \infty,$$

where $\tilde{x}_n = J^A_{\beta_n}(x_n + e_n)$.

This criterion is improved by Han and He [12] as follows

$$\|e_n\| \leq \eta_n \|\tilde{x_n} - x_n\| \quad \text{with} \quad \sum_{n=0}^{\infty} \eta_n^2 < \infty. \tag{II}$$

Under the criterion (I), Boikanyo and Morosanu proposed some iterative algorithms to ensure the strong convergence for the method of alternating resolvents [4, 6].

Let us turn our attention to the criterion (II). In the present paper, under the criterion (II), we will investigate the iterative algorithm as below

$$x_{2n+1} = \alpha_n u + \delta_n x_{2n} + \gamma_n J^A_{\beta_n}(x_{2n} + e_n), \quad n = 0, 1, \ldots, \tag{1.11}$$

$$x_{2n} = \lambda_n u + \rho_n x_{2n-1} + \sigma_n J^B_{\mu_n}(x_{2n-1} + e'_n), \quad n = 1, 2, \ldots, \tag{1.12}$$

where $u, x_0 \in H$, $\alpha_n, \delta_n, \gamma_n \in (0,1)$ are such that $\alpha_n + \delta_n + \gamma_n = 1$, $\lambda_n, \rho_n, \sigma_n \in (0,1)$ such that $\lambda_n + \rho_n + \sigma_n = 1$, and $\beta_n, \mu_n \in (0, \infty)$. This algorithm extends and unifies the algorithms of [8, 20].

## 2. Preliminaries

Let $H$ be a Hilbert space and let $x \in H$ and $(x_n)$ a sequence in $H$. In what follows, we always denote by '$x_n \to x$ and '$x_n \rightharpoonup x$' the strong and respectively, weak convergence to $x$ of the sequence $(x_n)$. Recall that an operator $A$ with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ in $H$ is said to monotone if

$$\langle u - v, x - y \rangle \geq 0, \quad x, y \in \mathcal{D}(A),\ u \in Ax,\ v \in Ay.$$

A monotone operator $A$ is said to maximal monotone if its graph

$$\mathcal{G}(A) = \{(x, y) : x \in \mathcal{D}(A),\ y \in Ax\}$$

is not properly contained in the graph of any other monotone operator.

Associated with a monotone operator $A$ is its resolvent of index $\beta > 0$ which is defined by $J_\beta^A$

$$J_\beta^A = (I + \beta A)^{-1}.$$

It is easily known that $J_\beta^A$ is single-valued.

Assume now $C$ is a nonempty closed and convex subset of $H$. We then use $P_C$ to denote the projection from $H$ to $C$. Thus, for $x \in H$, $P_C x$ is the unique point in $C$ with the property: $\|x - P_C x\| = \min_{y \in C} \|x - y\|$.

It is well known that $P_C x$ is characterized by

$$P_C x \in C, \quad \langle x - P_C x, z - P_C x\rangle \leq 0, \quad z \in C. \tag{2.1}$$

If $T : C \to H$ is a mapping, then say that

(a) $T$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;

(b) $T$ is firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2 \quad \text{for all } x, y \in C.$$

We next collect some nice properties of the resolvent of a monotone operator.

**Lemma 2.1.** *Let $A$ be a maximal monotone operator in $H$. Then*

(i) $J_\beta^A : H \to H$ *is single-valued and firmly nonexpansive;*

(ii) $Fix(J_\beta^A) = \{x \in \mathcal{D}(A) : 0 \in Ax\} = A^{-1}(0)$;

(iii) $\|x - J_\beta^A x\| \leq 2\|x - J_{\beta'}^A x\|$ *for all* $0 < \beta \leq \beta'$ *and for all* $x \in H$ [15].

**Lemma 2.2** ([10])**.** *Let $C$ be a nonempty closed convex subset of $H$ and $T : C \to H$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $(x_n)$ is a sequence in $C$ such that $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$, then $(I - T)x = 0$, i.e.$x \in Fix(T)$.*

**Lemma 2.3** ([22])**.** *Let $\{s_n\}, \{c_n\} \subset \mathbb{R}^+$, $\{\lambda_n\} \subset (0, 1)$ and $\{b_n\} \subset \mathbb{R}$ be sequences such that*

$$s_{n+1} \leq (1 - \lambda_n)s_n + b_n + c_n, \quad n \geq 0.$$

*If $\lambda_n \to 0$, $\sum_{n=0}^{\infty} \lambda_n = \infty$, $\sum_{n=0}^{\infty} c_n < \infty$ and $\limsup_{n\to\infty}(b_n/\lambda_n) \leq 0$, then $\lim_{n\to\infty} s_n = 0$.*

**Lemma 2.4** ([14])**.** *Let $(s_n)$ be a real sequence that does not decrease at the infinity in the sense that there exists a subsequence $(s_{n_k})$ so that*

$$s_{n_k} \leq s_{n_k+1}, \quad k \geq 0.$$

*For every $n > n_0$ define a sequence of integers, $(\tau(n))$, by*

$$\tau(n) = \max\{n_0 \leq k \leq n : s_k < s_{k+1}\}.$$

*Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n > n_0$,*

$$\max(s_{\tau(n)}, s_n) \leq s_{\tau(n)+1}. \tag{2.2}$$

**Lemma 2.5.** *Let $x, y \in H$ and let $t, s \geq 0$. Then*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;

(ii) $\|tx + sy\|^2 = t(t + s)\|x\|^2 + s(t + s)\|y\|^2 - st\|x - y\|^2$.

We now include two lemmas which play a role in proving the boundedness of the sequences generated by our iterative algorithms in Section 3.

**Lemma 2.6.** *Let $\beta > 0$ and let $(s_n)$ be a nonnegative real sequence so that, for $n \geq 0$,*

$$s_{n+1} \leq (1-\alpha_n)(1-\lambda_n)(1+\varepsilon_n)(1+\varepsilon'_n)s_n + [1-(1-\alpha_n)(1-\lambda_n)](1+\varepsilon_n)\beta, \tag{2.3}$$

*where $(\lambda_n) \subset (0,1)$, $(\alpha_n) \subset (0,1)$ and $(\varepsilon_n), (\varepsilon'_n) \in l_1$ are real sequences. Then $(s_n)$ is bounded; more precisely,*

$$s_n \leq \max\{\beta, s_0\} \exp\left(\sum_{k=0}^{\infty} \varepsilon_k + \sum_{k=0}^{\infty} \varepsilon'_k\right), \quad n \geq 0. \tag{2.4}$$

*Proof.* Seting $a_n = (1-\alpha_n)(1-\lambda_n) \in (0,1)$, we can rewrite (2.3) as

$$s_{n+1} \leq a_n(1+\varepsilon_n)(1+\varepsilon'_n)s_n + (1-a_n)(1+\varepsilon_n)\beta. \tag{2.5}$$

It turns out obviously that

$$\begin{aligned} s_{n+1} &\leq (1+\varepsilon_n)(1+\varepsilon'_n)(a_n s_n + (1-a_n)\beta) \\ &\leq (1+\varepsilon_n)(1+\varepsilon'_n)\max\{s_n, \beta\} \\ &\leq (1+\varepsilon_n)(1+\varepsilon'_n)\max\left\{(1+\varepsilon_{n-1})(1+\varepsilon'_{n-1})\max\{s_{n-1}, \beta\}, \beta\right\} \\ &\leq (1+\varepsilon_n)(1+\varepsilon'_n)(1+\varepsilon_{n-1})(1+\varepsilon'_{n-1})\max\{s_{n-1}, \beta\}. \end{aligned}$$

Continuing this way yields the following inequality from which (2.4) follows.

$$s_{n+1} \leq \max\{\beta, s_0\} \prod_{k=0}^{n}(1+\varepsilon_k)(1+\varepsilon'_k), \quad n \geq 0.$$

□

**Lemma 2.7.** *Let $\beta > 0$ and let $(s_n)$ be a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1-\alpha_n)(1-\lambda_n)(1+\varepsilon_n)(1+\varepsilon'_n)s_n + [1-(1-\alpha_n)(1-\lambda_n)](1+\varepsilon_n)\beta, \tag{2.6}$$

*where $(\alpha_n), (\lambda_n) \subset (0,1)$ and $(\varepsilon_n), (\varepsilon'_n) \subset [0,\infty)$ are real sequences. Set $b_n = 1-(1-\alpha_n)(1-\lambda_n)$. If there holds*

$$b_n\varepsilon_n + 2(1-b_n)(\varepsilon_n + \varepsilon'_n + \varepsilon_n\varepsilon'_n) \leq b_n, \tag{2.7}$$

*then $(s_n)$ is bounded. As a matter of fact, we have that $s_n \leq \max\{2\beta, s_0\}$.*

*Proof.* Denote $\tau_n = b_n - (1-b_n)(\varepsilon_n + \varepsilon'_n + \varepsilon_n\varepsilon'_n)$. Then $\tau_n \in (0,1)$ and it follows from (2.6) that

$$\begin{aligned} s_{n+1} &\leq b_n(1+\varepsilon_n)\beta + (1-b_n)(1+\varepsilon_n)(1+\varepsilon'_n)s_n \\ &= \tau_n\left(\frac{b_n(1+\varepsilon_n)}{\tau_n}\right)\beta + (1-\tau_n)s_n. \end{aligned} \tag{2.8}$$

Due to the assumption (2.7), we deduce that

$$\frac{b_n(1+\varepsilon_n)}{\tau_n} = \frac{b_n(1+\varepsilon_n)}{b_n-(1-b_n)(\varepsilon_n+\varepsilon'_n+\varepsilon_n\varepsilon'_n)} \leq 2.$$

Therefore, (2.8) yields that

$$s_{n+1} \leq 2\beta\tau_n + (1-\tau_n)s_n \leq \max\{2\beta, s_n\} \leq \cdots \leq \max\{2\beta, s_0\}.$$

□

## 3. Algorithms and strong convergence

In this section, we propose an iterative algorithm for finding a common zero of two maximal monotone operators $A$ and $B$. To this end, we assume the set of common zeros of $A$ and $B$, $S := A^{-1}(0) \cap B^{-1}(0)$, is nonempty. The feature of our algorithm is that it generate strongly convergent sequences under distinct accuracy criteria on the errors. We begin with the following lemma.

**Lemma 3.1** ([12, 20]). *Let $\eta \in (0, 1/2)$, $x, e \in H$ and $\tilde{x} := J_\beta^A(x+e)$. If $\|e\| \leq \eta\|x - \tilde{x}\|$, then*

$$\|\tilde{x} - z\|^2 \leq (1 + (2\eta)^2)\|x - z\|^2 - \frac{1}{2}\|\tilde{x} - x\|^2, \ \forall\ z \in S.$$

**Theorem 3.2.** *Let $x_0 \in H$. Consider the sequence $(x_n)$ generated by the algorithm:*

$$x_{2n+1} = \alpha_n u + \delta_n x_{2n} + \gamma_n J_{\beta_n}^A(x_{2n} + e_n), \quad n = 0, 1, ..., \tag{3.1}$$

$$x_{2n} = \lambda_n u + \rho_n x_{2n-1} + \sigma_n J_{\mu_n}^B(x_{2n-1} + e_n'), \quad n = 1, 2, ... \tag{3.2}$$

*where $u, x_0 \in H$, $\alpha_n, \delta_n, \gamma_n \in (0, 1)$ are such that $\alpha_n + \delta_n + \gamma_n = 1$, $\lambda_n, \rho_n, \sigma_n \in (0, 1)$ such that $\lambda_n + \rho_n + \sigma_n = 1$, and $\beta_n, \mu_n \in (0, \infty)$.*

*Suppose there hold the conditions:*

(i) $\beta_n \geq \beta > 0$, $\mu_n \geq \mu > 0$, $\gamma_n \geq \gamma > 0$, $\sigma_n \geq \sigma > 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \lambda_n = 0$;

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *or* $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(iv) $\|e_n\| \leq \eta_n \|J_{\beta_n}^A(x_{2n} + e_n) - x_{2n}\|$, $\sum_{n=0}^{\infty} \eta_n^2 < \infty$,

$\|e_n'\| \leq \eta_n' \|J_{\mu_n}^B(x_{2n-1} + e_n') - x_{2n-1}\|$, $\sum_{n=0}^{\infty} (\eta_n')^2 < \infty$.

*Then $(x_n)$ converges strongly to $P_S(u)$.*

*Proof.* Let $z = P_S(u)$. By our hypothesis, we may assume without loss of generality that $\eta_n \in (0, 1/2)$ and $\eta_n' \in (0, 1/2)$. Then by Lemma 3.1, we have,

$$\|J_{\beta_n}^A(x_{2n} + e_n) - z\|^2 \leq (1 + \varepsilon_n)\|x_{2n} - z\|^2 - \frac{1}{2}\|J_{\beta_n}^A(x_{2n} + e_n) - x_{2n}\|^2, \tag{3.3}$$

$$\|J_{\mu_n}^B(x_{2n-1} + e_n') - z\|^2 \leq (1 + \varepsilon_n')\|x_{2n-1} - z\|^2 - \frac{1}{2}\|J_{\mu_n}^B(x_{2n-1} + e_n') - x_{2n-1}\|^2, \tag{3.4}$$

where $\varepsilon_n := (2\eta_n)^2$ and $\varepsilon'_n := (2\eta'_n)^2$ satisfying $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and $\sum_{n=0}^{\infty} \varepsilon'_n < \infty$. By (3.3), (3.4) and (3.1), we deduce

$$\begin{aligned}
\|x_{2n+1} - z\|^2 &= \|\alpha_n u + \delta_n x_{2n} + \gamma_n J^A_{\beta_n}(x_{2n} + e_n) - z\|^2 \\
&= \|\alpha_n(u - z) + \delta_n(x_{2n} - z) + \gamma_n(J^A_{\beta_n}(x_{2n} + e_n) - z)\|^2 \\
&\le \alpha_n\|u - z\|^2 + (1 - \alpha_n)\|\frac{\delta_n}{\delta_n + \gamma_n}(x_{2n} - z) \\
&\quad + \frac{\gamma_n}{\delta_n + \gamma_n}(J^A_{\beta_n}(x_{2n} + e_n) - z)\|^2 \\
&\le \alpha_n\|u - z\|^2 + \delta_n\|x_{2n} - z\|^2 + \gamma_n\|J^A_{\beta_n}(x_{2n} + e_n) - z\|^2 \\
&\le \alpha_n\|u - z\|^2 + \delta_n\|x_{2n} - z\|^2 + \gamma_n(1 + \varepsilon_n)\|x_{2n} - z\|^2 \\
&\quad - \frac{\gamma_n}{2}\|J^A_{\beta_n}(x_{2n} + e_n) - x_{2n}\|^2 \\
&\le \alpha_n\|u - z\|^2 + [\delta_n + \gamma_n(1 + \varepsilon_n)]\|x_{2n} - z\|^2.
\end{aligned}$$

Similarly, we can also get

$$\|x_{2n} - z\|^2 \le \lambda_n\|u - z\|^2 + [\rho_n + \sigma_n(1 + \varepsilon'_n)]\|x_{2n-1} - z\|^2. \tag{3.5}$$

Hence,

$$\begin{aligned}
\|x_{2n+1} - z\|^2 &\le \alpha_n\|u - z\|^2 + [\delta_n + \gamma_n(1 + \varepsilon_n)][\lambda_n\|u - z\|^2 \\
&\quad + (\rho_n + \sigma_n(1 + \varepsilon'_n))\|x_{2n-1} - z\|^2] \\
&= \alpha_n\|u - z\|^2 + \lambda_n[\delta_n + \gamma_n(1 + \varepsilon_n)]\|u - z\|^2 \\
&\quad + [\delta_n + \gamma_n(1 + \varepsilon_n)][\rho_n + \sigma_n(1 + \varepsilon'_n)]\|x_{2n-1} - z\|^2 \\
&\le [\alpha_n + \lambda_n\delta_n + \lambda_n\gamma_n(1 + \varepsilon_n)]\|u - z\|^2 \\
&\quad + (\delta_n + \gamma_n)(\rho_n + \sigma_n)(1 + \varepsilon_n)(1 + \varepsilon'_n)\|x_{2n-1} - z\|^2 \\
&\le (\alpha_n + \lambda_n\delta_n + \lambda_n\gamma_n)(1 + \varepsilon_n)\|u - z\|^2 \\
&\quad + (1 - \alpha_n)(1 - \lambda_n)(1 + \varepsilon_n)(1 + \varepsilon'_n)\|x_{2n-1} - z\|^2 \\
&= (\alpha_n + \lambda_n - \alpha_n\lambda_n)(1 + \varepsilon_n)\|u - z\|^2 \\
&\quad + (1 - \alpha_n)(1 - \lambda_n)(1 + \varepsilon_n)(1 + \varepsilon'_n)\|x_{2n-1} - z\|^2 \\
&= [1 - (1 - \alpha_n)(1 - \lambda_n)](1 + \varepsilon_n)\|u - z\|^2 \\
&\quad + (1 - \alpha_n)(1 - \lambda_n)(1 + \varepsilon_n)(1 + \varepsilon'_n)\|x_{2n-1} - z\|^2.
\end{aligned}$$

Applying Lemma 2.6 to the last inequality, we conclude that $(x_{2n+1})$ is bounded. By (3.5), we find that $(x_{2n})$ is bounded as well. In a summary, $(x_n)$ is bounded.

It follows from Lemma 2.5 that,

$$\begin{aligned}
\|x_{2n+1} - z\|^2 &= \|\alpha_n u + \delta_n x_{2n} + \gamma_n J^A_{\beta_n}(x_{2n} + e_n) - z\|^2 \\
&= \|\alpha_n(u - z) + \delta_n x_{2n} + \gamma_n J^A_{\beta_n}(x_{2n} + e_n) - (\delta_n + \gamma_n)z\|^2 \\
&\le \|\delta_n x_{2n} + \gamma_n J^A_{\beta_n}(x_{2n} + e_n) - (\delta_n + \gamma_n)z\|^2 + 2\alpha_n\langle u - z, x_{2n+1} - z\rangle \\
&= \delta_n(\delta_n + \gamma_n)\|x_{2n} - z\|^2 + \gamma_n(\delta_n + \gamma_n)\|J^A_{\beta_n}(x_{2n} + e_n) - z\|^2
\end{aligned}$$

$$
\begin{aligned}
&- \delta_n\gamma_n\|J^A_{\beta_n}(x_{2n}+e_n)-x_{2n}\|^2 + 2\alpha_n\langle u-z, x_{2n+1}-z\rangle \\
&\le \delta_n(\delta_n+\gamma_n)\|x_{2n}-z\|^2 + \gamma_n(\delta_n+\gamma_n)[(1+\varepsilon_n)\|x_{2n}-z\|^2 \\
&\quad - \frac{1}{2}\|J^A_{\beta_n}(x_{2n}+e_n)-x_{2n}\|^2] - \delta_n\gamma_n\|J^A_{\beta_n}(x_{2n}+e_n)-x_{2n}\|^2 \\
&\quad + 2\alpha_n\langle u-z, x_{2n+1}-z\rangle \\
&= (1-\alpha_n)^2\|x_{2n}-z\|^2 + \varepsilon_n\gamma_n(\delta_n+\gamma_n)\|x_{2n}-z\|^2 \\
&\quad - \frac{1}{2}(3\gamma_n\delta_n+\gamma_n^2)\|J^A_{\beta_n}(x_{2n}+e_n)-x_{2n}\|^2 + 2\alpha_n\langle u-z, x_{2n+1}-z\rangle \\
&\le (1-\alpha_n)\|x_{2n}-z\|^2 + M_1\varepsilon_n \\
&\quad - \frac{1}{2}(3\gamma_n\delta_n+\gamma_n^2)\|J^A_{\beta_n}(x_{2n}+e_n)-x_{2n}\|^2 \\
&\quad + 2\alpha_n\langle u-z, x_{2n+1}-z\rangle,
\end{aligned}
$$

where $M_1 > 0$ such that $\gamma_n(\delta_n+\gamma_n)\|x_{2n}-z\|^2 \le M_1$. Similarly,

$$
\begin{aligned}
\|x_{2n}-z\|^2 &\le (1-\lambda_n)^2\|x_{2n-1}-z\|^2 + \varepsilon_n'\sigma_n(\rho_n+\sigma_n)\|x_{2n-1}-z\|^2 \\
&\quad - \frac{1}{2}(3\rho_n\sigma_n+\sigma_n^2)\|J^B_{\mu_n}(x_{2n-1}+e_n')-x_{2n-1}\|^2 + 2\lambda_n\langle u-z, x_{2n}-z\rangle \\
&\le (1-\lambda_n)\|x_{2n-1}-z\|^2 + M_2\varepsilon_n' \\
&\quad - \frac{1}{2}(3\rho_n\sigma_n+\sigma_n^2)\|J^B_{\mu_n}(x_{2n-1}+e_n')-x_{2n-1}\|^2 + 2\lambda_n\langle u-z, x_{2n}-z\rangle,
\end{aligned}
$$

where $M_2 > 0$ such that $\sigma_n(\rho_n+\sigma_n)\|x_{2n-1}-z\|^2 \le M_2$. Therefore,

$$
\begin{aligned}
\|x_{2n+1}-z\|^2 &\le (1-\alpha_n)[(1-\lambda_n)\|x_{2n-1}-z\|^2 + M_2\varepsilon_n' \\
&\quad - \frac{1}{2}(3\rho_n\sigma_n+\sigma_n^2)\|J^B_{\mu_n}(x_{2n-1}+e_n')-x_{2n-1}\|^2 \\
&\quad + 2\lambda_n\langle u-z, x_{2n}-z\rangle] + M_1\varepsilon_n \\
&\quad - \frac{1}{2}(3\gamma_n\delta_n+\gamma_n^2)\|J^A_{\beta_n}(x_{2n}+e_n)-x_{2n}\|^2 + 2\alpha_n\langle u-z, x_{2n+1}-z\rangle \\
&= (1-\alpha_n)(1-\lambda_n)\|x_{2n-1}-z\|^2 + M_2(1-\alpha_n)\varepsilon_n' \\
&\quad - \frac{1}{2}(1-\alpha_n)(3\rho_n\sigma_n+\sigma_n^2)\|J^B_{\mu_n}(x_{2n-1}+e_n')-x_{2n-1}\|^2 \\
&\quad + 2\lambda_n(1-\alpha_n)\langle u-z, x_{2n}-z\rangle + M_1\varepsilon_n \\
&\quad - \frac{1}{2}(3\gamma_n\delta_n+\gamma_n^2)\|J^A_{\beta_n}(x_{2n}+e_n)-x_{2n}\|^2 \\
&\quad + 2\alpha_n\langle u-z, x_{2n+1}-z\rangle \\
&\le (1-\alpha_n)(1-\lambda_n)\|x_{2n-1}-z\|^2 - \frac{1}{2}a\|J^A_{\beta_n}(x_{2n}+e_n)-x_{2n}\|^2 \\
&\quad - \frac{1}{2}b\|J^B_{\mu_n}(x_{2n-1}+e_n')-x_{2n-1}\|^2 + M(\varepsilon_n+\varepsilon_n') \\
&\quad + 2\lambda_n(1-\alpha_n)\langle u-z, x_{2n}-z\rangle + 2\alpha_n\langle u-z, x_{2n+1}-z\rangle,
\end{aligned}
$$

where $3\gamma_n\delta_n+\gamma_n^2\geq a>0$, $(1-\alpha_n)(3\rho_n\sigma_n+\sigma_n^2)\geq b>0$ (by condition (i) and (ii)), and $M=\max\{M_1,M_2\}$. It turns out that

$$\begin{aligned}\|x_{2n+1}-z\|^2-\|x_{2n-1}-z\|^2+(\alpha_n+\lambda_n-\alpha_n\lambda_n)\|x_{2n-1}-z\|^2\\+\frac{1}{2}a\|J^A_{\beta_n}(x_{2n}+e_n)-x_{2n}\|^2+\frac{1}{2}b\|J^B_{\mu_n}(x_{2n-1}+e'_n)-x_{2n-1}\|^2\\\leq 2\lambda_n(1-\alpha_n)\langle u-z,x_{2n}-z\rangle+2\alpha_n\langle u-z,x_{2n+1}-z\rangle+M(\varepsilon_n+\varepsilon'_n).\end{aligned}\tag{3.6}$$

Setting

$$s=\sum_{k=0}^{\infty}\varepsilon_k+\sum_{k=0}^{\infty}\varepsilon'_k<\infty,\quad t_n=s-\sum_{k=0}^{n-1}\varepsilon_k-\sum_{k=0}^{n-1}\varepsilon'_k,\quad s_n=\|x_{2n-1}-z\|^2+Mt_n,$$

we can rewrite (3.6) as

$$\begin{aligned}s_{n+1}-s_n+(\alpha_n+\lambda_n-\alpha_n\lambda_n)\|x_{2n-1}-z\|^2\\+\frac{1}{2}a\|J^A_{\beta_n}(x_{2n}+e_n)-x_{2n}\|^2+\frac{1}{2}b\|J^B_{\mu_n}(x_{2n-1}+e'_n)-x_{2n-1}\|^2\\\leq 2\lambda_n(1-\alpha_n)\langle u-z,x_{2n}-z\rangle+2\alpha_n\langle u-z,x_{2n+1}-z\rangle.\end{aligned}\tag{3.7}$$

It is obviously clear that $s_n\to 0\Leftrightarrow\|x_{2n-1}-z\|\to 0$.

We next verify that $s_n\to 0$ as $n\to\infty$ by considering two possible cases for the sequence $(s_n)$.

Case 1: $(s_n)$ is eventually decreasing (i.e. there exists $N>0$ such that $(s_n)$ is decreasing for $n\geq N$). In this case, $(s_n)$ is convergent. Then passing to the limit in (3.7), we get

$$\|J^A_{\beta_n}(x_{2n}+e_n)-x_{2n}\|\to 0,\quad \|J^B_{\mu_n}(x_{2n-1}+e'_n)-x_{2n-1}\|\to 0.$$

So it follows that

$$\begin{aligned}\|x_{2n+1}-x_{2n}\|&=\|\alpha_n u+\delta_n x_{2n}+\gamma_n J^A_{\beta_n}(x_{2n}+e_n)-x_{2n}\|\\&\leq\alpha_n\|u-x_{2n}\|+\gamma_n\|J^A_{\beta_n}(x_{2n}+e_n)-x_{2n}\|\to 0.\\\|x_{2n}-x_{2n-1}\|&=\|\lambda_n u+\rho_n x_{2n-1}+\sigma_n J^B_{\mu_n}(x_{2n-1}+e'_n)-x_{2n-1}\|\\&\leq\lambda_n\|u-x_{2n-1}\|+\sigma_n\|J^B_{\mu_n}(x_{2n-1}+e'_n)-x_{2n-1}\|\to 0.\end{aligned}$$

That is, $\|x_{n+1}-x_n\|\to 0$.

Now take a subsequence $(x_{n_k})$ of $(x_n)$ so that $(x_{n_k})$ converges weakly to $\hat{x}$ and

$$\limsup_{n\to\infty}\langle u-z,x_n-z\rangle=\lim_{k\to\infty}\langle u-z,x_{n_k}-z\rangle.\tag{3.8}$$

Since

$$\begin{aligned}\|J^A_{\beta_n}(x_{2n}+e_n)-(x_{2n}+e_n)\|&\leq\|J^A_{\beta_n}(x_{2n}+e_n)-x_{2n}\|+\|e_n\|\\&\leq\|J^A_{\beta_n}(x_{2n}+e_n)-x_{2n}\|+\eta_n\|J^A_{\beta_n}(x_{2n}+e_n)-x_{2n}\|\\&\to 0\end{aligned}$$

and by Lemma 2.1, $\|J^A_{\beta}(x_{2n}+e_n)-(x_{2n}+e_n)\|\leq 2\|J^A_{\beta_n}(x_{2n}+e_n)-(x_{2n}+e_n)\|$, we obtain that $\|J^A_{\beta}(x_{2n}+e_n)-(x_{2n}+e_n)\|\to 0$. Thus we can apply Lemma 2.2 to find that $\omega_w(x_{2n})\subset A^{-1}(0)$. By a similar argument, we have $\omega_w(x_{2n-1})\subset B^{-1}(0)$.

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, we conclude that $\omega_w(x_n)\subset S$; in particular, $\hat{x}\in S$. Moreover, we have by (3.8)

$$\limsup_{n\to\infty}\langle u-z, x_n-z\rangle = \langle u-P_S u, \hat{x}-P_S u\rangle \le 0. \tag{3.9}$$

Now from (3.6), we derive that

$$\begin{aligned}
&\|x_{2n+1}-z\|^2\\
&\le (1-\alpha_n)(1-\lambda_n)\|x_{2n-1}-z\|^2 + 2\lambda_n(1-\alpha_n)\langle u-z, x_{2n}-z\rangle\\
&\quad + 2\alpha_n\langle u-z, x_{2n+1}-z\rangle + M(\varepsilon_n+\varepsilon_n')\\
&= (1-\alpha_n)(1-\lambda_n)\|x_{2n-1}-z\|^2 + 2(\alpha_n+\lambda_n-\alpha_n\lambda_n)\langle u-z, x_{2n}-z\rangle\\
&\quad + 2\alpha_n\langle u-z, x_{2n+1}-x_{2n}\rangle + M(\varepsilon_n+\varepsilon_n')\\
&= (1-\alpha_n)(1-\lambda_n)\|x_{2n-1}-z\|^2 + (\alpha_n+\lambda_n-\alpha_n\lambda_n)[2\langle u-z, x_{2n}-z\rangle\\
&\quad + \frac{2\alpha_n}{\alpha_n+\lambda_n-\alpha_n\lambda_n}\langle u-z, x_{2n+1}-x_{2n}\rangle] + M(\varepsilon_n+\varepsilon_n')\\
&\le (1-\alpha_n)(1-\lambda_n)\|x_{2n-1}-z\|^2 + (\alpha_n+\lambda_n-\alpha_n\lambda_n)[2\langle u-z, x_{2n}-z\rangle\\
&\quad + 2\|u-z\|\|x_{2n+1}-x_{2n}\|] + M(\varepsilon_n+\varepsilon_n').
\end{aligned}$$

Conditions (ii) and (iii) trivially imply that

$$\lim_{n\to\infty}(\alpha_n+\lambda_n-\alpha_n\lambda_n)=0 \quad\text{and}\quad \sum_{n=0}^{\infty}(\alpha_n+\lambda_n-\alpha_n\lambda_n)=\infty.$$

Hence we can apply Lemma 2.3 to conclude that $\|x_{2n-1}-z\|\to 0$. According to (3.5), we also get $\|x_{2n}-z\|\to 0$, and therefore, $\|x_n-z\|\to 0$.

Case 2: $(s_n)$ is not eventually decreasing. In this case, we can find a subsequence $(s_{n_k})$ of $(s_n)$ so that $s_{n_k}\le s_{n_k+1}$ for all $k\ge 0$. Define a sequence of integers $(\tau(n))$ as in Lemma 2.4. Since $s_{\tau(n)}\le s_{\tau(n)+1}$ for all $n>n_0$ and by (3.7), we have

$$\|J^A_{\beta_{\tau(n)}}(x_{2\tau(n)}+e_{\tau(n)})-x_{2\tau(n)}\|\to 0, \quad \|J^B_{\mu_{\tau(n)}}(x_{2\tau(n)-1}+e'_{\tau(n)})-x_{2\tau(n)-1}\|\to 0.$$

On the other hand, from (3.1) and (3.2), we deduce that

$$\|x_{2\tau(n)+1}-x_{2\tau(n)}\|\to 0, \quad \|x_{2\tau(n)}-x_{2\tau(n)-1}\|\to 0.$$

By an analogous argument to the proof of (3.9), we have

$$\limsup_{n\to\infty}\langle u-z, x_{2\tau(n)-1}-z\rangle\le 0, \quad \limsup_{n\to\infty}\langle u-z, x_{2\tau(n)}-z\rangle\le 0. \tag{3.10}$$

Thus we get $\limsup_{n\to\infty}\langle u-z, x_{2\tau(n)+1}-z\rangle\le 0$. Noticing $s_{\tau(n)+1}-s_{\tau(n)}\ge 0$ and by (3.7), we deduce that

$$\begin{aligned}
&(\alpha_{\tau(n)}+\lambda_{\tau(n)}-\alpha_{\tau(n)}\lambda_{\tau(n)})\|x_{2\tau(n)-1}-z\|^2\\
&\quad\le 2\lambda_{\tau(n)}(1-\alpha_{\tau(n)})\langle u-z, x_{2\tau(n)}-z\rangle + 2\alpha_{\tau(n)}\langle u-z, x_{2\tau(n)+1}-z\rangle\\
&\quad= 2(\alpha_{\tau(n)}+\lambda_{\tau(n)}-\alpha_{\tau(n)}\lambda_{\tau(n)})\langle u-z, x_{2\tau(n)}-z\rangle\\
&\qquad + 2\alpha_{\tau(n)}\langle u-z, x_{2\tau(n)+1}-x_{2\tau(n)}\rangle\\
&\quad\le 2(\alpha_{\tau(n)}+\lambda_{\tau(n)}-\alpha_{\tau(n)}\lambda_{\tau(n)})\langle u-z, x_{2\tau(n)}-z\rangle
\end{aligned}$$

$$+ 2\alpha_{\tau(n)}\|u - z\|\|x_{2\tau(n)+1} - x_{2\tau(n)}\|.$$

It turns out that

$$\begin{aligned}\|x_{2\tau(n)-1} - z\|^2 &\le 2\langle u - z, x_{2\tau(n)} - z\rangle \\ &\quad + 2\frac{\alpha_{\tau(n)}}{\alpha_{\tau(n)} + \lambda_{\tau(n)} - \alpha_{\tau(n)}\lambda_{\tau(n)}}\|u - z\|\|x_{2\tau(n)+1} - x_{2\tau(n)}\| \\ &\le 2\langle u - z, x_{2\tau(n)} - z\rangle + 2\|u - z\|\|x_{2\tau(n)+1} - x_{2\tau(n)}\|.\end{aligned}$$

This implies that $\limsup_{n\to\infty} \|x_{2\tau(n)-1} - z\| \le 0$ and hence

$$\lim_{n\to\infty} \|x_{2\tau(n)-1} - z\| = 0 \quad \text{or} \quad \lim_{n\to\infty} s_{\tau(n)} = 0.$$

Similarly, by (3.7) and noticing the fact that $s_{\tau(n)+1} - s_{\tau(n)} \ge 0$, we can also derive that $\lim_{n\to\infty}(s_{\tau(n)+1} - s_{\tau(n)}) = 0$ so that $\lim_{n\to\infty} s_{\tau(n)+1} = 0$. Now by (2.2) in Lemma 2.4, we obtain $s_n \to 0$, yielding

$$\lim_{n\to\infty}(\|x_{2n-1} - z\| + Mt_n) = 0.$$

This together with the fact that $t_n \to 0$ immediately implies that $\lim_{n\to\infty}\|x_{2n-1} - z\| = 0$ which in turns implies from (3.5) that $\lim_{n\to\infty}\|x_{2n} - z\| = 0$. Therefore, $\lim_{n\to\infty}\|x_n - z\| = 0$. □

Next we consider the strong convergence of the algorithm (3.1)-(3.2) under an accuracy criterion on the errors distinct from condition (iv) of Theorem 3.2.

**Theorem 3.3.** *Let $(x_n)$ be generated by the algorithm (3.1)-(3.2). Assume the same conditions (i)-(iii) in Theorem 3.2. Assume, in addition, condition (iv) in Theorem 3.2 is replaced with the following condition:*

(iv)′ $\|e_n\| \le \eta_n\|J^A_{\beta_n}(x_{2n} + e_n) - x_{2n}\|$, $\lim_{n\to\infty}\frac{\eta_n^2}{\alpha_n} = 0$,
$\|e'_n\| \le \eta'_n\|J^B_{\mu_n}(x_{2n-1} + e'_n) - x_{2n-1}\|$, $\lim_{n\to\infty}\frac{(\eta'_n)^2}{\lambda_n} = 0$.

*Then $(x_n)$ converges in norm to $P_S(u)$.*

*Proof.* Let $z = P_S(u)$. Repeating the argument for estimating $\|x_{2n+1} - z\|^2$ in the proof of Theorem 3.2, we can get

$$\begin{aligned}\|x_{2n+1} - z\|^2 &\le [1 - (1 - \alpha_n)(1 - \lambda_n)](1 + \varepsilon_n)\|u - z\|^2 \\ &\quad + (1 - \alpha_n)(1 - \lambda_n)(1 + \varepsilon_n)(1 + \varepsilon'_n)\|x_{2n-1} - z\|^2,\end{aligned}$$

where $\varepsilon_n := (2\eta_n)^2$ and $\varepsilon'_n := (2\eta'_n)^2$ which are easily seen to satisfy two conditions:

$$\frac{\varepsilon_n}{\alpha_n} \to 0 \quad \text{and} \quad \frac{\varepsilon'_n}{\lambda_n} \to 0.$$

Set $b_n = 1 - (1 - \alpha_n)(1 - \lambda_n)$. Without loss of generality, we assume that

$$b_n\varepsilon_n + 2(1 - b_n)(\varepsilon_n + \varepsilon'_n + \varepsilon_n\varepsilon'_n) \le b_n.$$

We claim that the sequence $(x_n)$ is bounded. In fact, the boundedness of $(x_{2n+1})$ is guaranteed by Lemma 2.7 and the boundedness of $(x_{2n})$ is then a consequence of (3.5). Further from (3.6), we obtain that

$$\begin{aligned}(3.11)\qquad & s_{n+1} - s_n + (\alpha_n + \lambda_n - \alpha_n\lambda_n)s_n \\ &\quad + \frac{1}{2}a\|J^A_{\beta_n}(x_{2n} + e_n) - x_{2n}\|^2 + \frac{1}{2}b\|J^B_{\mu_n}(x_{2n-1} + e'_n) - x_{2n-1}\|^2 \\ &\quad \le 2\lambda_n(1-\alpha_n)\langle u - z, x_{2n} - z\rangle + 2\alpha_n\langle u - z, x_{2n+1} - z\rangle + M(\varepsilon_n + \varepsilon'_n),\end{aligned}$$

where we define $s_n := \|x_{2n-1} - z\|^2$.

To see the strong convergence of $(x_n)$, we again distinguish two cases for $(s_n)$.

Case 1: $(s_n)$ is eventually decreasing (i.e. there exists $N \ge 0$ such that $(s_n)_{n\ge N}$ is decreasing); thus $(s_n)$ must converge. We have

$$\begin{aligned}&\frac{1}{2}a\|J^A_{\beta_n}(x_{2n} + e_n) - x_{2n}\|^2 + \frac{1}{2}b\|J^B_{\mu_n}(x_{2n-1} + e'_n) - x_{2n-1}\|^2 \\ &\quad \le 2\lambda_n(1-\alpha_n)\langle u - z, x_{2n} - z\rangle + 2\alpha_n\langle u - z, x_{2n+1} - z\rangle \\ &\quad + M\alpha_n\left(\frac{\varepsilon_n}{\alpha_n}\right) + M\lambda_n\left(\frac{\varepsilon'_n}{\lambda_n}\right) + (s_n - s_{n+1}) - (\alpha_n + \lambda_n - \alpha_n\lambda_n)s_n \\ &\le \lambda_n M' + \alpha_n M'' + M\alpha_n\left(\frac{\varepsilon_n}{\alpha_n}\right) + M\lambda_n\left(\frac{\varepsilon'_n}{\lambda_n}\right) \\ &\quad + (s_n - s_{n+1}) - (\alpha_n + \lambda_n - \alpha_n\lambda_n)s_n \to 0,\end{aligned}$$

where $M' > 0$ and $M'' > 0$ are constants such that

$$2(1-\alpha_n)\|u - z\|\|x_{2n} - z\| \le M' \quad \text{and} \quad 2\|u - z\|\|x_{2n+1} - z\| \le M''.$$

It turns out that

$$(3.12)\qquad \|J^A_{\beta_n}(x_{2n} + e_n) - x_{2n}\| \to 0, \quad \|J^B_{\mu_n}(x_{2n-1} + e'_n) - x_{2n-1}\| \to 0,$$

and consequently, $\|x_{2n+1} - x_{2n}\| \to 0$ and $\|x_{2n} - x_{2n-1}\| \to 0$. Namely, we have proven that $\|x_{n+1} - x_n\| \to 0$. We also get by (3.12)

$$\begin{aligned}\|J^A_{\beta_n}(x_{2n} + e_n) - (x_{2n} + e_n)\| &\le \|J^A_{\beta_n}(x_{2n} + e_n) - x_{2n}\| + \|e_n\| \\ &\le \|J^A_{\beta_n}(x_{2n} + e_n) - x_{2n}\| + \eta_n\|J^A_{\beta_n}(x_{2n} + e_n) - x_{2n}\| \\ &\to 0.\end{aligned}$$

Similarly, we also have

$$\|J^B_{\mu_n}(x_{2n-1} + e'_n) - (x_{2n-1} + e'_n)\| \to 0.$$

Therefore,

$$\omega_w(x_{2n}) \subset A^{-1}(0) \quad \text{and} \quad \omega_w(x_{2n-1}) \subset B^{-1}(0).$$

This together with the fact that $\|x_{n+1} - x_n\| \to 0$ yields that $\omega_w(x_n) \subset S$. Analogous to the proof of (3.9) for Theorem 3.2, we have

$$\limsup_{n\to\infty}\langle u - z, x_n - z\rangle \le 0.$$

It now turns out that

$$s_{n+1} \le (1-\alpha_n)(1-\lambda_n)s_n + 2\lambda_n(1-\alpha_n)\langle u - z, x_{2n} - z\rangle$$

$$
\begin{aligned}
&+2\alpha_n\langle u-z, x_{2n+1}-z\rangle + M(\varepsilon_n+\varepsilon_n') \\
&= (1-\alpha_n)(1-\lambda_n)s_n + 2(\alpha_n+\lambda_n-\alpha_n\lambda_n)\langle u-z, x_{2n}-z\rangle \\
&\quad + 2\alpha_n\langle u-z, x_{2n+1}-x_{2n}\rangle + M(\varepsilon_n+\varepsilon_n') \\
&\le (1-\alpha_n)(1-\lambda_n)s_n + 2(\alpha_n+\lambda_n-\alpha_n\lambda_n)\langle u-z, x_{2n}-z\rangle \\
&\quad + 2\alpha_n\|u-z\|\|x_{2n+1}-x_{2n}\| + M(\varepsilon_n+\varepsilon_n') \\
&= (1-\alpha_n)(1-\lambda_n)s_n + (\alpha_n+\lambda_n-\alpha_n\lambda_n)\bigg[2\langle u-z, x_{2n}-z\rangle \\
&\quad + 2\frac{\alpha_n}{\alpha_n+\lambda_n-\alpha_n\lambda_n}\|u-z\|\|x_{2n+1}-x_{2n}\| \\
&\quad + M\frac{\varepsilon_n}{\alpha_n+\lambda_n-\alpha_n\lambda_n} + M\frac{\varepsilon_n'}{\alpha_n+\lambda_n-\alpha_n\lambda_n}\bigg] \\
&\le (1-\alpha_n)(1-\lambda_n)s_n + (\alpha_n+\lambda_n-\alpha_n\lambda_n)\bigg[2\langle u-z, x_{2n}-z\rangle \\
&\quad + 2\|u-z\|\|x_{2n+1}-x_{2n}\| + M\frac{\varepsilon_n}{\alpha_n} + M\frac{\varepsilon_n'}{\lambda_n}\bigg].
\end{aligned}
$$

Again we have the trivial relations from conditions (ii) and (iii)

$$
\lim_{n\to\infty}(\alpha_n+\lambda_n-\alpha_n\lambda_n) = 0 \quad \text{and} \quad \sum_{n=0}^{\infty}(\alpha_n+\lambda_n-\alpha_n\lambda_n) = \infty.
$$

Applying Lemma 2.3, we get $s_n \to 0$, that is, $\|x_{2n-1}-z\| \to 0$, which together with (3.5) yields $\|x_{2n}-z\| \to 0$; hence, $\|x_n - z\| \to 0$ and $x_n \to z$.

Case 2: $(s_n)$ is not eventually decreasing. In this case, define a sequence $(\tau(n))$ of integers as in Lemma 2.4. Since $s_{\tau(n)} \le s_{\tau(n)+1}$ for all $n > n_0$, it follows from (3.11) that

$$
\begin{aligned}
&\|J^A_{\beta_{\tau(n)}}(x_{2\tau(n)}+e_{\tau(n)}) - x_{2\tau(n)}\| \to 0, \\
&\|J^B_{\mu_{\tau(n)}}(x_{2\tau(n)-1}+e'_{\tau(n)}) - x_{2\tau(n)-1}\| \to 0.
\end{aligned}
$$

Furthermore, repeating the main argument for Case 2 of the proof of Theorem 3.2, we get

$$
\begin{aligned}
&\|x_{2\tau(n)+1}-x_{2\tau(n)}\| \to 0 \quad \text{and} \quad \|x_{2\tau(n)}-x_{2\tau(n)-1}\| \to 0, \\
&\limsup_{n\to\infty}\langle u-z, x_{2\tau(n)+1}-z\rangle \le 0 \quad \text{and} \quad \limsup_{n\to\infty}\langle u-z, x_{2\tau(n)}-z\rangle \le 0.
\end{aligned}
$$

We deduce from (3.11), for all $n > n_0$,

$$
\begin{aligned}
&(\alpha_{\tau(n)}+\lambda_{\tau(n)}-\alpha_{\tau(n)}\lambda_{\tau(n)})s_{\tau(n)} \\
&\le 2\lambda_{\tau(n)}(1-\alpha_{\tau(n)})\langle u-z, x_{2\tau(n)}-z\rangle + 2\alpha_{\tau(n)}\langle u-z, x_{2\tau(n)+1}-z\rangle \\
&\quad + M(\varepsilon_{\tau(n)}+\varepsilon'_{\tau(n)}) \\
&= 2(\alpha_{\tau(n)}+\lambda_{\tau(n)}-\alpha_{\tau(n)}\lambda_{\tau(n)})\langle u-z, x_{2\tau(n)}-z\rangle \\
&\quad + 2\alpha_{\tau(n)}\langle u-z, x_{2\tau(n)+1}-x_{2\tau(n)}\rangle + M(\varepsilon_{\tau(n)}+\varepsilon'_{\tau(n)}) \\
&\le 2(\alpha_{\tau(n)}+\lambda_{\tau(n)}-\alpha_{\tau(n)}\lambda_{\tau(n)})\langle u-z, x_{2\tau(n)}-z\rangle
\end{aligned}
$$

$$+ 2\alpha_{\tau(n)}\|u-z\|\|x_{2\tau(n)+1} - x_{2\tau(n)}\| + M(\varepsilon_{\tau(n)} + \varepsilon'_{\tau(n)}).$$

Consequently,

$$\begin{aligned} s_{\tau(n)} &\le 2\langle u-z, x_{2\tau(n)} - z\rangle \\ &\quad + 2\frac{\alpha_{\tau(n)}}{\alpha_{\tau(n)} + \lambda_{\tau(n)} - \alpha_{\tau(n)}\lambda_{\tau(n)}}\|u-z\|\|x_{2\tau(n)+1} - x_{2\tau(n)}\| \\ &\quad + M\Big(\frac{\varepsilon_{\tau(n)}}{\alpha_{\tau(n)} + \lambda_{\tau(n)} - \alpha_{\tau(n)}\lambda_{\tau(n)}} + \frac{\varepsilon'_{\tau(n)}}{\alpha_{\tau(n)} + \lambda_{\tau(n)} - \alpha_{\tau(n)}\lambda_{\tau(n)}}\Big). \\ &\le 2\langle u-z, x_{2\tau(n)} - z\rangle + 2\|u-z\|\|x_{2\tau(n)+1} - x_{2\tau(n)}\| + M\left(\frac{\varepsilon_{\tau(n)}}{\alpha_{\tau(n)}} + \frac{\varepsilon'_{\tau(n)}}{\lambda_{\tau(n)}}\right). \end{aligned}$$

We arrive at $\lim_{n\to\infty} s_{\tau(n)} = 0$. As $\lim_{n\to\infty} \varepsilon_n = 0$ and $\lim_{n\to\infty} \varepsilon'_n = 0$, and by (3.11), we find that $\lim_{n\to\infty}(s_{\tau(n)+1} - s_{\tau(n)}) = 0$. Hence,

$$\lim_{n\to\infty} s_{\tau(n)+1} = 0.$$

Finally, by (2.2) in Lemma 2.4, we obtain $\|x_{2n-1} - z\| \to 0$, which together with (3.5) immediately implies that $\|x_{2n} - z\| \to 0$, and so $x_n \to z$, as required. □

## References


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