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ON CHARACTERIZING THE BLUNT MINIMIZERS OF EPSILON CONVEX PROGRAMS

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This paper is dedicated to Professor Sompong Dhompongsa on the occasion of his 65th birthday.

Abstract. In this paper, we study the minimization of a differentiable epsilon convex function over a convex set and provide several new and simple characterizations of the set of all epsilon blunt minimizers of the extremum problem. By using the basic properties of the differentiable epsilon convex functions, we characterize the set of all epsilon blunt minimizers. We also present some characterizations of epsilon straight functions and characterizations of the epsilon blunt minimizers of epsilon straight functions over a convex set are obtained. The results of this paper extend and give approximate version of various results present in literature.

1. Introduction

Consider the nonlinear optimization problem

$$(\mathrm{P}) \min f(x) \text{ subject to } x \in K,$$

where K is a nonempty convex subset of R^n and f is a real-valued derivable function defined on an open subset $D \supseteq K$. A vector $\bar{x} \in K$ is said to be an optimal solution of the problem (P), if and only if $f(\bar{x}) \leq f(x)$ for all $x \in K$.

The characterization of the optimal solutions of the problem (P) is an important study in optimization and is useful for understanding the behavior of solution methods. In 1988, Mangasarian [7] presented several characterizations of the problem (P) involving convex objective function and used these characterizations to study monotone linear complementarity problems. In 1991, Burke and Ferris [2] extended the results in [7] for nondifferentiable convex objective functions. Later, Penot [8] extended the known characterizations in the convex case to a much wider class of quasiconvex functions using subdifferentials.

On the other hand, Jeyakumar et al. [4] characterized the solution set of a convex minimization problem involving explicit convex inequality constraints in terms of the Lagrange multipliers. Later, Dinh et al. [3] established Lagrange multiplier characterizations of the solution set of the minimization of a pseudolinear function over a closed convex set subject to explicit linear inequality constraints and derived corresponding results for fractional programming problems. Further, Xu and Wu [9] derived various simple Lagrange multiplier based characterizations

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of the solution set of a programming problem under inclusion constraints. Recently, Yang [10] studied the minimization of a pseudoinvex function over an invex set and provided several new and simple characterizations of the solution set of pseudoinvex extremum problems.

In this paper, we use the notion of epsilon convexity for nonconvex and differentiable functions to introduce the concept of epsilon straight functions. We present some characterizations of epsilon convex and straight functions. The characterization of the solution set of epsilon blunt minimizers of a nonconvex and differentiable scalar-valued epsilon convex and epsilon straight functions are obtained. The results of this paper extend and give approximate version of various results present in literature.

2. Preliminaries

In this section, we recall some known definitions and results which will be used in the sequel.

Definition 2.1 (See [6]). A set K is said to be *convex*, iff for any $x, y \in K$ and $\lambda \in [0,1]$, one has

$$x + \lambda(y - x) \in K.$$

Definition 2.2 (See [5]). Let $K \subseteq R^n$ be a convex set and let $\epsilon > 0$ be given. A function $f : K \to R$ is said to be $\epsilon-$*convex* on K, iff for any $x, y \in K$ and $\lambda \in [0,1]$, one has

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + \epsilon\lambda(1-\lambda)\|x - y\|.$$

If $-f$ is $\epsilon-$convex on K, then f is said to be $\epsilon-$concave on K. If f is both $\epsilon-$convex and $\epsilon-$concave on K, then f is said to be $\epsilon-$straight on K.

Lemma 2.3 (See [5]). *Let $K \subseteq R^n$ be an open convex set and let $\epsilon > 0$ be given. Then, $f : K \to R$ is differentiable $\epsilon-$convex on K if and only if for any $x, y \in K$, one has*

$$f(x) - f(y) \geq \langle \nabla f(y), x - y \rangle - \epsilon\|x - y\|.$$

Lemma 2.4 (See [5]). *Let $K \subseteq R^n$ be an open convex set and let $\epsilon > 0$ be given. Then, $f : K \to R$ is differentiable $\epsilon-$convex on K if and only if for any $x, y \in K$, one has*

$$\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq -2\epsilon\|y - x\|.$$

Definition 2.5 (See [1]). Let $\epsilon > 0$ be given. A vector $\bar{x} \in K$ is said to be an $\epsilon-$*blunt minimizer* of $f : K \to R$ over K, iff for any $x \in K$, one has

$$f(\bar{x}) - \epsilon\|\bar{x} - x\| \leq f(x).$$

3. Variational inequalities

In this section, we characterize epsilon blunt minimizers of a differentiable epsilon convex function over a convex set using variational inequalities of Stampacchia and Minty type.

Theorem 3.1. *Let $\epsilon, \epsilon_1 > 0$ and let $K \subseteq R^n$ be an open convex set. Let $f : K \to R$ be $\epsilon-$convex on K. Then, the following implications hold:*

(i) *If* $\bar{x} \in K$ *is an* ϵ_1-*blunt minimizer of* f *over* K, *then*

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq -\epsilon_1 \|x - \bar{x}\|, \forall x \in K;$$

(ii) *If*

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq -\epsilon_1 \|x - \bar{x}\|, \forall x \in K,$$

then $\bar{x}$ *is an* $(\epsilon + \epsilon_1)-$*blunt minimizer of* f *over* K.

Proof. (i) Suppose that $\bar{x} \in K$ is an ϵ_1-blunt minimizer of f over K. Then, for any $x \in K$, one has

$$f(\bar{x}) - \epsilon_1 \|\bar{x} - x\| \leq f(x).$$

Since K is a convex set, for any $x \in K$ and $\lambda \in]0,1[$, one has

$$\bar{x} + \lambda(x - \bar{x}) \in K,$$

which implies that, for any $x \in K$ and $\lambda \in]0,1[$, one has

$$f(\bar{x} + \lambda(x - \bar{x})) - f(\bar{x}) \geq -\epsilon_1 \lambda \|x - \bar{x}\|.$$

Dividing throughout by λ and passing to the limits as λ tends to 0, one has

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq -\epsilon_1 \|x - \bar{x}\|.$$

(ii) Suppose that, for any $x \in K$, one has

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq -\epsilon_1 \|x - \bar{x}\|.$$

By $\epsilon-$convexity of f at $\bar{x}$ over K, one has

$$f(x) - f(\bar{x}) \geq \langle \nabla f(\bar{x}), x - \bar{x} \rangle - \epsilon \|x - \bar{x}\|,$$

which implies that

$$f(\bar{x}) - (\epsilon + \epsilon_1) \|x - \bar{x}\| \leq f(x),$$

that is, $\bar{x}$ is an $(\epsilon + \epsilon_1)-$blunt minimizer of f over K. This completes the proof.

□

Based on Theorem 3.1, for $\epsilon = \epsilon_1 = 0$, we have the following result.

Corollary 3.2. *Let* $K \subseteq R^n$ *be an open convex set and let* $f : K \to R$ *be a convex function on* K. *Then,* $\bar{x} \in K$ *is an optimal solution of the problem (P) if and only if*

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0, \forall x \in K.$$

Theorem 3.3. *Let* $\epsilon, \epsilon_1 > 0$ *and let* $K \subseteq R^n$ *be an open convex set. Let* $f : K \to R$ *be* $\epsilon-$*convex on* K. *Then, the following implications hold:*

(i) *If* $\bar{x} \in K$ *is an* ϵ_1-*blunt minimizer of* f *over* K, *then*

$$\langle \nabla f(x), x - \bar{x} \rangle \geq -(2\epsilon + \epsilon_1) \|x - \bar{x}\|, \forall x \in K;$$

(ii) *If*

$$\langle \nabla f(x), x - \bar{x} \rangle \geq -(2\epsilon + \epsilon_1) \|x - \bar{x}\|, \forall x \in K,$$

then $\bar{x}$ *is an* $(3\epsilon + \epsilon_1)-$*blunt minimizer of* f *over* K.

Proof. (i) Suppose that $\bar{x} \in K$ is an ϵ_1-blunt minimizer of f over K. Then, from Theorem 3.1 (i), it follows that

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq -\epsilon_1 \|x - \bar{x}\|, \forall x \in K.$$

Since f is differentiable $\epsilon-$convex on K, by Lemma 2.4, it follows that

$$\langle \nabla f(x), x - \bar{x} \rangle \geq -(2\epsilon + \epsilon_1) \|x - \bar{x}\|, \forall x \in K.$$

(ii) Suppose that

$$\langle \nabla f(x), x - \bar{x} \rangle \geq -(2\epsilon + \epsilon_1) \|x - \bar{x}\|, \forall x \in K.$$

Since K is an open convex set, it follows that

$$\langle \nabla f(\bar{x} + \lambda(x - \bar{x})), x - \bar{x} \rangle \geq -(2\epsilon + \epsilon_1) \|x - \bar{x}\|, \forall x \in K, \forall \lambda \in (0, 1).$$

Letting $\lambda \to 0^+$, it follows that

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq -(2\epsilon + \epsilon_1) \|x - \bar{x}\|, \forall x \in K.$$

From Theorem 3.1 (ii), it follows that, $\bar{x}$ is an $(3\epsilon + \epsilon_1)-$blunt minimizer of f over K.

□

For $\epsilon = \epsilon_1 = 0$, we have the following result.

Corollary 3.4. *Let $K \subseteq R^n$ be an open convex set and let $f : K \to R$ be convex on K. Then, $\bar{x} \in K$ is a minimizer of f over K if and only if*

$$\langle \nabla f(x), x - \bar{x} \rangle \geq 0, \forall x \in K.$$

Example 3.5. Consider the optimization problem as follows:

$$\min f(x) \text{ s.t.} x \in \mathbb{R},$$

where $f(x) := x^3 - x^2$. It is easy to see that, for any $\epsilon > 0$, there exists $0 < \delta_1 < \left|\frac{1-\sqrt{1+4\epsilon}}{2}\right|$ such that, f is $\epsilon-$convex at $\bar{x} := 0$ over $B(\bar{x}; \delta_1)$. Now, for any $\epsilon > 0$, one has

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle + \epsilon \|x - \bar{x}\| = \epsilon |x| \geq 0, \forall x \in \mathbb{R}.$$

By Theorem 3.1, it follows that, for any $\epsilon > 0$, there exists $0 < \delta_1 < \left|\frac{1-\sqrt{1+4\epsilon}}{2}\right|$ such that, $\bar{x} := 0$ is an $\epsilon-$blunt minimizer of f over $B(\bar{x}; \delta_1)$.

Also, for any $\epsilon > 0$, there exists $0 < \delta_2 < \left|\frac{2-\sqrt{4+12\epsilon}}{6}\right|$ such that

$$\langle \nabla f(x), x - \bar{x} \rangle + \epsilon \|x - \bar{x}\| = 3x^3 - 2x^2 + \epsilon |x| \geq 0, \forall x \in B(\bar{x}; \delta_2).$$

Setting $\delta := \min \{\delta_1, \delta_2\}$, by Theorem 3.3, it follows that, $\bar{x} := 0$ is an $\epsilon-$blunt minimizer of f over $B(\bar{x}; \delta)$.

Example 3.6. Consider the optimization problem as follows:

$$\min f(x) \text{ s.t.} x \in \mathbb{R},$$

where $f(x) := x^3 - x^2$. It is easy to see that, for any $\epsilon > 0$, there exists $0 < \delta_1$ such that, f is $\epsilon-$convex at $\bar{x} := 1$ over $B(\bar{x}; \delta_1)$. Now, for any $\epsilon \geq 1$, one has

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle + \epsilon \|x - \bar{x}\| = x - 1 + \epsilon |x - 1| \geq 0, \forall x \in \mathbb{R}.$$

By Theorem 3.1, it follows that, for any $\epsilon \geq 1$, there exists $0 < \delta_1$ such that, $\bar{x} := 1$ is an $\epsilon-$blunt minimizer of f over $B(\bar{x};\delta_1)$.

Also, for any $\epsilon \geq 1$, there exists $0 < \delta_2$ such that

$$\langle \nabla f(x), x - \bar{x} \rangle + \epsilon \|x - \bar{x}\| = 3x^3 - 5x^2 + 2x + \epsilon|x-1| \geq 0, \forall x \in B(\bar{x};\delta_2).$$

Setting $\delta := \min\{\delta_1, \delta_2\}$, by Theorem 3.3, it follows that, for any $\epsilon \geq 1$, $\bar{x} := 0$ is an $\epsilon-$blunt minimizer of f over $B(\bar{x};\delta)$.

4. Characterizations of the solution sets

In this section, we characterize epsilon blunt minimizers of a differentiable epsilon convex function over a convex set. Throughout this paper, the solution set of (P) is denoted by

$$\bar{S} := \arg\min_{x \in K} f(x).$$

For any $\epsilon > 0$, the set of all $\epsilon-$blunt minimizers of f over K is denoted by $\bar{S}(\epsilon)$. For $\epsilon = 0$, $\bar{S}(\epsilon) = \bar{S}$.

Theorem 4.1. *Let $\epsilon, \epsilon_1, \epsilon_2 > 0$ be given and let $K \subseteq R^n$ be an open convex set. If $f : K \to R$ is differentiable $\epsilon-$convex on K and $\bar{x} \in S(\epsilon_1), \bar{y} \in S(\epsilon_2)$, then*

$$-\epsilon_1 \|\bar{y} - \bar{x}\| \leq \langle \nabla f(\bar{x}), \bar{y} - \bar{x} \rangle \leq (\epsilon_2 + 2\epsilon) \|\bar{y} - \bar{x}\|$$

and

$$-\epsilon_2 \|\bar{x} - \bar{y}\| \leq \langle \nabla f(\bar{y}), \bar{x} - \bar{y} \rangle \leq (\epsilon_1 + 2\epsilon) \|\bar{x} - \bar{y}\|.$$

Proof. Suppose that $\bar{x} \in S(\epsilon_1)$ and $\bar{y} \in S(\epsilon_2)$. Then, by Theorem 3.1 (i), it follows that

$$\langle \nabla f(\bar{x}), \bar{y} - \bar{x} \rangle \geq -\epsilon_1 \|\bar{y} - \bar{x}\|$$

and

$$\langle \nabla f(\bar{y}), \bar{x} - \bar{y} \rangle \geq -\epsilon_2 \|\bar{x} - \bar{y}\|.$$

By $\epsilon-$convexity of f on K, by Lemma 2.4, one has

$$\langle \nabla f(\bar{x}) - \nabla f(\bar{y}), \bar{x} - \bar{y} \rangle \geq -2\epsilon \|\bar{x} - \bar{y}\|,$$

which implies that

$$\langle \nabla f(\bar{y}), \bar{x} - \bar{y} \rangle \leq (\epsilon_1 + 2\epsilon) \|\bar{x} - \bar{y}\|$$

and

$$\langle \nabla f(\bar{x}), \bar{y} - \bar{x} \rangle \leq (\epsilon_2 + 2\epsilon) \|\bar{y} - \bar{x}\|.$$

Hence, one has

$$-\epsilon_1 \|\bar{y} - \bar{x}\| \leq \langle \nabla f(\bar{x}), \bar{y} - \bar{x} \rangle \leq (\epsilon_2 + 2\epsilon) \|\bar{y} - \bar{x}\|$$

and

$$-\epsilon_2 \|\bar{x} - \bar{y}\| \leq \langle \nabla f(\bar{y}), \bar{x} - \bar{y} \rangle \leq (\epsilon_1 + 2\epsilon) \|\bar{x} - \bar{y}\|.$$

This completes the proof. □

For $\epsilon = \epsilon_1 = \epsilon_2 = 0$, from Theorem 4.1, we have the following result.

Corollary 4.2. *Let $K \subseteq R^n$ be an open convex set. If $f : K \to R$ is differentiable convex on K and $\bar{x}, \bar{y} \in \bar{S}$, then*

$$\langle \nabla f(\bar{x}), \bar{y} - \bar{x} \rangle = \langle \nabla f(\bar{y}), \bar{x} - \bar{y} \rangle = 0.$$

Theorem 4.3. *Let $\epsilon, \epsilon_1, \epsilon_2 > 0$ be given and let $K \subseteq R^n$ be an open convex set. Let $f : K \to R$ be differentiable $\epsilon-$convex on K and $\bar{x} \in \bar{S}(\epsilon_1)$, then*

$$\bar{S}(\epsilon_2) \subseteq \tilde{S}(\epsilon_3),$$

where

$$\epsilon_3 := \max\{2\epsilon + \epsilon_1, \epsilon_2\}$$

and

$$\tilde{S}(\epsilon_3) := \{x \in K : -\epsilon_3 \|\bar{x} - x\| \leq \langle \nabla f(x), \bar{x} - x\rangle \leq \epsilon_3 \|\bar{x} - x\|\}.$$

Proof. Suppose that $\bar{y} \in S(\epsilon_2)$. Then, from Theorem 3.1 (i), it follows that

$$\langle \nabla f(\bar{y}), \bar{x} - \bar{y}\rangle \geq -\epsilon_2 \|\bar{x} - \bar{y}\|.$$

Also, from Theorem 3.3 (i), it follows that

$$\langle \nabla f(\bar{y}), \bar{y} - \bar{x}\rangle \geq -(2\epsilon + \epsilon_1) \|\bar{y} - \bar{x}\|.$$

From the above inequalities, one has

$$-\epsilon_2 \|\bar{x} - \bar{y}\| \leq \langle \nabla f(\bar{y}), \bar{x} - \bar{y}\rangle \leq (\epsilon + \epsilon_1) \|\bar{y} - \bar{x}\|.$$

Setting $\epsilon_3 := \max\{2\epsilon + \epsilon_1, \epsilon_2\}$, it follows that

$$-\epsilon_3 \|\bar{x} - \bar{y}\| \leq \langle \nabla f(\bar{y}), \bar{x} - \bar{y}\rangle \leq \epsilon_3 \|\bar{y} - \bar{x}\|,$$

that is, $\bar{y} \in \tilde{S}(\epsilon_3)$, where

$$\tilde{S}(\epsilon_3) := \{x \in K : -\epsilon_3 \|\bar{x} - x\| \leq \langle \nabla f(x), \bar{x} - x\rangle \leq \epsilon_3 \|\bar{x} - x\|\}$$

and hence

$$\bar{S}(\epsilon_2) \subseteq \tilde{S}(\epsilon_3).$$

□

For $\epsilon = \epsilon_1 = \epsilon_2 = 0$, we have the following result.

Corollary 4.4. *Let $K \subseteq R^n$ be an open convex set and let $f : K \to R$ be differentiable convex on K. If $\bar{x} \in \bar{S}$, then*

$$\bar{S} \subseteq \tilde{S},$$

where

$$\tilde{S} := \{x \in K : \langle \nabla f(x), \bar{x} - x\rangle = 0\}.$$

Theorem 4.5. *Let $\epsilon, \epsilon_1, \epsilon_3, \epsilon_4 > 0$ be given and let $K \subseteq R^n$ be an open convex set. Let $f : K \to R$ be differentiable $\epsilon-$convex on K and $\bar{x} \in \bar{S}(\epsilon_1)$, then*

$$\tilde{S}(\epsilon_3) \subseteq \bar{\bar{S}}(\epsilon_4),$$

where

$$\epsilon_4 := \max\{\epsilon_3 + \epsilon, \epsilon_1\}$$

and

$$\bar{\bar{S}}(\epsilon_4) := \{x \in K : -\epsilon_4 \|x - \bar{x}\| \leq f(x) - f(\bar{x}) \leq \epsilon_4 \|x - \bar{x}\|\}.$$

Proof. Suppose that $\bar{y} \in \tilde{S}(\epsilon_3)$. Then, by $\epsilon-$convexity of f on K, it follows that

$$\begin{aligned} f(\bar{x}) - f(\bar{y}) &\geq \langle \nabla f(\bar{y}), \bar{x} - \bar{y} \rangle - \epsilon \|\bar{x} - \bar{y}\| \\ &\geq -(\epsilon_3 + \epsilon) \|\bar{x} - \bar{y}\|. \end{aligned} \tag{4.1}$$

Also, $\bar{x} \in \bar{S}(\epsilon_1)$, it follows that

$$f(\bar{x}) - f(\bar{y}) \leq \epsilon_1 \|\bar{x} - \bar{y}\|.$$

From the above inequalities, one has

$$-\epsilon_1 \|\bar{y} - \bar{x}\| \leq f(\bar{y}) - f(\bar{x}) \leq (\epsilon + \epsilon_3) \|\bar{y} - \bar{x}\|.$$

Setting $\epsilon_4 := \max\{\epsilon_1, \epsilon + \epsilon_3\}$, it follows that

$$-\epsilon_4 \|\bar{y} - \bar{x}\| \leq f(\bar{y}) - f(\bar{x}) \leq \epsilon_4 \|\bar{y} - \bar{x}\|,$$

which implies that

$$\tilde{S}(\epsilon_3) \subseteq \bar{\bar{S}}(\epsilon_4),$$

where

$$\bar{\bar{S}}(\epsilon_4) := \{x \in K : -\epsilon_4 \|x - \bar{x}\| \leq f(x) - f(\bar{x}) \leq \epsilon_4 \|x - \bar{x}\|\}.$$

This completes the proof. □

For $\epsilon = \epsilon_1 = \epsilon_3 = \epsilon_4 = 0$, we have the following result.

Corollary 4.6. *Let $K \subseteq R^n$ be an open convex set and let $f : K \to R$ be differentiable convex on K. If $\bar{x} \in \bar{S}$, then*

$$\tilde{S} \subseteq \bar{S}.$$

5. Some characterizations of epsilon straight functions

In this section, we characterize epsilon blunt minimizers under the assumptions of epsilon straight functions.

Theorem 5.1. *Let $\epsilon > 0$ and let $K \subseteq R^n$ be an open convex set. Let $f : K \to R$ be $\epsilon-$straight on K. Then, for any $x, y \in K$, one has*

$$-2\epsilon \|x - y\| \leq f(x) - f(y) \leq 2\epsilon \|x - y\|$$

if and only if

$$-\epsilon \|x - y\| \leq \langle \nabla f(y), x - y \rangle \leq \epsilon \|x - y\|.$$

Proof. Let $\epsilon > 0$ and $\bar{x} \in K \subseteq R^n$ be arbitrary. Let f be $\epsilon-$starshaped at $\bar{x}$ over K, that is, for all $x \in K$ and $\lambda \in]0, 1[$, one has

$$f(\bar{x} + \lambda(x - \bar{x})) \leq f(\bar{x}) + \lambda(f(x) - f(\bar{x})) + \epsilon\lambda(1 - \lambda) \|x - \bar{x}\|.$$

Dividing throughout by λ and passing to the limit as λ tends to 0, it follows that

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \epsilon \|x - \bar{x}\|, \forall x \in K.$$

Since, for all $x \in K$, $-\epsilon \|x - \bar{x}\| \leq \langle \nabla f(\bar{x}), x - \bar{x} \rangle$, it follows that

$$-2\epsilon \|x - \bar{x}\| \leq f(x) - f(\bar{x}), \forall x \in K.$$

Similarly, by $\epsilon-$starshapedness of $-f$ at $\bar{x}$ over K, one has

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq f(x) - f(\bar{x}) - \epsilon \|x - \bar{x}\|, \forall x \in K.$$

Since, for all $x \in K$, $\langle \nabla f(\bar{x}), x - \bar{x} \rangle \leq \epsilon \|x - \bar{x}\|$, it follows that

$$f(x) - f(\bar{x}) \leq 2\epsilon \|x - \bar{x}\|, \forall x \in K.$$

From the above inequalities, for any $x \in K$, one has

$$-\epsilon \|x - \bar{x}\| \leq \langle \nabla f(\bar{x}), x - \bar{x} \rangle \leq \epsilon \|x - \bar{x}\|$$

implies

$$-2\epsilon \|x - \bar{x}\| \leq f(x) - f(\bar{x}) \leq 2\epsilon \|x - \bar{x}\|.$$

Conversely, suppose that, for any $x, y \in K$, one has

$$-2\epsilon \|x - y\| \leq f(x) - f(y) \leq 2\epsilon \|x - y\|.$$

We have to show that

$$-\epsilon \|x - y\| \leq \langle \nabla f(y), x - y \rangle \leq \epsilon \|x - y\|.$$

We first show that, for any $\lambda \in]0, 1[$, one has

$$-\epsilon\lambda \|x - y\| \leq f(y + \lambda(x - y)) - f(y) \leq \epsilon\lambda \|x - y\|.$$

If $f(y + \lambda(x - y)) - f(y) > \epsilon\lambda \|x - y\|$, then, by the $\epsilon-$convexity of f on K, one has

$$f(y) - f(y + \lambda(x - y)) \geq \langle \nabla f(y + \lambda(x - y)), -\lambda(x - y) \rangle + \epsilon\lambda \|x - y\|,$$

which implies that

$$2\epsilon(1 - \lambda) \|x - y\| < \langle \nabla f(y + \lambda(x - y)), x - (y + \lambda(x - y)) \rangle.$$

By $\epsilon-$convexity of f on K, one has

$$\langle \nabla f(y + \lambda(x - y)), x - (y + \lambda(x - y)) \rangle \leq f(x) - f(y + \lambda(x - y)) - \epsilon(1 - \lambda) \|x - y\|,$$

which implies that

$$3\epsilon(1 - \lambda) \|x - y\| < f(x) - f(y + \lambda(x - y)),$$

a contradiction to the assumption that

$$f(x) - f(y + \lambda(x - y)) \leq \epsilon(1 - \lambda)(x - y).$$

Similarly, $f(y + \lambda(x - y)) - f(y) < -\epsilon\lambda \|x - y\|$ leads to a contradiction by using the $\epsilon-$concavity of f on K.

Hence, for any $\lambda \in]0, 1[$, one has

$$-\epsilon\lambda \|x - y\| \leq f(y + \lambda(x - y)) - f(y) \leq \epsilon\lambda \|x - y\|.$$

Dividing throughout by λ and passing to the limits as λ tends to 0, one has

$$-\epsilon \|x - y\| \leq \langle \nabla f(y), x - y \rangle \leq \epsilon \|x - y\|.$$

This completes the proof. □

Theorem 5.2. *Let $\epsilon > 0$ and let $K \subseteq R^n$ be an open convex set. Let $f : K \to R$ be $\epsilon-$straight on K and let $S(2\epsilon)$ be the set of all $2\epsilon-$blunt minimizers of f over K given by*

$$S(2\epsilon) := \{x \in K : f(x) - 2\epsilon \|x - y\| \leq f(y), \forall y \in K\}.$$

If $\bar{x} \in S(2\epsilon)$, then $S(2\epsilon) = S_1(\epsilon) = S_2(\epsilon)$, where

$$S_1(\epsilon) := \{x \in K : -\epsilon \|x - \bar{x}\| \leq \langle \nabla f(\bar{x}), x - \bar{x} \rangle \leq \epsilon \|x - \bar{x}\|\}$$

and

$$S_2(\epsilon) := \{x \in K : -\epsilon \|x - \bar{x}\| \leq \langle \nabla f(x), x - \bar{x} \rangle \leq \epsilon \|x - \bar{x}\|\}.$$

Proof. The point $x \in S(2\epsilon)$ if and only if $-2\epsilon \|x - \bar{x}\| \leq f(x) - f(\bar{x}) \leq 2\epsilon \|x - \bar{x}\|$. Then, from the above discussion, one has

$$-2\epsilon \|x - \bar{x}\| \leq f(x) - f(\bar{x}) \leq 2\epsilon \|x - \bar{x}\|$$

if and only if

$$-\epsilon \|x - \bar{x}\| \leq \langle \nabla f(\bar{x}), x - \bar{x} \rangle \leq \epsilon \|x - \bar{x}\|.$$

Also, one has

$$-2\epsilon \|x - \bar{x}\| \leq f(\bar{x}) - f(x) \leq 2\epsilon \|x - \bar{x}\|$$

if and only if

$$-\epsilon \|x - \bar{x}\| \leq \langle \nabla f(x), x - \bar{x} \rangle \leq \epsilon \|x - \bar{x}\|.$$

This completes the proof. □

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