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STRONG AND Δ-CONVERGENCE OF MOUDAFI'S ITERATIVE SCHEME IN CAT(0) SPACES

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Abstract. In this paper, we first introduce an extension of Moudafi's iterative scheme in CAT(0) spaces. Then, we prove several strong and Δ-convergence theorems of the proposed iterative scheme for two quasi-nonexpansive mappings in CAT(0) spaces by using a new method. Our results generalize the recent results due to Iemoto and Takahashi (2009), and Kim (2012).

1. Introduction

Let (X, d) be a metric space and C be a nonempty subset of X. Then a mapping T of C into itself is called *nonexpansive* iff $d(Tx, Ty) \leqslant d(x, y)$ for all $x, y \in C$. We denote by $F(T)$ the set of all fixed points of T, i.e., $F(T) := \{x \in C : Tx = x\}$. A mapping T from C into C is also called *quasi-nonexpansive* iff the set $F(T)$ of fixed points of T is nonempty and $d(Tx, p) \leqslant d(x, p)$ for all $x \in C$ and $p \in F(T)$.

Approximating fixed points of nonexpansive mappings by iterative sequences has been investigated by several authors, see e.g., [14, 18, 21–23]. In 2009, Iemoto and Takahashi [8] studied the approximation of common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space by using Moudafi's iterative scheme. Recently, Kim [11] generalized results of Iemoto and Takahashi to quasi-nonexpansive mapping and proved several convergence theorems of Moudafi's iterative scheme in Hilbert spaces. Uniform convexity of Hilbert spaces plays important role in the convergence of iterative schemes. One of the successful attempts to extend uniform convexity to metric spaces is due to Khamsi and Khan [9]. They introduced the notion of uniformly convex metric spaces and studied some fundamental properties of such spaces. A nice subclass of uniformly convex metric spaces is the class of CAT(0) spaces which includes all Hilbert spaces.

Motivated by the above works, we introduce an extension of Moudafi's iterative scheme in CAT(0) spaces. By using a new method, we prove convergence theorems of this algorithm for quasi-nonexpansive mappings. Our results generalize the recent results due to Iemoto and Takahashi [8], and Kim [11].

2. Preliminaries

For a metric space (X, d), suppose that there exists a family $\mathfrak{F}$ of metric segments such that any two points x, y in X are endpoints of a unique metric segment $[x, y] \in \mathfrak{F}$ ($[x, y]$ is an isometric image of the real line interval $[0, d(x, y)]$). We shall denote

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by $\lambda x \oplus (1-\lambda)y$ the unique point z of $[x,y]$ which satisfies

$$d(z,x) = (1-\lambda)d(x,y) \quad \text{and} \quad d(z,y) = \lambda d(x,y). \tag{2.1}$$

Such metric spaces are usually called *convex metric spaces* [15]. Moreover, if we have

$$d\left(\frac{1}{2}u \oplus \frac{1}{2}x, \frac{1}{2}u \oplus \frac{1}{2}y\right) \leqslant \frac{1}{2}d(x,y).$$

for all u, x, y in X, then X is said to be a *hyperbolic metric space* (see [19]). Obviously, normed linear spaces are hyperbolic spaces. One can consider, as non-linear examples, the Hadamard manifolds [3], the Hilbert open unit ball equipped with the hyperbolic metric [7].

A hyperbolic metric space X is said to be *uniformly convex* if for any $a \in X$, for every $r > 0$, and for each $\varepsilon > 0$,

$$\delta(r,\varepsilon) = \tag{2.2}$$

$$\inf\left\{1 - \frac{1}{r}d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \;:\; d(x,a) \leqslant r, d(y,a) \leqslant r, d(x,y) \geqslant r\varepsilon\right\} > 0.$$

Let us observe that $\delta(r,0) = 0$, and $\delta(r,\varepsilon)$ is an increasing function of ε for every fixed r (for more properties of δ, see [9]).

A metric space (X,d) is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. For other equivalent definitions and basic properties, we refer the reader to standard texts such as [1, 2]. Complete CAT(0) spaces are often called Hadamard spaces. A subset C of a CAT(0) space is convex if $[x,y] \subseteq C$ for all $x, y \in C$. we need the following lemmas.

Lemma 2.1 ([2, Proposition 2.2])**.** *Let X be a CAT(0) space, $p, q, r, s \in X$ and $\lambda \in [0,1]$. Then*

$$d(\lambda p \oplus (1-\lambda)q, \lambda r \oplus (1-\lambda)s) \leqslant \lambda d(p,r) + (1-\lambda)d(q,s).$$

Lemma 2.2 ([6, Lemma 2.4])**.** *Let X be a CAT(0) space, $x, y, z \in X$ and $\lambda \in [0,1]$. Then*

$$d(\lambda x \oplus (1-\lambda)y, z) \leqslant \lambda d(x,z) + (1-\lambda)d(y,z).$$

Lemma 2.3 ([6, Lemma 2.5])**.** *Let X be a CAT(0) space, $x, y, z \in X$ and $\lambda \in [0,1]$. Then*

$$d^2(\lambda x \oplus (1-\lambda)y, z) \leqslant \lambda d^2(x,z) + (1-\lambda)d^2(y,z) - \lambda(1-\lambda)d^2(x,y).$$

For CAT(0) spaces, it follows from Lemma 2.3 that

$$\delta(r,\varepsilon) = \delta(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \tag{2.3}$$

and thus CAT(0) spaces are uniformly convex. From now on, we assume that X is a CAT(0) space.

Lemma 2.4. *Let X be a CAT(0) space and $a \in X$. Then for any $r > 0$ and $\varepsilon > 0$ there exists $\delta(r, \epsilon) > 0$ such that if $x, y \in X$ with $d(x, a) \leqslant r$, $d(y, a) \leqslant r$ and $d(x, y) \geqslant r\varepsilon$, then*

$$d(\lambda x \oplus (1-\lambda)y, a) \leqslant r\left[1 - 2\min\{\lambda, (1-\lambda)\}\delta(\epsilon)\right]$$

for all $\lambda \in [0, 1]$.

Proof. Let $\lambda \leqslant 1/2$, $u = \lambda x \oplus (1-\lambda)y$ and

$$v = 2\lambda\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) \oplus (1-2\lambda)y.$$

Then, by (2.1), we have $d(v, y) = 2\lambda d\left(\frac{1}{2}x \oplus \frac{1}{2}y, y\right) = \lambda d(x, y) = d(u, y)$. Uniqueness property in (2.1) implies that $v = u$. Also, by (2.2),

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leqslant r(1 - \delta(\epsilon)).$$

This together with Lemma 2.2 implies that

$$\begin{aligned} d(\lambda x \oplus (1-\lambda)y, a) &= d(v, a) \leqslant 2\lambda d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) + (1-2\lambda)d(y, a) \\ &\leqslant 2\lambda r(1-\delta(\epsilon)) + (1-2\lambda)r = r(1 - 2\lambda\delta(\epsilon)) \\ &= r\left[1 - 2\min\{\lambda, (1-\lambda)\}\delta(\epsilon)\right]. \end{aligned}$$

In the case that $\lambda > 1/2$, we put $\alpha = 1 - \lambda < 1/2$ and apply the proved case. □

Lemma 2.5. *Let X be a CAT(0) space, $a \in X$, $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $\{\lambda_n\}$ be a sequence in $[0, 1]$. If $\liminf_{n\to\infty} \lambda_n(1-\lambda_n) > 0$,*

$$\limsup_{n\to\infty} d(x_n, a) \leqslant R, \quad \limsup_{n\to\infty} d(y_n, a) \leqslant R \quad \text{and} \quad \lim_{n\to\infty} d(\lambda_n x_n \oplus (1-\lambda_n)y_n, a) = R$$

for some $R \in [0, \infty)$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Proof. Without loss of generality, we may assume that $R > 0$. Assume that the conclusion is not true. Then, there exist $\varepsilon > 0$ and subsequence $\{n_i\}$ of $\{n\}$ such that $d(x_{n_i}, y_{n_i}) \geqslant (R+1)\varepsilon$ for all $i \geqslant 1$. Let $\gamma \in (0, 1)$ be arbitrarily chosen. There exists subsequence $\{n_j\}$ of $\{n_i\}$ such that $d(x_{n_j}, a) \leqslant R + \gamma$ and $d(y_{n_j}, a) \leqslant R + \gamma$ for all $j \geqslant 1$. Since $\liminf_{n\to\infty} \lambda_n(1-\lambda_n) > 0$, there exist $\lambda > 0$ and subsequence $\{n_k\}$ of $\{n_j\}$ such that $\lambda_{n_k}(1-\lambda_{n_k}) \geqslant \lambda$ for all $k \geqslant 1$. It follows from Lemma 2.4 that

$$0 < 2\lambda\delta(\varepsilon) \leqslant 2\min\{\lambda_{n_k}, (1-\lambda_{n_k})\}\delta(\varepsilon) \leqslant 1 - \frac{1}{R+\gamma} d(\lambda_{n_k} x_{n_k} \oplus (1-\lambda_{n_k})y_{n_k}, a).$$

Since $\lim_{k\to\infty} d(\lambda_{n_k} x_{n_k} \oplus (1-\lambda_{n_k})y_{n_k}, a) = R$, we obtain

$$0 < 2\lambda\delta(\varepsilon) \leqslant \frac{\gamma}{R+\gamma}.$$

Letting $\gamma \to 0$, we get a contradiction. □

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from Proposition 7 of [5] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

A sequence $\{x_n\} \subset X$ is said to Δ-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Uniqueness of asymptotic center implies that CAT(0) space X satisfies Opial's property, i.e., for given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and given $y \in X$ with $y \neq x$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

Since it is not possible to formulate the concept of demiclosedness in a CAT(0) setting, as stated in linear spaces, let us formally say that "$I - T$ is demiclosed at zero" if the conditions, $\{x_n\} \subseteq C$ Δ- converges to x and $d(x_n, Tx_n) \to 0$ imply $x \in F(T)$.

We need the following lemmas in the sequel.

Lemma 2.6 ([13]). *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Lemma 2.7 ([4]). *If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.8 ([22]). *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} b_n < \infty$ and*

$$a_{n+1} \leqslant a_n + b_n$$

for all $n \geqslant 1$. Then $\lim_{n\to\infty} a_n$ exists.

3. Convergence theorems

Let C be a nonempty closed convex subset of a CAT(0) space X. Let $S, T : C \to C$ be two mapping. Define the iterative sequence $\{x_n\}$ as follows:

$$\begin{cases} x_1 \in C, \text{ chosen arbitrary}, \\ y_n = (1-\alpha_n)x_n \oplus \alpha_n Sx_n, \\ z_n = (1-\alpha_n)x_n \oplus \alpha_n Tx_n, \\ x_{n+1} = \beta_n y_n \oplus (1-\beta_n) z_n, \quad n \geqslant 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$. If X is a linear space such as Hilbert space, then iterative scheme (3.1) reduces to Moudafi's iterative scheme [16]: $x_1 \in C$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n[\beta_n Sx_n + (1-\beta_n)Tx_n], \quad n \geqslant 1. \tag{3.2}$$

The following theorem extends Theorem 3.1(i), (ii) of Kim [11] and hence Theorem 4.1(i), (iii) of Iemoto and Takahashi [8] to CAT(0) spaces.

Theorem 3.1. *Let X be a complete CAT(0) space and C be a nonempty, closed and convex subset of X, and let S, T be two quasi-nonexpansive mappings of C into itself such that $I - S$, $I - T$ are demiclosed at zero with $F(S) \cap F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by (3.1). Then the following hold:*

(i) *If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, then $\{x_n\}$ Δ-converges to a fixed point of S.*

(ii) *If $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$, then $\{x_n\}$ Δ-converges to a common fixed point of S and T.*

Proof. Let $p \in F(S) \cap F(T)$. Since S and T are quasi-nonexpansive, by Lemma 2.2, we have

$$\begin{aligned} d(x_{n+1}, p) &= d\left(\beta_n y_n \oplus (1 - \beta_n) z_n, p\right) \\ &\leqslant \beta_n d(y_n, p) + (1 - \beta_n) d(z_n, p) \\ &\leqslant \beta_n \left[(1 - \alpha_n) d(x_n, p) + \alpha_n d(Sx_n, p)\right] \\ &\quad +(1 - \beta_n)\left[(1 - \alpha_n) d(x_n, p) + \alpha_n d(Tx_n, p)\right] \\ &\leqslant \beta_n \left[(1 - \alpha_n) d(x_n, p) + \alpha_n d(x_n, p)\right] \\ &\quad +(1 - \beta_n)\left[(1 - \alpha_n) d(x_n, p) + \alpha_n d(x_n, p)\right] \\ &= d(x_n, p), \end{aligned} \tag{3.3}$$

which implies that $\lim_{n\to\infty} d(x_n, p)$ exists. Repeating (3.3), we obtain

$$d(x_n, p) \leqslant d(x_1, p)$$

for all $n \geqslant 1$. Therefore,

$$d(Sx_n, Tx_n) \leqslant d(Sx_n, p) + d(Tx_n, p) \leqslant 2d(x_n, p) \leqslant 2d(x_1, p).$$

(i) Utilizing (2.1) and Lemma 2.1, we have

$$\begin{aligned} d(x_{n+1}, y_n) &= d\left(\beta_n y_n \oplus (1 - \beta_n) z_n, y_n\right) \\ &= (1 - \beta_n) d(y_n, z_n) \\ &= (1 - \beta_n) d((1 - \alpha_n)x_n \oplus \alpha_n Sx_n, (1 - \alpha_n)x_n \oplus \alpha_n Tx_n) \\ &\leqslant (1 - \beta_n)\alpha_n d(Sx_n, Tx_n) \\ &\leqslant (1 - \beta_n) d(Sx_n, Tx_n). \end{aligned}$$

Since $\sum_{n=1}^{\infty}(1 - \beta_n) < \infty$, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} d(x_{n+1}, y_n) &\leqslant \sum_{n=1}^{\infty}(1 - \beta_n) d(Sx_n, Tx_n) \\ &\leqslant 2d(x_1, p)\sum_{n=1}^{\infty}(1 - \beta_n) < \infty, \end{aligned} \tag{3.4}$$

which implies that $\lim_{n\to\infty} d(x_{n+1}, y_n) = 0$ and thus

$$\lim_{n\to\infty} d(y_n, p) = \lim_{n\to\infty} d(x_n, p). \tag{3.5}$$

From Lemma 2.3, we have

$$d^2(y_n, p) \leqslant (1 - \alpha_n) d^2(x_n, p) + \alpha_n d^2(Sx_n, p) - \alpha_n(1 - \alpha_n) d^2(x_n, Sx_n)$$

$$\leqslant \quad d^2(x_n, p) - \alpha_n(1-\alpha_n)d^2(x_n, Sx_n).$$

It follows that

$$\alpha_n(1-\alpha_n)d^2(x_n, Sx_n) \leqslant d^2(x_n, p) - d^2(y_n, p).$$

Since $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, it follows from (3.5) that

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0. \tag{3.6}$$

Since $\{x_n\}$ is bounded, by Lemma 2.6, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which Δ-converges to a point y. By Lemma 2.7, $y \in C$. Since $I - S$ is demiclosed at zero, it follows from (3.6) that $y \in F(S)$. If $\{x_{n_j}\}$ is another subsequence of $\{x_n\}$ which Δ-converges to a point $z \in C$, then by using the same argument as in the proof above, we get $z \in F(S)$. We show that for any $q \in F(S)$, $\lim_{n\to\infty} d(x_n, q)$ exists. We note that

$$\begin{aligned} d(y_n, q) &\leqslant (1-\alpha_n)d(x_n, q) + \alpha_n d(Sx_n, q) \\ &\leqslant d(x_n, q) \\ &\leqslant d(y_{n-1}, q) + d(x_n, y_{n-1}). \end{aligned}$$

It follows from (3.4) and Lemma 2.8 that $\lim_{n\to\infty} d(y_n, q)$ exists which together with $\lim_{n\to\infty} d(x_n, y_{n-1}) = 0$ implies that $\lim_{n\to\infty} d(x_n, q)$ exists. Next, we show $y = z$. If not, By Opial's condition,

$$\begin{aligned} \lim_{n\to\infty} d(x_n, y) &= \lim_{i\to\infty} d(x_{n_i}, y) \\ &< \lim_{i\to\infty} d(x_{n_i}, z) = \lim_{n\to\infty} d(x_n, z) \\ &= \lim_{j\to\infty} d(x_{n_j}, z) < \lim_{j\to\infty} d(x_{n_j}, y) \\ &= \lim_{n\to\infty} d(x_n, y). \end{aligned} \tag{3.7}$$

This contradiction shows $y = z$ and hence $\{x_n\}$ Δ-converges to $y \in F(S)$.

(ii) For any $p \in F(S) \cap F(T)$, by (3.3), we know that $\lim_{n\to\infty} d(x_n, p)$ exists. Let

$$\lim_{n\to\infty} d(x_n, p) = R. \tag{3.8}$$

Moreover, the inequalities $d(Sx_n, p) \leqslant d(x_n, p)$ and $d(Tx_n, p) \leqslant d(x_n, p)$ imply that

$$\limsup_{n\to\infty} d(Sx_n, p) \leqslant R \quad \text{and} \quad \limsup_{n\to\infty} d(Tx_n, p) \leqslant R. \tag{3.9}$$

Next, we show that

$$\lim_{n\to\infty} d(y_n, p) = R \quad \text{and} \quad \lim_{n\to\infty} d(z_n, p) = R. \tag{3.10}$$

Using (3.1) and Lemma 2.2, we have

$$d(y_n, p) \leqslant d(x_n, p) \quad \text{and} \quad d(z_n, p) \leqslant d(x_n, p).$$

Also,

$$\begin{aligned} d(x_{n+1}, p) &\leqslant \beta_n d(y_n, p) + (1-\beta_n)d(z_n, p) \\ &\leqslant \beta_n d(y_n, p) + (1-\beta_n)d(x_n, p) \end{aligned}$$

and

$$d(x_{n+1}, p) \leqslant \beta_n d(x_n, p) + (1-\beta_n)d(z_n, p).$$

Since $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then

$$\frac{d(x_{n+1},p)-d(x_n,p)}{\beta_n} + d(x_n,p) \leqslant d(y_n,p)$$

and

$$\frac{d(x_{n+1},p)-d(x_n,p)}{(1-\beta_n)} + d(x_n,p) \leqslant d(z_n,p)$$

for sufficiently large numbers n. Taking $\liminf_{n\to\infty}$ in both sides, we obtain

$$\lim_{n\to\infty} d(x_n,p) \leqslant \liminf_{n\to\infty} d(y_n,p) \quad \text{and} \quad \lim_{n\to\infty} d(x_n,p) \leqslant \liminf_{n\to\infty} d(z_n,p).$$

The inequalities (3.8)-(3.10) together with Lemma 2.5 imply that

$$\lim_{n\to\infty} d(x_n,Sx_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(x_n,Tx_n) = 0. \tag{3.11}$$

Since $\{x_n\}$ is bounded, by Lemma 2.6, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which Δ-converges to a point y. By Lemma 2.7, $y \in C$. Since $I-S$ and $I-T$ are demiclosed at zero, it follows from (3.11) that $y \in F(S)\cap F(T)$. If $\{x_{n_j}\}$ is another subsequence of $\{x_n\}$ which Δ-converges to a point $z \in C$, then $z \in F(S)\cap F(T)$. Since for any $q \in F(S)\cap F(T)$, $\lim_{n\to\infty} d(x_n,q)$ exists, by the same argument as in (3.7) we conclude that $y = z$. Hence $\{x_n\}$ Δ-converges to $y \in F(S)\cap F(T)$. □

The next theorem is a generalization of Theorem 3.2 of Kim [11] and hence Theorem 4.1(ii) of Iemoto and Takahashi [8].

Theorem 3.2. *Let X be a complete CAT(0) space and C be a nonempty, closed and convex subset of X, and let S be a quasi-nonexpansive mappings of C into itself and T be a nonexpansive mappings of C into itself such that $F(S)\cap F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by (3.1). If $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, then $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof. For any $p \in F(S)\cap F(T)$, it follows from (3.3) that $\lim_{n\to\infty} d(x_n,p)$ exists and $d(x_n,p) \leqslant d(x_1,p)$ for all $n \geqslant 1$. Utilizing (2.1) and Lemma 2.1, we have

$$d(x_{n+1},z_n) = \beta_n d(y_n,z_n) \leqslant \beta_n\alpha_n d(Sx_n,Tx_n) \leqslant \beta_n d(Sx_n,Tx_n) \leqslant 2d(x_1,p)\beta_n.$$

Since $\sum_{n=1}^{\infty} \beta_n < \infty$, then $\lim_{n\to\infty} d(x_{n+1},z_n) = 0$ and thus

$$\lim_{n\to\infty} d(z_n,p) = \lim_{n\to\infty} d(x_n,p). \tag{3.12}$$

From Lemma 2.3, we have

$$\begin{aligned}
d^2(z_n,p) &\leqslant (1-\alpha_n)d^2(x_n,p) + \alpha_n d^2(Tx_n,p) - \alpha_n(1-\alpha_n)d^2(x_n,Tx_n) \\
&\leqslant d^2(x_n,p) - \alpha_n(1-\alpha_n)d^2(x_n,Tx_n) \\
&\leqslant (d(x_n,z_{n-1}) + d(z_{n-1},p))^2 - \alpha_n(1-\alpha_n)d^2(x_n,Tx_n) \\
&\leqslant (2d(x_1,p)\beta_{n-1} + d(z_{n-1},p))^2 - \alpha_n(1-\alpha_n)d^2(x_n,Tx_n).
\end{aligned}$$

It follows that

$$\alpha_n(1-\alpha_n)d^2(x_n,Tx_n) \leqslant (2d(x_1,p)\beta_{n-1} + d(z_{n-1},p))^2 - d^2(z_n,p).$$

Since $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\lim_{n\to\infty} d(z_n, p)$ exists, then

$$\sum_{n=2}^{\infty} \alpha_n(1-\alpha_n)d^2(x_n, Tx_n) < \infty.$$

This together with $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ implies that

$$\liminf_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.13}$$

Moreover,

$$\begin{aligned} d(x_{n+1}, Tx_{n+1}) &\leqslant d(x_{n+1}, Tx_n) + d(Tx_{n+1}, Tx_n) \\ &\leqslant \beta_n d(y_n, Tx_n) + (1-\beta_n)d(z_n, Tx_n) + d(x_{n+1}, x_n) \\ &\leqslant \beta_n\left[(1-\alpha_n)d(x_n, Tx_n) + \alpha_n d(Sx_n, Tx_n)\right] \\ &\quad +(1-\beta_n)(1-\alpha_n)d(x_n, Tx_n) \\ &\quad +\beta_n\alpha_n d(Sx_n, x_n) + (1-\beta_n)\alpha_n d(Tx_n, x_n) \\ &\leqslant d(x_n, Tx_n) + \beta_n\left[d(Sx_n, x_n) + d(Sx_n, Tx_n)\right] \\ &\leqslant d(x_n, Tx_n) + 4\beta_n d(x_1, p). \end{aligned}$$

Since $\sum_{n=1}^{\infty} \beta_n < \infty$, it follows from Lemma 2.8 that $\lim_{n\to\infty} d(x_n, Tx_n)$ exists. Hence, by (3.13), we obtain

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.14}$$

Since $\{x_n\}$ is bounded, by Lemma 2.6, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which Δ-converges to a point y. By Lemma 2.7, $y \in C$. we show that $y \in F(T)$. If $Ty \neq y$, by Opial's condition and (3.14), we have

$$\begin{aligned} \limsup_{n\to\infty} d(x_n, y) &< \limsup_{n\to\infty} d(x_n, Ty) \\ &\leqslant \limsup_{n\to\infty}[d(x_n, Tx_n) + d(Tx_n, Ty)] \\ &\leqslant \limsup_{n\to\infty} d(x_n, y). \end{aligned}$$

This is a contradiction. Hence we obtain $Ty = y$. By the same argument as in the proof of (i) in Theorem 3.1, $\{x_n\}$ Δ-converges to $y \in F(T)$. □

Two mappings $S, T : C \to C$ are said to satisfy Condition **A** [10, 20] iff there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that

$$f(d(x, F)) \leqslant \frac{1}{2}(d(x, Sx) + d(x, Tx))$$

for all $x \in C$, where $F = F(S) \cap F(T) \neq \emptyset$ and $d(x, F) = \inf\{d(x, y) : y \in F\}$.

Theorem 3.3. *Let X be a complete CAT(0) space and C be a nonempty, closed and convex subset of X, and let S, T be two quasi-nonexpansive mappings of C into itself satisfying Condition **A** with $F = F(S) \cap F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by (3.1). If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. By taking infimum over all $p \in F$ on both sides of (3.3), we see that

$$d(x_{n+1}, F) \leqslant d(x_n, F),$$

which implies that $\lim_{n\to\infty} d(x_n, F)$ exists. We claim that $\lim_{n\to\infty} d(x_n, F) = 0$. If not, there exist $\varepsilon_0 > 0$ and a subsequence $\{n_k\}$ of $\{n\}$ such that $d(x_{n_k}, F) > \varepsilon_0$ for all $k \geqslant 1$. Using Condition **A** and (3.11), we obtain

$$0 < f(\varepsilon_0) \leqslant f(d(x_{n_k}, F)) \leqslant \frac{1}{2}(d(x_{n_k}, Sx_{n_k}) + d(x_{n_k}, Tx_{n_k})) \to 0$$

as $k \to \infty$, which is a contradiction. Moreover, $\{x_n\}$ is Cauchy. To see this let $n, m \geqslant k \geqslant 1$. Then, by (3.3), we have

$$d(x_n, x_m) \leqslant d(x_n, p) + d(x_m, p) \leqslant 2d(x_k, p)$$

and thus

$$d(x_n, x_m) \leqslant 2d(x_k, F).$$

Since $\lim_{k\to\infty} d(x_k, F) = 0$, then $\{x_n\}$ is Cauchy and converges to some $q \in C$. Since F is closed, then $q \in F$. This completes the proof. □

The following is an example of a quasi-nonexpansive mapping in a non-Hilbert CAT(0) space which is not a nonexpansive mapping.

Example 3.4. Consider $\mathbb{R}^2$ with the usual Euclidean meter d. Let $X = \mathbb{R}^2$ be an $\mathbb{R}$-tree with the radial meter d_r, where $d_r(x, y) = d(x, y)$ if x and y are situated on a Euclidean straight line passing through the origin and $d_r(x, y) = d(x, \mathbf{0}) + d(y, \mathbf{0})$ otherwise (see [12] and [17, page 65]). We put

$$C = \{(t, 0) : t \in [0, 3/2]\} \cup \{(0, t) : t \in [0, 3/2]\} \subset \mathbb{R}^2$$

and define $T : C \to C$ by

$$T(t, 0) = \left(0, \frac{t^2}{2}\right) \quad \text{and} \quad T(0, t) = \left(\frac{t^2}{2}, 0\right)$$

for all $t \in [0, 3/2]$. Clearly, $F(T) = \{(0, 0)\}$. Let $x = (t, 0)$ and $\mathbf{0} = (0, 0)$. Since x and $\mathbf{0}$ are situated on a Euclidean straight line passing through the origin, we have

$$d_r(Tx, \mathbf{0}) = d(Tx, \mathbf{0}) = \frac{t^2}{2} \leqslant t = d_r(x, \mathbf{0}).$$

Similarly, for $y = (0, t)$, $d_r(Ty, \mathbf{0}) \leqslant d_r(y, \mathbf{0})$. Therefore, T is quasi-nonexpansive. But it is not a nonexpansive mapping. In fact, if $x = (5/4, 0)$ and $y = (3/2, 0)$, then we have

$$d_r(Tx, Ty) = \frac{11}{32} > \frac{1}{4} = d_r(x, y).$$

Also, taking $S = T$, we see that S and T satisfy Condition **A** with the function $f : [0, \infty) \to [0, \infty)$ defined by $f(r) = r$. Note that for $x = (t, 0)$ we have $d_r(x, F) = d(x, \mathbf{0}) = t$ and

$$d_r(x, Tx) = d(x, \mathbf{0}) + d(Tx, \mathbf{0}) = t + \frac{t^2}{2}.$$

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