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GENERALIZED NONEXPANSIVE MAPPINGS ON UNBOUNDED SETS

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Dedicated to S. Dhompongsa on occasion of his 65th birthday

Abstract. Let C be a convex closed unbounded set in a Banach space X and $T : C \to C$ a mapping which satisfies a condition weaker than nonexpansivity defined by T. Suzuki. Assume that $\{x_n\}$ is the sequence of the Ishikawa iterates of T, i.e. $x_n = S^n x$ where S is an average of the mapping T and the identity. In this paper, we prove that the limit of $\|x_n - Tx_n\|$ is equal to the minimal displacement of T, similarly as in the case of nonexpansive mappings as proved in [3]. By using this result and some weak additional conditions concerning asymptotic contractivenes, we prove the existence of fixed points for some applications.

1. Introduction

Several generalizations of the notion of nonexpansivity have appeared in the last years and some fixed point results have been proved for these classes of mappings (see, for instance, [2], [12], [21], [23]). One of the most relevant extension was defined by T. Suzuki in [22]:

Definition 1.1. Let M be a metric space. A mapping $T : M \to M$ is said to satisfy condition (C) if

$$\frac{1}{2}d(x, Tx) \le d(x, y) \Rightarrow d(Tx, Ty) \le d(x, y).$$

Besides [22], some other papers [1], [6], [8], [9], have studied the behavior of mappings which are defined on **bounded** convex subsets of a Banach space and satisfy condition (C). Many relevant contributions on this type of mappings have also been achieved by S. Dhompongsa and his research group (see, for instance, [4], [5], [17]). In this paper we are going to show some results for mappings satisfying condition (C) on **unbounded** domains.

When T is a nonexpansive mapping defined on a bounded convex subset of a Banach space, S. Ishikawa [15] proved that the average of T and the identity (i.e. the mapping $S = (1-\alpha)I + \alpha T$, $\alpha \in (0, 1)$) is asymptotically regular, which means that $\lim_n \|S^n x - S^{n-1}x\| = 0$ for every $x \in C$, and, in consequence, the sequence $S^n x$ is an approximate fixed point sequence for T. The same was proved for mappings satisfying condition (C) and defined on a bounded convex domain (lemma 6 in

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[22]), and, in fact, this is the main tool to obtain fixed point results for this class of mappings.

In Section 2 we study Ishikawa iterations for mappings satisfying condition (C) on unbounded domains. In this case, similarly to the case of nonexpansive mappings on unbounded domains, the minimal displacement of the mapping can be positive and so, it is not possible to obtain asymptotic regularity for the Ishikawa iterates. However, Ishikawa's result can be also understood as follows: the limit of $\|x_n - Tx_n\|$ is equal to the minimal displacement of T, $x_n = S^n x$ being the iterates of the average of the nonexpansive mapping T and the identity. The same can be said for mappings which satisfy condition (C) according to lemma 6 in [22]. Thus, in the case of a nonexpansive mapping defined on an unbounded domain, with a non-null minimal displacement, we can ask if the convergence of $\|x_n - Tx_n\|$ to the minimal displacement still holds. In [3] it is proved that this is, in general, the case. (In fact, the result in [3] shows the convergence for Kranosels'kii-Milman iterates). By using some technical lemmas we can prove that the same is true for Ishikawa iterates of mappings satisfying condition (C) which are defined on unbounded domains.

In Section 3 we use the results of Section 2 to prove fixed point results for mappings satisfying condition (C) on unbounded sets. Since, in the unbounded setting, mappings satisfying condition (C) (even nonexpansive mappings) can be fixed point free (consider, for instance a displacement) we need to assume some additional conditions. In the case of nonexpansive mappings it is usual to assume that the mapping is, in addition, asymptotically contractive (see, for instance, [16], [13]). We define the notion of scalar asymptotic contractiveness which is strictly weaker than asymptotic contractiveness (as we show in Example 3.3), and we prove that this condition suffices to prove the existence of a fixed point. Thus, our results in this section extend those in [16], [13] in two different directions, replacing nonexpasivity by condition (C) and asymptotic contractiveness by scalar asymptotic contractiveness.

2. Ishikawa iterations for mappings satisfying condition (C)

In the following K will be a closed convex subset of a Banach space X and $T : K \to K$ a mapping which satisfies condition (C). For some $\alpha \in (1/2, 1)$ we denote $S = (1-\alpha)I + \alpha T$. A more general condition than condition (C) is defined in [9] as follows:

Definition 2.1. Let M be a metric space. A mapping $V : M \to M$ is said to satisfy condition (C_λ) if for some $\lambda \in (0, 1)$

$$\lambda d(x, Vx) \leq d(x, y) \Rightarrow d(Vx, Vy) \leq d(x, y).$$

Note that S satisfies condition C_r where $r = 1/(2\alpha) < 1$ [9].

Without loss of generality we assume that $0 \in K$. For a given mapping $V : K \to K$, we denote by $r_K(V) = \inf\{\|x - Vx\| : x \in K\}$ the minimal displacement of the mapping V on K. For an arbitrary $x_0 \in K$ we denote $x_n = S(x_{n-1})$ and $y_n = Tx_n$.

The following lemma shows some basic properties of the sequences $\{x_n\}$ and $\{y_n\}$ which are implicitly in [22]. Since we will use these properties in several proofs, we prefer to state them explicitly.

Lemma 2.2. (a) *For every $n \in \mathbb{N}$ we have $\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\|$.*

(b) *The sequences* $\{\|x_n - y_n\|\}$ *and* $\{\|x_{n+1} - x_n\|\}$ *are nonincreasing.*

Proof. Since

$$\frac{1}{2}\|x_n - Tx_n\| = \frac{1}{2}\|x_n - y_n\| < \alpha\|x_n - y_n\| = \|x_n - x_{n+1}\|$$

condition (C) implies

$$\|y_{n+1} - y_n\| = \|Tx_{n+1} - Tx_n\| \leq \|x_{n+1} - x_n\|.$$

To prove (b) note that

$$\begin{aligned} \|x_{n+1} - y_{n+1}\| &= \|(1-\alpha)(x_n - y_n) + y_n - y_{n+1}\| \\ &\leq (1-\alpha)\|x_n - y_n\| + \|x_n - x_{n+1}\| \\ &= (1-\alpha)\|x_n - y_n\| + \|x_n - (1-\alpha)x_n - \alpha y_n\| \\ &= (1-\alpha)\|x_n - y_n\| + \alpha\|x_n - y_n\| = \|x_n - y_n\|. \end{aligned}$$

Finally, since $x_{n+1} - x_n = \alpha(y_n - x_n)$ the monotonicity of $\|x_{n+1} - x_n\|$ is now obvious. □

Note that the above lemma implies that the sequence $\{y_n\}$ is bounded if and only if $\{x_n\}$ is bounded. In this case, the following lemma shows that the minimal displacement must be null.

Proposition 2.3. *Assume that the sequence* $\{x_n\}$ *is bounded. Then,* $\lim_n \|x_n - y_n\| = 0$*. In particular* $r_K(T) = 0$ *.*

Proof. As in the proof of lemma 6 in [22], note that $\{x_n\}$ and $\{y_n\}$ are bounded sequences which satisfy $x_{n+1} = \alpha y_n + (1-\alpha)x_n$ and $\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\|$ for every $n \in \mathbb{N}$. Thus, by lemma 3 in [22] (see also [10]) $\lim_n \|y_n - x_n\| = 0$. □

The following lemma (for non-expansive mappings) appears in [11] (inequality 9.12 in lemma 9.4).

Lemma 2.4. *For every* $n, k \in \mathbb{N}$ *we have*

$$\|y_{n+k} - x_n\| \geq (1-\alpha)^{-k}(\|y_{n+k} - x_{n+k}\| - \|y_n - x_n\|) + (1 + k\alpha)\|y_n - x_n\|. \tag{2.1}$$

Proof. The proof is similar to that of inequality 9.12 in [11] for nonexpansive mappings using the previous lemma to replace some conditions derived from the nonexpansivity. We include the proof for the sake of the completeness.

We proceed by induction on k. If $k = 0$ (2.1) is trivial for every n. Assuming that (2.1) holds for a given k and all n. Replacing n with $n+1$ in (2.1) yields

$$\begin{aligned} \|y_{n+k+1} - x_{n+1}\| \geq\ & (1-\alpha)^{-k}[\|y_{n+k+1} - x_{n+k+1}\| - \|y_{n+1} - x_{n+1}\|] \\ & + (1 + k\alpha)\|y_{n+1} - x_{n+1}\|. \end{aligned}$$

Also, by lemma 2.2 (a)

$$\begin{aligned} \|y_{n+k+1} - x_{n+1}\| &\leq (1-\alpha)\|y_{n+k+1} - x_n\| + \alpha\|y_{n+k+1} - y_n\| \\ &\leq (1-\alpha)\|y_{n+k+1} - x_n\| + \alpha\sum_{i=0}^{k}\|y_{n+i+1} - y_{n+i}\| \end{aligned}$$

$$\leq \quad (1-\alpha)\|y_{n+k+1}-x_n\| + \alpha \sum_{i=0}^{k} \|x_{n+i+1}-x_{n+i}\|.$$

Now, combining the above two inequalities:

$$\begin{aligned}\|y_{n+k+1}-x_n\| \quad &\geq \quad (1-\alpha)^{-(k+1)}[\|y_{n+k+1}-x_{n+k+1}\| - \|y_{n+1}-x_{n+1}\|] \\ &\quad +(1-\alpha)^{-1}(1+k\alpha)\|y_{n+1}-x_{n+1}\| \\ &\quad -\alpha(1-\alpha)^{-1}\sum_{i=0}^{k}\|x_{n+i+1}-x_{n+i}\|.\end{aligned}$$

Since $\|x_{n+i+1}-x_{n+i}\| = \alpha\|y_{n+i}-x_{n+i}\|$, and since the sequence $\{\|y_n - x_n\|\}$ is nonincreasing (by lemma 2.2 (b)) and $1+k\alpha \leq (1-\alpha)^{-k}$, we have

$$\begin{aligned}\|y_{n+k+1}-x_n\| \quad &\geq \quad (1-\alpha)^{-(k+1)}[\|y_{n+k+1}-x_{n+k+1}\| - \|y_{n+1}-x_{n+1}\|] \\ &\quad +(1-\alpha)^{-1}(1+k\alpha)\|y_{n+1}-x_{n+1}\| \\ &\quad -\alpha^2(1-\alpha)^{-1}(k+1)\|y_n-x_n\| \\ &= \quad (1-\alpha)^{-(k+1)}[\|y_{n+k+1}-x_{n+k+1}\| - \|y_n-x_n\|] \\ &\quad +[(1-\alpha)^{-1}(1+k\alpha)-(1-\alpha)^{-(k+1)}]\|y_{n+1}-x_{n+1}\| \\ &\quad +[(1-\alpha)^{-(k+1)}-\alpha^2(1-\alpha)^{-1}(k+1)]\|y_n-x_n\| \\ &\geq \quad (1-\alpha)^{-(k+1)}[\|y_{n+k+1}-x_{n+k-1}\| - \|y_n-x_n\|] \\ &\quad +[(1-\alpha)^{-1}(1+k\alpha)-(1-\alpha)^{-(k+1)}]\|y_n-x_n\| \\ &\quad +[(1-\alpha)^{-(k+1)}-\alpha^2(1-\alpha)^{-1}(k+1)]\|y_n-x_n\| \\ &= \quad (1-\alpha)^{-(k+1)}[\|y_{n+k+1}-x_{n+k+1}\| - \|y_n-x_n\|] \\ &\quad +(1+(k+1)\alpha)\|y_n-x_n\|.\end{aligned}$$

Thus (2.1) holds for $k+1$, completing the proof. □

The following lemma is inspired on Lemma 6.4 in [19]

Lemma 2.5. *Denote $L = \lim\|y_n - x_n\|$ and let ϵ be an arbitrary positive number. Choose n such that $|\|x_n - y_n\| - L| < \epsilon/2$. Then, for every $k \in \mathbb{N}$ one has*

$$\frac{\|y_n - y_{n+k}\|}{k\alpha} > L - \frac{\epsilon}{k\alpha(1-\alpha)^k}.$$

Proof. By lemmas 2.2 and 2.4 we obtain

$$\begin{aligned}\|y_{n+k}-y_n\| \quad &\geq \quad \|y_{n+k}-x_n\| - \|x_n-y_n\| \\ &\geq \quad (1-\alpha)^{-k}(\|y_{n+k}-x_{n+k}\| - \|x_n-y_n\|) \\ &+ \quad (1+k\alpha)\|x_n-y_n\| - \|x_n-y_n\| \\ &= \quad (1-\alpha)^{-k}(\|y_{n+k}-x_{n+k}\| - \|x_n-y_n\|) + k\alpha\|x_n-y_n\| \\ &\geq \quad k\alpha\|x_n-y_n\| - \frac{\epsilon}{2(1-\alpha)^k}.\end{aligned}$$

Thus,

$$\frac{\|y_n - y_{n+k}\|}{k\alpha} > \|x_n - y_n\| - \frac{\epsilon}{2k\alpha(1-\alpha)^k} \geq L - \frac{\epsilon}{k\alpha(1-\alpha)^k}.$$

□

If T is a non-expansive mapping, we have $\|T^k x - T^k y\| \leq \|x - y\|$ and so, the distance between $T^k x$ and $T^k y$ keeps bounded by the distance between x and y. This is not true for mappings satisfying condition C. However, the following lemma shows that, in this more general case, the distance between two orbits keeps bounded (even if the orbits do not).

Lemma 2.6. *For every $x, y \in K$ and every $k \in \mathbb{N}$ one has*

$$\|S^k x - S^k y\| \leq a(x,y) =: \|x - y\| + 2\|x - Sx\| + \|y - Sy\|.$$

In particular, $S^k x$ is bounded if and only if $S^k y$ is.

Proof. The inequality is obvious for $k = 0$. By induction, assume that it holds for $k = 0, 1, \dots, n$. If $\|S^n x - S^{n+1} x\| \leq \|S^n x - S^n y\|$ condition C_r for S implies that $\|S^{n+1} x - S^{n+1} y\| \leq \|S^n x - S^n y\| \leq a(x,y)$. Otherwise, by using lemma 2.2 we obtain $\|S^n x - S^n y\| < \|S^n x - S^{n+1} x\| \leq \|Sx - x\|$ and we have

$$\begin{aligned}
\|S^{n+1}x - S^{n+1}y\| &\leq \|S^{n+1}x - S^n x\| + \|S^n x - S^n y\| + \|S^n y - S^{n+1} y\| \\
&\leq 2\|x - Sx\| + \|y - Sy\| \leq a(x,y).
\end{aligned}$$

□

Lemma 2.7. *Assume that the sequence $\{x_n\}$ is unbounded. Then, $L =: \lim_n \|x_n - y_n\|$ does not depend on the initial value x_0.*

Proof. Consider two initial values x, x^* and denote L, L^* the corresponding limits and x_n, y_n x_n^* , y_n^* the corresponding iterates. Assume $L^* < L$ and choose d such that $L^* + d < L$. Denote $M = \frac{2(\|Tx - x\| + a(x,x^*))}{\alpha}$ and choose k such that $M/k < d/3$ and ϵ, n such that $\frac{\epsilon}{k\alpha(1-\alpha)^k} < d/3$, $|L - \|x_n - y_n\|| < \epsilon$ and $|L^* - \|x_n^* - y_n^*\|| < \epsilon$. We apply lemma 2.5 and lemma 2.6 to obtain

$$\begin{aligned}
L &\leq \frac{\|y_n - y_{n+k}\|}{k\alpha} + \frac{\epsilon}{k\alpha(1-\alpha)^k} \\
&\leq \frac{\|x_n - x_{n+k}\| + \|y_n - x_n\| + \|y_{n+k} - x_{n+k}\|}{k\alpha} + d/3 \\
&\leq \frac{2\|Tx - x\| + +\|x_n - x_{n+k}\|}{k\alpha} + d/3 \\
&\leq \frac{2\|Tx - x\| + \|x_n^* - x_{n+k}^*\| + \|x_n - x_n^*\| + \|x_{n+k} - x_{n+k}^*\|}{k\alpha} + d/3 \\
&\leq \frac{M}{k\alpha} + \frac{\sum_{i=0}^{k-1} \|x_{n+i+1}^* - x_{n+i}^*\|}{k\alpha} + d/3 \\
&\leq 2d/3 + \frac{\sum_{i=0}^{k-1} \|x_{n+i}^* - y_{n+i}^*\|}{k} \\
&\leq 2d/3 + \|x_n^* - y_n^*\| < 2d/3 + L^* + \epsilon < L^* + d < L.
\end{aligned}$$

□

Theorem 2.8. *Let K be a closed convex subset of a Banach space and $T : K \to K$ a mapping which satisfies condition (C). For some $\alpha \in (1/2, 1)$ let x_n be the Ishikawa iterates. Then,* $\lim_n \|x_n - Tx_n\| = r_K(T)$.

Proof. If $\{x_n\}$ is bounded, Proposition 2.3 implies that $r_K(T) = 0 = \lim_n \|x_n - Tx_n\|$. If $\{x_n\}$ is unbounded, the result is an easy consequence of lemma 2.7 and lemma 2.2 (b). □

3. Fixed point for generalized non-expansive mappings on unbounded sets

The following notion has been used in [20] and has proved to be very useful to obtain fixed points of nonexpansive mappings on unbounded domains (see [16], [13])

Definition 3.1. Let C be a subset of a Banach space X. A mapping $f : C \to X$ is said to be asymptotically contractive on C if there exists $x_0 \in C$ such that

$$\limsup_{x \in C, \|x\| \to \infty} \frac{\|f(x) - f(x_0)\|}{\|x - x_0\|} < 1.$$

We are going to consider a weaker condition:

Definition 3.2. Let C be a subset of a Banach space X. A mapping $T : C \to C$ is said to be an asymptotically strongly pseudo-contractive mapping on C if there is $x_0 \in C$ such that

$$\lim_{\|x\| \to \infty, x \in C} \inf_{j \in J(x - x_0)} \frac{j(Tx - Tx_0)}{\|x - x_0\|^2} < 1,$$

where $J(x - x_0) = \{j \in X^* \ : \ \|j\| = \|x - x_0\| \text{ and } j(x - x_0) = \|x - x_0\|^2\}$.

The following example shows that asymptotically strongly pseudo-contractiveness is a strict generalization of asymptotic contractiveness

Example 3.3. Let $C = \{(x.y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$. Easily, C is a convex and closed subset of $\mathbb{R}^2$. Let $f : C \to C$ such that

$$f(x, y) = \left(0, \frac{\|(x, y)\|}{2} + \frac{x}{2}\right).$$

Then,

- f is a non-expansive mapping. Indeed, let $(x, y), (x', y') \in C$

$$\begin{aligned} \|f(x, y) - f(x', y')\| &= \left\| \left(0, \frac{\|(x, y)\| - \|(x', y')\|}{2} + \frac{x - x'}{2}\right) \right\| \\ &\leq \left| \frac{\|(x, y)\| - \|(x', y')\|}{2} \right| + \left| \frac{x - x'}{2} \right| \\ &\leq \|(x - x', y - y')\|. \end{aligned}$$

- f is not asymptotically contractive. Indeed, otherwise, we have (x_0, y_0) such that
$$\limsup_{\|(x,y)\|\to\infty} \frac{\|f(x,y) - f(x_0,y_0)\|}{\|(x,y)-(x_0,y_0)\|} < 1$$
which is a contradiction, because
$$1 > \limsup_{x\to\infty} \frac{\|f(x,0) - f(x_0,y_0)\|}{\|(x,0)-(x_0,y_0)\|} = \limsup_{x\to\infty} \frac{\|(0,x) - f(x_0,y_0)\|}{\|(x,0)-(x_0,y_0)\|} = 1.$$
- f is asymptotically strongly pseudo-contractive. Indeed,

$$\begin{aligned}\frac{\langle f(x,y) - f(0,0), (x,y)-(0,0)\rangle}{\|(x,y)\|^2} &= \frac{\|(x,y)\|y + xy}{2\|(x,y)\|^2} \\ &\leq \frac{\|(x,y)\|^2 + (1/2)\|(x,y)\|^2}{2\|(x,y)\|^2} = 3/4.\end{aligned}$$

Theorem 3.4. *Let K be a closed convex locally weakly compact subset of a Banach space X with normal structure and $T : K \to K$ an asymptotically strongly pseudo-contractive mapping which satisfies condition (C). Then, T has a fixed point.*

Proof. For $\alpha \in (1/2, 1)$ denote $S = (1-\alpha)I + \alpha T$. We claim that $\{S^n x\}$ is a bounded sequence. Indeed, otherwise there exists $a < 1$ such that for large enough n there exists $j \in J(S^{n-1}x - x_0)$ such that

$$\begin{aligned} a &> \frac{j(S^n x - Sx_0)}{\|S^{n-1}x - x_0\|^2} \\ &= \frac{j(S^n x - S^{n-1}x) + j(S^{n-1}x - x_0) + j(x_0 - Sx_0)}{\|S^{n-1}x - x_0\|^2} \\ &\geq 1 - \frac{\|S^n x - S^{n-1}x\|}{\|S^{n-1}x - x_0\|} - \frac{\|Sx_0 - x_0\|}{\|S^{n-1}x - x_0\|}. \end{aligned}$$

Taking limits as n tends to infinity and using that the sequence $\|S^n x - S^{n-1}x\|$ is nonincreasing (lemma 2.2 (b)) we obtain the contradiction $a > 1$. By Proposition 2.3 there exists an approximate fixed point sequence $\{x_n\}$ for T in K. The same argument as above shows that $\{x_n\}$ is bounded. Let C be the asymptotic center of $\{x_n\}$. Since K is locally weakly compact, C is convex weakly compact and nonempty. The existence of a fixed point follows from Proposition 3.4 and Theorem 4.4 in [18]. □

In fact, we have proved above that asymptotically strongly pseudo-contractiveness implies that there exists a bounded closed subset of K which is invariant for T. Thus, the above theorem could be also stated in the following more general (and abstract) form:

Theorem 3.5. *Let K be a closed convex locally weakly compact subset of a Banach space X and $T : K \to K$ an asymptotically strongly pseudo-contractive mapping which satisfies condition (C). Assume that T belongs to a type of mapping such that* *any convex weakly compact set has the FPP for this class. Then, T has a fixed point.*

Remark 3.6. In [14] a weaker asymptotic condition is considered. To do that, it is assumed that C is a nonempty unbounded closed convex subset with $0 \in C$, and $G : X \times X \to \mathbb{R}$ is a mapping which satisfies the conditions

(g1) $G(\lambda x, y) = \lambda G(x, y)$ for any $x, y \in X$ and $\lambda > 0$,

(g2) $\|x\|^2 \leq G(x, x)$ for any $x \in X$.

For a mapping $T : C \to C$ the following asymptotic condition is assumed

$$\limsup_{\|x\|\to\infty} \frac{G(Tx, x)}{\|x\|^2} < 1.$$

It is easy to check that asymptotically strongly pseudo-contractiveness can be replaced by the above condition and Theorem 3.4 still holds. However, in [7] the following asymptotic condition is considered to obtain fixed point results for pseudo-contractive mappings: "There exists $R > 0$ such that for every $x \in C$ with $\|x\| > R$ the inequality $j(T(x)) \leq \|x\|^2$ holds for every $j \in J(x)$". It is a open problem to know if Theorem 3.4 still holds whenever this condition replaces asymptotically strongly pseudo-contractiveness.

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