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WEAK AND STRONG CONVERGENCE THEOREMS FOR STRICTLY PSEUDONONSPREADING MAPPINGS AND EQUILIBRIUM PROBLEM IN HILBERT SPACES

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Abstract. The purpose of this paper is to prove a convergence theorem for finding common fixed point of a countable family of k-srictly pseudononspreading mappings and equilibrium point of a bifunction. Moreover, some numerical example of the proposed method is also given. The main results of the paper improve and extend those in the literature.

1. Introduction

For $f : C \times C \to \mathbb{R}$ as a bifunction with $\mathbb{R}$ being set of real numbers and C being a nonempty closed convex subset of a real Hilbert space, the equilibrium problem of f is to find $x \in C$ such that $f(x,) \geq 0$ for all $y \in C$. The solution set of such a problem is denoted by $EP(f)$ and the following four condition are assumed to be satisfied by the bifunction $f : C \times C \to \mathbb{R}$:

$(A1)$ $f(x, x) = 0,\ \forall x \in C$;
$(A2)$ f is monotone, i.e. $f(x, y) + f(y, x) \leq 0,\ \forall x, y \in C$;
$(A3)$ $\forall x, y, z \in C$, $\lim_{t\to 0^+} f(tz + (1 - t)x, y) \leq f(x, y)$;
$(A4)$ $\forall x \in C, y \mapsto f(x, y)$ is convex and lower semicontinuous.

We know the following theorem; see, for instance, [2, 4].

Theorem 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$. Let $r > 0$ and $x \in H$, Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \ for\ all\ y \in C. \tag{1.1}$$

Further, for any $r > 0$ and $x \in H$, define $T_r : H \to C$ by $z = T_r x$. Then, the following holds:

(i) $T_r$is single valued;
(ii) $T_r$is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(iii) $F(T_r) = EP(f)$;

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(iv) $EP(f)$ is closed and convex.

In addition, Wataru Takahashi, Jen-Chih Yao and Fumiaki Kohsaka mentioned in their published work "The fixed point property and unbounded sets in Banach spaces" the following nonlinear mapping. Let E be a smooth strictly convex and reflexive Banach space, let J be the duality mapping of E and let C be a nonempty closed convex subset fo E. Then the mapping $T : C \to C$ is said to be nonspreading if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2$, $\forall x, y \in E$. They considered the class of nonspreading mappings to study the resolvents of a maximal monotone operators in the Banach space. In the case when E is a Hilbert space, we know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. So, a nonspreading mapping $T : C \to C$ in a Hilbert space H is defined as follows:

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \tag{1.2}$$

for all $x, y \in C$.

Recently, Iemoto and Takahashi [5] proved and defined the follwing nonlinear mapping $T : C \to C$ called hybrid which is also deduced from a firmly nonexpansive mapping :

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle \tag{1.3}$$

for all $x, y \in C$. Now, we know that a nonspreading mapping is deduced from a firmly nonexpansive mapping; see [6, 7, 10]. Moreover, a strong convergence theorem of the hybrid type for nonspreeading mappings have been proved by Matsushita and Takahashi [8], followikng the terminology of Browder-Petryshyn [3]. It clearly mentioned and proved that a mapping $T : D(T) \subseteq H :\to H$ is $k-$ *strictly pseudononspreading* if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle, \tag{1.4}$$

for all $x, y \in D(T)$. While it is clear that every nonspreading mapping is k-strictly psedononspreading. We also say that the class of k-strictly pseudononspreading mapping provides more general approach than the class of nonspreading mappings (see example [9]).

Recently, in 2013, it was then acknowledged by Zhao and Chang [13] after they proposed an iterative algorithm for equilibrium problem and a class of strictly psuedononspreading mappings which is more general than the class of nonspreading mapping which was studid in 2010 by Kurokawa and Takahashi. Zhao and Chang explored an auxiliary mapping in their theorems and proofs and under suitable conditions, some weak and strong convergence theorems are proved. They obtained the following result :

For each $S_i : C \to C, i = 1, 2, \ldots$ is a k_i - strictly pseudononexpansive mapping with $k := sup_{i\geq 1} k_i \in (0, 1)$. For given $\beta \in [k, 1)$, denoted by $S_{i,\beta} := \beta I + (1 - \beta)S_i$.

$$\begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \geq 0, \ \ \forall y \in C, \\ x_{n+1} = \alpha_{0,n} u_n + \sum_{i=0}^{\infty} \alpha_{i,n} S_{i,\beta} u_n, \end{cases} \tag{1.5}$$

where $\{\alpha_{i,n}\} \subset (0,1)$ and $\{r_n\}$ satisfy the following conditions:

(a) $\sum_{i=1}^{\infty} \alpha_{i,n} = 1$, for each $n \geq 1$;
(b) for each $i \geq 1, \liminf_{n\to\infty} \alpha_{0,n}\alpha_{i,n} > 0$;
(c) $\{r_n\} \subset (0,\infty)$ and $\liminf_{n\to\infty} r_n > 0$.
(I) If $\mathbb{F} := (\bigcap_{i=1}^{\infty} F(S_i)) \bigcap EP(f) \neq \phi$, then both $\{x_n\}$ and $\{u_n\}$ converge weakly to some point $x* \in \mathbb{F}$;
(II) In addition, if there exists some positive integer m such that S_m is semi-compact, then both $\{x_n\}$ and $\{u_n\}$ converge strongly to $x* \in \mathbb{F}$.

We see that each step of (1.5) the infinite series $\sum_{i=0}^{\infty} \alpha_{i,n} S_{i,\beta} u_n$ have to be computed before we calculate the value of x_n. This makes the calculation more complicated. In this research, the purposes are to improve and simplify the iteration process of Zhao and Chang. Moreover, we provide some numerical examples of the studied problem.

2. Preliminaries and lemmas

Throughout this paper, we denote by H a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We also denote by $\mathbb{N}$ the set of natural numbers. In a Hilbert space, it is known that

Lemma 2.1 ([9])**.** *Let H be a real Hilbert space. Then the following well known results hold:*

(i) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$, *for all $x, y \in H$ and for all $t \in [0,1]$.*
(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle$ *for all $x, y \in H$.*
(iii) *If $\{x_n\}_{n=1}^{\infty}$ is a sequence in H which converges weakly to $z \in H$ then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} (\|x_n - z\|^2 + \|z - y\|^2), \ \ \forall y \in H.$$

Lemma 2.2 ([1, 11])**.** *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers satisfying the condition*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\beta_n, \ \ n \geq 0,$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are real sequences such that

(i) $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \beta_n \leq 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 ([9])**.** *Let C be nonempty closed convex subset of a real Hilbert space H. and let $T : C \to C$ be k-strictly pseudononspreading mapping. If $F(T) \neq \emptyset$, then it is closed and convex.*

Lemma 2.4 ([9])**.** *Let C be nonempty closed convex subset of a real Hilbert space H. and let $T : C \to C$ be k-strictly pseudononspreading mapping. Then $(I-T)$ is demiclosed at 0.*

Lemma 2.5 ([13])**.** *Let $T : C \to C$ be strictly pseudononspreading mapping with $k \in [0,1)$ Denote by $T_\beta : \beta I + (I-\beta)T$, where $\beta \in [k,1)$, then*

(i) $F(T) = F(T_\beta)$;

(ii) *the following inequality holds:*

$$\|T_\beta x - T_\beta y\|^2 \leq \|x-y\|^2 + \frac{2}{1-\beta}\langle x - T_\beta x, y - T_\beta y\rangle, \forall x, y \in C; \tag{2.1}$$

(iii) T_β *is an quasinonexpansive mapping, that is,*

$$\|T_\beta x - p\|^2 \leq \|x-p\|^2, \forall x \in C, p \in F(T). \tag{2.2}$$

3. Main results

We first prove some useful lemma for our main result. The following result was proved in [12].

Lemma 3.1 ([12])**.** *Let $p > 1, r > 0$ to be fixed numbers. Then a Banach spaces X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty), g(0) = 0$, such that*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda)g\|x-y\|, \tag{3.1}$$

for all x, y in $B_r = \{x \in X : \|x\| \leq r\}, \lambda \in [0,1]$, where $w_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

So if $p = 2$ we have $\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g\|x-y\|$,

By above lemma we can prove a new lemma as follows.

Lemma 3.2. *Let X be a uniformly convex Banach space and $r > 0$ is constant. Then there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty), g(0) = 0$, such that*

$$\|\Sigma_{i=1}^n \alpha_i x_i\|^2 \leq \Sigma_{i=1}^n \alpha_i \|x_i\|^2 - \alpha_1\alpha_2 g(\|x_1 - x_2\|), \tag{3.2}$$

for all $n \geq 2$ such that $x_i \in B_r = \{x \in X : \|x\| \leq r\}, \alpha_i \in (0,1)$ and $\Sigma_{i=1}^n \alpha_i = 1$

Proof. Let X be a uniformly convex Banach space and $r > 0$ is constant. According to Lemma 3.1, there exists a strictly increasing, and convex function $g : [0,\infty) \to [0,\infty), g(0) = 0$, such that

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 - \lambda(1-\lambda)g(\|x-y\|), \text{ for all } x, y \in B_r.$$

Assume $\left\|\Sigma_{i=1}^{n-1}\beta_i y_i\right\| \leq \Sigma_{i=1}^{n-1}\beta_i\|y_i\|^2 - \beta_1\beta_2 g\left(\|y_1 - y_2\|\right)$, for all $y_i \in B_i = \{x \in X : \|x\| \leq\}, \beta_i \in (0,1)$ and $\Sigma_{i=1}^{n-1}\beta_i = 1$. Let $\alpha_i \in (0,1)$ such that $\Sigma_{i=1}^n \alpha_i = 1$ and $\{x_j\} \subset B_r$. Then $\Sigma_{i=1}^{n-1}\frac{\alpha_i}{1-\alpha_n}x_i \in B_r$. By Lemma 3.1 we have

$$\begin{aligned}
\|\Sigma_{i=1}^n \alpha_i x_i\|^2 &= \left\|(1-\alpha_n)\Sigma_{i=1}^{n-1}\frac{\alpha_i}{1-\alpha_n}x_i + \alpha_n x_n\right\|^2 \\
&\leq (1-\alpha_n)\left\|\Sigma_{i=1}^{n-1}\frac{\alpha_i}{1-\alpha_n}x_n\right\|^2 + \alpha_n\|x_n\|^2 \\
&\leq (1-\alpha_n)\Sigma_{i=1}^{n-1}\frac{\alpha_i}{1-\alpha_n}\|x_i\|^2 - \alpha_1\alpha_2 g(\|x_1-x_2\|) + \alpha_n\|x_n\|^2 \\
&= \Sigma_{i=1}^n \alpha_i\|x_i\| - \alpha_1\alpha_2 g(\|x_1 - x_2\|). \qquad (3.3)
\end{aligned}$$

By mathematical induction, there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty), g(0) = 0$, such that

$$\|\Sigma_{i=1}^n \alpha_i x_i\|^2 \leq \Sigma_{i=1}^n \alpha_i\|x_i\|^2 - \alpha_1\alpha_2 g(\|x_1 - x_2\|),$$

for all $n \geq 2$ such that $x_i \in B_r = \{x \in X : \|x\| \leq r\}, \alpha_i \in (0,1)$ and $\Sigma_{i=1}^{n}\alpha_i = 1$ $\square$

Next we prove our main result.

Theorem 3.3. *Let H be a Hilbert space, C be a closed and convex subset of H, $\{S_i\}$ is a class of k_i-strictly pseudononspreading mappings, $\phi : C \times C \to \mathbb{R}$ be a bi-function that satisfies $(A1)-(A4)$ and $S_{i,\beta_i} = \beta_i I + (1-\beta_i)S_i$ where $\beta_i \in [k,1)$. Let $\{x_n\}$ and $\{u_n\}$ be a sequence defined by*

$$(3.4) \qquad \begin{cases} x_1 \in C \\ \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_{0,n}u_n + \Sigma_{i=1}^{n}\alpha_{i,n}S_{i,\beta}u_n, \end{cases}$$

where $\{\alpha_{i,n}\} \subset (0,1)$ and $\{r_n\}$ satisfies the following condition:

(a) $\sum_{i=1}^{n}\alpha_{i,n} = 1$, *for each* $n \geq 1$;
(b) *for each* $i \geq 1, \liminf_{n\to\infty} \alpha_{0,n}\alpha_{i,n} > 0$;
(c) $\{r_n\} \subset (0,\infty)$ *and* $\liminf_{n\to\infty} r_n > 0$.
(I) *If* $\mathbb{F} := (\bigcap_{i=1}^{\infty} F(S_i)) \bigcap EP(f) \neq \phi$, *then both* $\{x_n\}$ *and* $\{u_n\}$ *converge weakly to some point* $x* \in \mathbb{F}$;
(II) *In addition, if there exists some positive integer m such that S_m is semi-compact, then both $\{x_n\}$ and $\{u_n\}$ converge strongly to $x* \in \mathbb{F}$.*

Proof. Let $p \in \mathcal{F}$. By Lemma 3.1, $u_n = T_{r_n}x_n, p \in F(T_{r_n})$. Since T_{r_n} is quasi-nonexpansive, we have $\|u_n - p\| \leq \|T_{r_n}x_n - p\| \leq \|x_n - p\|$. By Lemma 2.5, we have $S_{i,\beta}$ is quasi-nonexpansive and $p \in F(S_{i,\beta_i})$, that is $\|S_{i,\beta_i}x - p\| \leq \|x - p\|$ for all $x \in C$. Then

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_{0,n}u_n + \Sigma_{i=1}^{\infty}\alpha_{i,n}S_{i,\beta_i}u_n - p\| \\ &\leq \alpha_{0,n}\|u_n - p\| + \Sigma_{i=1}^{\infty}\alpha_{i,n}\|S_{i,\beta_i}u_n - p\| \\ &\leq \alpha_{0,n}\|u_n - p\| + \Sigma_{i=1}^{\infty}\alpha_{i,n}\|u_n - p\| \\ &= \|u_n - p\| \\ &\leq \|x_n - p\|, \ \forall n \in \mathbb{N}. \qquad (3.5) \end{aligned}$$

That is $\|x_{p+1} - p\| \leq \|u_n - p\| \leq \|x_n - p\|$. So $\lim_{n\to\infty}\|x_n - p\|$ exists. Moreover, $\lim_{n\to\infty}\|x_n - p\| = \lim_{n\to\infty}\|u_n - p\|$. Hence $\{x_n\}$ and $\{u_n\}$ are bounded. And since$\|S_{i,\beta_i}x_n - p\| \leq \|x_n - p\|$ and $\|S_{i,\beta_i}u_n - p\| \leq \|u_n - p\|$, then $\{S_{i,\beta_i x_n}$ and $\{S_{i,\beta_i}u_n\}$ are also bounded. Let $r = \sup\|u_n - p\|$. Then $u_n - p, S_{i,\beta_i}u_n \in B_r$ for all $i, n \geq 1$. So by Lemma 3.1, there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty), g(0) = 0$ such that

$$\begin{aligned} \|\alpha_{0,n}(u_n - p) + \Sigma_{i=1}^{n}\alpha_{i,n}\left(S_{i,\beta_i}u_n - p\right)\|^2 \ \leq\ & \alpha_{0,n}\|u_n - p\|^2 + \Sigma_{i=1}^{n}\alpha_{i,n}\|S_{i,\beta_i}u_n - p\|^2 \\ & -\alpha_{0,n}\alpha_{1,n}g\left(\|u_n - S_{i,\beta_i}u_n\|\right) \end{aligned}$$

for all $n \geq 2$. Then,

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_{0,n}(u_n - p) + \Sigma_{i=1}^{n}\alpha_{i,n}\left(S_{i,\beta_i}u_n - p\right)\|^2 \\ &\leq \alpha_{0,n}\|u_n - p\|^2 + \Sigma_{i=1}^{n}\alpha_{i,n}\|S_{i,\beta_i}u_n - p\|^2 \\ &\quad -\alpha_{0,n}\alpha_{i,n}g\left(\|u_n - S_{i,\beta_i}u_n\|\right) \\ &\leq \alpha_{0,n}\|u_n - p\|^2 + \Sigma_{i=1}^{n}\alpha_{i,n}\|u_n - p\|^2 \end{aligned}$$

$$
\begin{aligned}
&\quad -\alpha_{0,n}\alpha_{i,n} g\left(\|u_n - S_{i,\beta_i}u_n\|\right) \\
&= \|u_n - p\|^2 - \alpha_{0,n}\alpha_{i,n} g\left(\|u_n - S_{i,\beta_i}u_n\|\right) \\
&\leq \|x_n - p\|^2 - \alpha_{0,n}\alpha_{i,n} g\left(\|u_n - S_{i,\beta_i}u_n\|\right). \qquad (3.6)
\end{aligned}
$$

So $\alpha_{0,n}\alpha_{i,n} g\left(\|u_n - S_{i,\beta_i}u_n\|\right) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \to \infty.$ when $n \to \infty$. By condition(b), $g\left(\|u_n - S_{i,\beta_i}\|\right) \to 0$ when $n \to \infty$. Since g is strictly increasing function such that $g(0) = 0$ we have,

$$\lim_{n\to\infty} \|u_n - S_{i,\beta_i}u_n\| = \lim_{n\to\infty} \frac{1}{1-\beta_i} \|u_n - S_{i,\beta_i}\| = 0.$$

Since T_{r_n} is firmly nonexpansive, we have

$$
\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\|^2 \\
&\leq \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle \\
&= \langle u_n - p, x_n - p\rangle \\
&= \frac{1}{2}\left\{\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2\right\} \qquad (3.7)
\end{aligned}
$$

Hence $\|u_n - x_n\|^2 \leq \|x_n - p\|^2 - \|u_n - p\|^2$. That is $\lim_{n\to\infty}\|u_n - x_n\| = 0$
Then

$$
\begin{aligned}
\|x_n - S_{i,\beta_i} &\leq \|x_n - u_n\| + \|u_n - S_{i,\beta_i}u_n\| + \|S_{i,\beta_i}u_n - S_{i,\beta_i}x_n\| \\
&\leq \|x_n - u_n\| + \|u_n - S_{i,\beta_i}u_n\| \\
&\quad + \left\{\|x_n - u_n\|^2 + \frac{2}{1-\beta_i}\left|\langle u_n - S_{i,\beta_i}, x_n - S_{i,\beta_i}x_i\rangle\right|\right\}^{1/2} \\
&\to 0.
\end{aligned}
$$

Hence we have,

$$\lim_{n\to\infty}\|x_n - S_i x_n\| = \lim_{n\to\infty}\frac{1}{1-\beta_i}\|x_n - S_{i,\beta_i}x_n\| = 0.$$

Since $\{x_n\}$ is bounded, there exists $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightharpoonup w$. Since S_{i,β_i} is demiclosed, we have $w \in \bigcap_{i=1}^{\infty} F\left(S_{i,\beta_i}\right)$. Suppose $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup w^*$. and $w \neq w^*$. So $w^* \in \bigcap_{i=1}^{\infty} F\left(S_{i,\beta_i}\right)$. Since $\lim_{n\to\infty}\|x_n - w\|$ and $\lim_{n\to\infty}\|x_n - w^*\|$ exist, by opial's property of H, we have,

$$
\begin{aligned}
\liminf_{n_j\to\infty}\|x_{n_j} - w\| &< \liminf_{n_j\to\infty}\|x_{n_j} - w^*\| \\
&= \lim_{n\to\infty}\|x_n - w^*\| \\
&= \lim_{n_j\to\infty}\|x_{n_i} - w^*\| \\
&= \liminf_{n_i\to\infty}\|x_{n_j} - w\| \\
&= \lim_{n\to\infty}\|x_n - w\| \\
&= \lim_{n_j\to\infty}\|x_{n_j} - w\|
\end{aligned}
$$

This is a contradiction. So for each weak convergent subsequence, it converges weakly to w. So we have $x_n \rightharpoonup w$ and also $u_n \rightharpoonup w$. Since $u_n = T_{r_n}x_n$ by (A2), we have $\langle y-u_{n_i},\frac{1}{r_{n_i}}(u_{n_i}-x_{n_i})\rangle \geq \phi(y,u_{n_i}),\ y\in C$. Since $(1/r_{n_i})(u_{n_i}-x_{n_i})\to 0$ and $u_{n_i}\rightharpoonup w$, by (A4), we have $\phi(y,w)\leq 0,\ y\in C$. For each $t\in(0,1),\ y\in C$, let $y_t=ty+(1-t)w$. Then $y_t\in C$. From (A1) and (A4), $0=\phi(y_t,y)+(1-t)\phi(y_t,w)\leq t\phi(y_t,y)$. So $\phi(y_t,y)\geq 0$. Let $t\to 0$, by (A3), we have $\phi(w,y)\geq 0,\ \forall y\in C$. Hence $w\in EP(\phi)$. That is $x_n\rightharpoonup w$ and $u_n\rightharpoonup w$ such that $w\in\mathcal{F}$. Thus the proof of (I) is completed. To prove (II), let S_1 is semi-compact. Since $\|x_n-S_1x_n\|\to 0$, then there exists $\{x_{n_j}\}\subset\{x_n\}$ such that $x_{n_j}\rightharpoonup u^*\in C$. Since $x_{n_j}\rightharpoonup w$, we have $u^*=w$. So $x_{n_j}\to w$. Hence $\lim_{n\to\infty}\|u_n-w\|=\lim_{n\to\infty}\|x_n-w\|=0$ □

4. Example and numerical results

In this section, we give an example and numerical result for our main theorem.

Example 4.1. Let $H=\mathbb{R}$ and $C=[-9,3]$. And let

$S_1:[-9,3]\to\{0,1\}$ be defined by

$$S_2x=\begin{cases}0, & [-9,2];\\ 1, & (2,3].\end{cases}$$

$S_2:[-9,3]\to[-9,3]$ be defined by

$$S_2x=\begin{cases}x, & [-9,0);\\ -3x, & [0,3].\end{cases}$$

$f(x,y)=y^2+xy-2x^2$. Find $\widehat{x}\in[-9,3]$ such that $\widehat{x}\in F(S_1)\cap F(S_2)\cap EP(f)$.

Solution. To see that S_2 is k-strictly pseudononspreading, if $x,y\in[-9,2]$, then

$$\begin{aligned}|x-y|^2+&k|x-S_1x-(y-S_1)|^2+2\langle x-S_1,y-S_1y\rangle\\ &=|x-y|^2+k|x-S_1x-(y-Ty)|^2+2\langle x,y\rangle\\ &=x^2+y^2+k|x-S_1x-(y-S_1y)|^2\\ &\geq 0=|S_1x-S_1y|^2\end{aligned}$$

for all $k\in[0,1)$. If $x,y\in(2,3]$, then

$$\begin{aligned}|x-y|^2+&k|x-S_1x-(y-S_1)|^2+2\langle x-S_1,y-S_1y\rangle\\ &=|x-y|^2+k|x-1-(y-1)|^2+2\langle x-1,y-1\rangle\\ &=(k+1)|x-y|^2+2(x-1)(y-1)\\ &\geq 0=|S_1x-S_1y|^2\end{aligned}$$

for all $k\in[0,1)$. If $x\in[-9,2)$ and $y\in(2,3]$, then

$$\begin{aligned}|x-y|^2+&k|x-S_1x-(y-S_1)|^2+2\langle x-S_1,y-S_1y\rangle\\ &=|x-y|^2+k|x-(y-1)|^2+2\langle x,y-1\rangle\\ &=x^2-2xy+y^2+k|x(y-1)|^2+2x(y-1)\\ &=x^2-2xy+y^2+k|x-(y-1)|^2+2xy-2x\\ &\geq(x-1)^2+y^2+k|x-(y-1)|^2-1\\ &\geq(x-1)^2+y^2-1\end{aligned}$$

$$> 3 > 1 = |S_1x - S_1y|^2$$

for all $k \in [0,1)$. So S_1 is k-strictly nonspreading for all $k \in [0,1)$ and we can let $\beta_1 = \frac{1}{2}$. To see that S_2 is $k-$strictly pseudononspreading, if $x, y \in [-9,0)$, then

$$\begin{aligned} |S_2x - S_2y|^2 &= |x-y|^2 \\ &= |x-y|^2 + k|x - x - (y-y)|^2 + 2\langle x - x, y - y\rangle \\ &= |x-y|^2 + k|x - S_2x - (y - S_2y)|^2 + 2\langle x - S_2x, y - S_2y\rangle \end{aligned}$$

for all $k \in [0,1)$. For all $x, y \in [0,3]$, we have $|S_2x - S_2y|^2 = 9|x-y|^2$, $|x - S_2x - (y - S_2y)|^2 = 16|x-y|^2$ and $2\langle x - S_2x, y - S_2y\rangle = 32xy \geq 0$. Thus

$$\begin{aligned} |S_2x - S_2y|^2 &= 9|x-y|^2 \\ &= |x-y|^2 + \frac{1}{2}|x - S_2x - (y - S_2y)|^2 \\ &\leq |x-y|^2 + \frac{1}{2}|x - S_2x - (y - S_2y)|^2 + 2\langle x - S_2x, y - S_2y\rangle. \end{aligned}$$

If $x \in [-9,0)$ and $y \in [0,3]$ we have $|S_2x - S_2y|^2 = |x+3y|^2 = x^2 + 6xy + 9y^2$, $2\langle x - S_2x, y - S_2y\rangle = 0$, and $\frac{1}{2}|x - S_2x - (y - S_2y)|^2 = 8y^2$. Hence

$$\begin{aligned} |S_2x - S_2y|^2 &= |x+3y|^2 \\ &= (x+3y)^2 \\ &= x^2 + 6xy + 9y^2 \\ &\leq x^2 + 6xy + 9y^2 - 8xy \\ &= x^2 - 2xy + 9y^2 \\ &= |x-y|^2 + \frac{1}{2}|x - S_2x - (y - S_2y)|^2 + 2\langle x - S_2x, y - S_2y\rangle. \end{aligned}$$

Hence, for all $x, y \in [-9,3]$, we obtain

$$|S_2x - S_2y|^2 \leq |x-y|^2 + \frac{1}{2}|x - S_2x - (y - S_2y)|^2 + 2\langle x - S_2x, y - S_2y\rangle.$$

Thus S_2 is $\frac{1}{2}-$strictly pseudononspreading, and we can let $\beta_2 = \frac{1}{2}$.

For $r > 0$ and $z \in [-9,3]$, by Theorem 1.1 , there exists $x = T_{r_n}z \in [-9,3]$ that is for each $y \in [-9,3]$ that is $f(x,y) + \frac{1}{r}\langle y - x, x - z\rangle \geq 0$.
Hence

$$\begin{aligned} ry^2 + (rx + x - z)y - (2rx^2 + x^2 - xz) &= ry^2 + rxy - 2rx^2 + xy - x^2 - yz + xz \\ &= r\left(y^2 + xy - 2x^2 + \frac{1}{r}(y-x)(x-z)\right) \\ &= r\left(f(x,y) + \frac{1}{r}\langle y - x, x - z\rangle\right) \\ &\geq 0 \end{aligned}$$

Put $G(y) = ry^2 + (rx + x - z)y - (2rx^2 + x^2 - xz)$. Then G is a quadratic function of y with coefficient $a = r, b = rx + x - z$ and $c = -(2rx^2 + x^2 - xz)$. Consider:

$$\begin{aligned} b^2 - 4ac &= (rx + x - z)^2 + 4r(2rx^2 + x^2 - xz) \\ &= z^2 - 2(rx + x)z + (rx + x)^2 + 8rx^2 + 4rx^2 - 4rxz \end{aligned}$$

$$
\begin{aligned}
&= z^2 - 2rxz - 2xz + r^2x^2 + 2rx^2 + x^2 + 8r^2x^2 + 4rx^2 - 4rxz \\
&= z^2 - 6rxz - 2xz + 9r^2x^2 + 6rx^2 + x^2 \\
&= z^2 - 2(3rx + x) + (9r^2 + 6r + 1)x^2 \\
&= [z - (3r + 1)x]^2 \geq 0.
\end{aligned}
$$

Since $G(y) \geq 0$ for all $y \in [-9, 3]$, we have $b^2 - 4ac \leq 0$. Hence $z - (3r+1)x = 0$ and $z = 3rx + x$. So we have $x = T_r z = \frac{z}{3r+1}$. Let $\alpha_{i,n} = \frac{1}{2^{i+1}} + \frac{1}{n2^{n-1}}$. We can see that $\Sigma_{i=1}^{n} \alpha_{i,n} = 1$ for all $n \geq 1$ and $\liminf_{n\to\infty} \alpha_{i,n} = \liminf_{n\to\infty} \frac{1}{2^{i+1}} + \frac{1}{n2^{n-1}} = \frac{1}{2^{i+1}} > 0$. Let $r_n = \frac{n}{n+1} \in (0, \infty)$. Now we give numerical result for our algorithm. Let $\beta_i = \frac{1}{2}, \alpha_{i,n} = \frac{1}{2^{i+1}} + \frac{1}{n2^{n-1}}, r_n = \frac{n}{n+1} and S_i = S_2$ for all $i = 3, 4, \ldots$. Then algorithm becomes

$$
(4.1) \qquad \begin{cases} x_1 = 1, \\ u_n = \frac{x_n}{3r_n+1}, \\ x_{n+1} = \alpha_{0,n} u_n + \Sigma_{i=1}^{n} \alpha_{i,n} S_{i,\beta_i} u_n. \end{cases}
$$

n	x_n	z_n
1	1.000	0.400
2	0.300	0.100
3	0.051	0.015
4	0.009	0.002
5	0.003	0.001
6	0.002	0.000
⋮	⋮	⋮

Table 1:

Conclusion. Table 1 show that the sequence $\{x_n\}$ and $\{z_n\}$ converge to 0 which solves both the equilibrium problem of f and the fixed point problem of $\{S_i\}$.

References

[1] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyada, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal. **67** (2007), 2350–2360.

[2] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, The Mathematics Student. **63** (1994), 123–145.

[3] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in a Hilbert spaces*, J. Math. Anal. Appl. **20** (1967), 197–228.

[4] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), 117–136.

[5] S. Iemmoto and W. Takahashi, *Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space*, Nonlinear Anal. **71** (2009), 2080–2089.

[6] T. Igarashi, W. Takahashi and K. Tanaka, *Weak convergence theorems for nonspreading mappings and equilibrium problems*, in Nonlinear Analysis and Optimization, S. Akashi, W. Takahashi, T. Tanaka (eds), Yokohama Publishers, Yokohama, 2009, pp. 75–85.

[7] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings relate to maximal monotone operators in Banach spaces*, Arch. Math. (Basel) **91** (2008), 166–177.

[8] S. Matsushita and W. Takahashi, *Weak and strong convergence theorems for relatively nonexpansive mappings in Banach space*, Fixed Point Theory Appl. (2004), 37–47.

[9] M. O. Osilike and F. O. Isiogugu, *Weak and strong convergence theorems for nonspreading-type mappings in Hilbert space*, Nonlinear Anal. **125** (2010), 3641–3645.

[10] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.

[11] H. K. Xu, *Iterative algorithm for nonlinear operators*, J. London. Math. Soc. **2** (2002), 240–256.

[12] H. K. Xu, *Inequality in Banach spaces with apploications*, Nonlinear Anal. **16** (1991) 1127–1138.

[13] Y. H. Zhao and S.-S. Chang, *Weak and strong convergence theorems for strictly Pseudononspreading mappings and equilibrium problem in Hilbert spaces*, Abstract and Applies Anal. 2013, Article ID 169206

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