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# FIXED POINT THEOREMS FOR INWARD MAPPINGS IN ℝ-TREES

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*Dedicated to Professor Sompong Dhompongsa on the occasion of his 65th birthday*

Abstract. In an ℝ-tree setting, we develop fixed point theorems for multivalued mappings that are strictly contractive, nonexpansive or upper semicontinuous and satisfy an inward condition. As applications, we obtain common fixed point theorems for a point-valued and a multivalued mapping that commute.

## 1. Introduction

The study of metric spaces without linear structure has played a crucial role in various branches of pure and applied sciences. One of such space is an ℝ-tree, whose study found applications in mathematics, biology/medicine and computer science. It should be worth mentioning that all the edges are assumed to have same length in the notion of an ordinary tree and so the metric structure is not required, which limit use of it in many areas. To overcome this problem, an ℝ-tree was introduced as a generalization of an ordinary tree where edges are of different length. The notion of an ℝ-tree (metric tree or T-theory) was given by Tits [20] and Dress [7] and further investigation was made by Mayer, Mohler, Oversteegen, and Tymchatyn [15], Mayer and Oversteegen [16], Kirk [12], etc. For details on application, we refer the reader to [4, 10, 18]. Fixed point theory in metric spaces without linear structure has been developed by a number of authors. For results in ℝ-trees, see e.g., [1, 2, 9, 10, 11] and the references cited therein. This paper continues the development of fixed point results for inward type mappings in metric spaces without linear structure. Previous fixed point results in such spaces were obtained by Bae [3] for a complete metric space and multivalued mappings that are inward and weakly contractive, while Dhompongsa, Kaewkhao, and Panyanak [6] considered a CAT(0) space and multivalued mappings that are inward and nonexpansive. Here we develop similar results in ℝ-trees for multivalued inward mappings that are strictly contractive, nonexpansive or upper semicontinuous. Our results weaken the inward condition so that the values of the multivalued mapping need only intersect the inward set. We make use of our results to obtain common fixed point theorems for a point-valued and a multivalued mapping that commute.

Our result for upper semicontinuous mappings depends on the following best approximation result of Kirk and Panyanak.

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**Theorem 1.1** ([13]). *Let $X$ be a closed convex subset of a complete $\mathbb{R}$-tree $(M,d)$, and assume that $X$ is geodesically bounded. Let $F : X \to 2^M$ be an upper semi-continuous mapping whose values are nonempty closed convex subsets of $M$. Then there exists a point $x \in X$ such that*

$$d(x, F(x)) = \inf_{y \in X} d(y, F(x)).$$

## 2. Preliminaries

For any pair of points $x, y$ in a metric space $(M,d)$, a *geodesic path* joining these points is a map $c$ from a closed interval $[0,r] \subset \mathbb{R}$ to $M$ such that $c(0) = x$, $c(r) = y$ and $d(c(t), c(s)) = |t-s|$ for all $s,t \in [0,r]$. The mapping $c$ is an isometry and $d(x,y) = r$. The image of $c$ is called a *geodesic segment* joining $x$ and $y$ which when unique is denoted by $[x,y]$. For any $x,y \in M$, denote the point $z \in [x,y]$ such that $d(x,z) = \alpha d(x,y)$ by $z = (1-\alpha)x \oplus \alpha y$, where $0 \leq \alpha \leq 1$. The space $(M,d)$ is called a geodesic space if any two points of $X$ are joined by a geodesic, and $M$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ and $y$ for each $x,y \in M$. A subset $X$ of $M$ is called convex if $X$ includes every geodesic segment joining any two of its points.

A geodesic triangle $\Delta(x_1,x_2,x_3)$ in a geodesic metric space $(M,d)$ consists of three points in $M$ (the vertices of $\Delta$) and a geodesic segment between each pair of points (the edges of $\Delta$). A comparison triangle for $\Delta(x_1,x_2,x_3)$ in $(M,d)$ is a triangle $\overline{\Delta}(x_1,x_2,x_3) := \Delta(\overline{x_1},\overline{x_2},\overline{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i},\overline{x_j}) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$.

A geodesic metric space $M$ is called a *CAT(0) space* if all geodesic triangles of appropriate size satisfy the following comparison axiom [5]:
Let $\Delta$ be a geodesic triangle in $M$ and let $\overline{\Delta}$ in $\mathbb{R}^2$ be its comparison triangle. Then $\Delta$ is said to satify the CAT(0) inequality if for all $x,y \in \Delta$ and all comparison points $\overline{x},\overline{y}$ in $\overline{\Delta}$, $d(x,y) \leq d_{\mathbb{R}^2}(\overline{x},\overline{y})$. A subset $X$ of $(M,d)$ is said to be gated [8] if for any point $x \notin X$ there exists a unique $z \in X$ such that for any $y \in X$,

$$d(x,y) = d(x,z) + d(z,y).$$

The point $z$ is called the gate of $x$ in $X$.

The following properties of gated sets are useful [8, 9].

(i) Gated sets in a complete geodesic space are always closed and convex.
(ii) Gated subsets of a complete geodesic space $(M,d)$ are proximinal nonexpansive retracts of $M$.
(iii) The family of gated sets in a complete geodesic space $(M,d)$ has the *Helly property*, that is, if $X_1, .., X_n$ is a collection of gated sets in $M$ with pairwise nonempty intersection, then $\cap_{i=1}^n X_i \neq \emptyset$.
(iv) Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of nonempty gated subsets of a complete geodesic space $(M,d)$ which is downward directed by set inclusion. If $M$ (or more generally, some $X_\alpha$) does not contain a geodesic ray (that is, is geodesically bounded), then $\cap_{\alpha \in I} X_\alpha \neq \emptyset$.

There are many equivalent definitions of $\mathbb{R}$-tree. Here we include the following definition.

An $\mathbb{R}$-*tree* is a metric space $M$ such that:

(i) there is a unique geodesic segment $[x, y]$ joining each pair of points $x, y \in M$.
(ii) if $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.
It follows from (i) and (ii) that
(iii) if $u, v, w \in M$, then $[u, v] \cap [u, w] = [u, z]$ for some $z \in M$.

Examples of $\mathbb{R}$-trees can be found in [10].

The following properties of an $\mathbb{R}$-tree are useful [5, 9, 10].

(i) An $\mathbb{R}$-tree is a CAT(0) space.
(ii) The metric $d$ in an $\mathbb{R}$-tree is *convex,* that is, it satisfies the inequality

$$d(\alpha x \oplus (1-\alpha)y, \alpha u \oplus (1-\alpha)v) \leq \alpha d(x, u) + (1-\alpha)d(y, v)$$

for any points $x, y, u, v \in M$.
(iii) A metric space is a complete $\mathbb{R}$-tree if and only if it is hyperconvex and has unique geodesic segments.
(iv) In an $\mathbb{R}$-tree, the gated subsets are precisely its closed and convex subsets.

For any subset $X$ in a metric space $M$, we define $d(x, X) = \inf_{y \in X} d(x, y)$. Denote the nonempty subsets of a metric space $M$ by $2^M$. In a metric space $M$, a mapping $F : X \to 2^M$ with closed bounded values is called *strictly contractive* if $H(F(x), F(y)) \leq kd(x, y)$ for a fixed $k \in [0, 1)$ and any pair $x, y \in X$, where $H$ denotes the Hausdorff metric derived from the metric $d$. $F$ is called *nonexpansive* if $H(F(x), F(y)) \leq d(x, y)$ for any pair $x, y \in M$. The mapping $F$ is said to be *upper semicontinuous* at a point $x \in X$ if for any open set $U$ containing $F(x)$ there is an open set $V$ containing $x$ such that $y \in V$ implies $F(y) \subset U$. $F$ is upper semicontinuous on $X$ if it is so for every $x \in X$. The mappings $f : M \to M$ and $F : M \to 2^M$ are said to *commute* [17] if $f(F(x)) \subset F(f(x))$ for all $x \in M$. If $X$ is a closed convex subset of a metric space $M$, a mapping $F : X \to 2^M$ is said to be *inward* if for each $x \in X$, $F(x) \cap I_X(x) \neq \emptyset$, where $I_X(x)$ is the *metrically inward set* of $X$ at $x$ defined by

$$I_X(x) = \{z \in M : z = x \text{ or there exists } y \in X \text{ such that } y \neq x \text{ and}$$
$$d(x, z) = d(x, y) + d(y, z)\}.$$

In a metric space $M$, the *$\epsilon$-fixed point set* of a mapping $F : M \to 2^M$ is defined as $\{x \in M : d(x, F(x)) \leq \epsilon\}$.

## 3. Fixed points for inward mappings

The following result shows that the metrically inward set of a gated subset of a complete $\mathbb{R}$-tree is also gated.

**Theorem 3.1.** *Let $(M, d)$ be a complete $\mathbb{R}$-tree and $U$ a closed convex subset of $M$. Then $I_U(x)$ is a closed convex set for each $x \in U$.*

*Proof.* For $x \in U$, assume $\{x_n\}$ is a Cauchy sequence in $I_U(x)$ that converges to a point $p$. If $p \in U$, then $p \in I_U(x)$ since $U \subset I(x)$. Otherwise assume $p \notin U$, and let $d = d(p, U)$. Choose a natural number $m$ sufficiently large so that $d(p, x_m) < d$, and let $d_1 = d(x_m, U)$. Since the sets $B(p, d)$, $B(x_m, d_1)$ and $U$ have pairwise nonempty intersections, by the Helly property we have

$$B(p, d) \cap B(x_m, d_1) \cap U \neq \emptyset. \tag{3.1}$$

Since the sets $B(p, d)$ and $B(x_m, d_1)$ each intersect $U$ in a single point, the nonemptiness of the set in (3.1) implies that the two points of intersection are the same point $q \in U$. Since $x_m \in I_U(x)$ and $x_m \notin U$, by the definition of $I_U(x)$, there is a $z \in U$ such that $z \neq x$ and

$$d(x, x_m) = d(x, z) + d(z, x_m),$$

which implies that

$$z \in [x, x_m]. \tag{3.2}$$

Since $q$ is the gate of $x_m$ in $U$ and $z \in U$, we have

$$q \in [z, x_m]. \tag{3.3}$$

The inclusions (3.2) and (3.3) imply that $q \neq x$. Then, by the fact that $q$ is also the gate of $p$ in $U$, it follows that

$$d(x, p) = d(x, q) + d(q, p),$$

and therefore, that $p \in I_U(x)$.

For any pair of points $u, v \in I_U(x)$, we construct a geodesic connecting them that lies in $I_U(x)$. Considering the metric segments $[u, x]$ and $[v, x]$, by the definition of an $\mathbb{R}$-tree there is a point $w \in M$ such that $[u, x] \cap [v, x] = [w, x]$. Then since $[u, w] \cap [v, w] = \{w\}$, we have $[u, w] \cup [v, w] = [u, v]$, and by its construction, $[u, v] \subset I_U(x)$. □

**Remark 3.2.** In a complete $\mathbb{R}$-tree $M$, if $U$ is a gated subset of $M$, then $I_U(x)$ is also gated for each $x \in U$.

**Theorem 3.3.** *Let $M$ be a complete R-tree, $X$ a closed convex subset and $F : X \to 2^M$ a strictly contractive mapping with values that are nonempty closed bounded convex subsets of $M$ and that satisfies*

$$F(x) \cap I_X(x) \neq \emptyset \text{ for } x \in X.$$

*Then $F$ has a fixed point in $X$, that is, there is a $z \in X$ such that $z \in F(z)$.*

*Proof.* Let $P$ be the proximinal nonexpansive retraction of $M$ onto $X$. By considering the mapping $F(P(.)) : M \to 2^M$ which agrees with $F$ on the subset $X$, we can apply Theorem 3.2 of [14] to obtain a strictly contractive selection $f : X \to M$ of the multivalued mapping $F$. Define $g = P \circ f$. Then $g$ is a strictly contractive mapping from $X$ into $X$. Let $u \in X$ be the unique fixed point of $g$. Then $f(u) \in F(u)$ and $d(u, f(u)) = d(f(u), X)$. Let $v$ be the unique closest point in $F(u)$ to $u$. Then $v$ is the gate of $u$ in $F(u)$, and therefore $v$ lies on the unique metric segment connecting $u$ and $f(u)$. This implies that $u$ is the closest point to $v$ in the set $X$. Choosing a point $z \in F(u) \cap I_X(u)$, the convexity of $F(u)$ implies that $[v, z] \subset F(u)$. However, $u$ is the closest point to $v$ in $X$, and $z \in I_X(u)$, which implies that

$$[v, z] = [v, u] \cup [u, z] \subset F(u).$$

It follows that $u \in F(u)$. □

**Theorem 3.4.** *Let $(M, d)$ be a complete $\mathbb{R}$-tree, $X$ a closed convex subset of $M$ that is geodesically bounded, and $F : X \to 2^M$ an upper semicontinuous mapping with values that are nonempty closed convex subsets of $M$. If $F$ satisfies the condition:*

$$F(x) \cap I_X(x) \neq \emptyset \text{ for } x \in X,$$

*then $F$ has a fixed point in $X$.*

*Proof.* By Theorem 1.1, there is a point $u \in X$ such that

$$d(u, F(u)) = \inf_{y \in X} d(y, F(u)).$$

Let $v$ be the unique closest point in $F(u)$ to $u$. Since

$$F(u) \cap I_X(u) \neq \emptyset,$$

we can choose a point $z \in F(u) \cap I_X(u)$, where by convexity of $F(u)$, $[v, z] \subset F(u)$. However, $u$ is the closest point in $X$ to $v$, and $z \in I_X(u)$, which implies that

$$[v, z] = [v, u] \cup [u, z] \subset F(u).$$

It follows that $u \in F(u)$. □

**Corollary 3.5.** *Let $(M, d)$ be a complete $\mathbb{R}$-tree, $X$ a closed convex subset of $M$ that is geodesically bounded, and $F : X \to 2^M$ a nonexpansive mapping with values that are nonempty closed convex subsets of $M$. If $F$ satisfies the condition:*

$$F(x) \cap I_X(x) \neq \emptyset \text{ for } x \in X,$$

*then $F$ has a fixed point in $X$.*

**Theorem 3.6.** *Let $(M, d)$ be a complete $\mathbb{R}$-tree, $X$ a closed convex subset of a $M$ that is geodesically bounded, and $F : X \to 2^M$ an upper semicontinuous mapping with values that are nonempty closed convex subsets of $M$. Then there exist $x_0 \in X$ and $y_0 \in F(x_0)$ with*

$$d(y_0, x_0) = d(y_0, X) = d(y_0, I_X(x_0)).$$

*Proof.* Theorem 1.1 guarantees existence of points $x_0 \in X$ and $y_0 \in F(x_0)$ such that

$$d(y_0, x_0) = d(y_0, X).$$

It remains to show that these same points satisfy

$$d(y_0, X) = d(y_0, I_X(x_0)).$$

Since $x_0$ is the unique closest point in $X$ to $y_0$, for any $y \in I_X(x_0)$, we have

$$d(y_0, y) = d(y_0, x_0) + d(x_0, y),$$

which implies that

$$d(y_0, y) \geq d(y_0, x_0)$$

and, therefore, that

$$d(y_0, X) = d(y_0, I_X(x_0)).$$

□

**Theorem 3.7.** *Let $(M,d)$ be a complete $\mathbb{R}$-tree, $X$ a closed convex subset of $M$ that is geodesically bounded, and $F : X \to 2^M$ an upper semicontinuous mapping with values that are nonempty closed convex subsets of $M$. If $F$ satisfies any one of the following conditions:*

(i) *for any $x \in X$ we have $d(y,z) < d(y,x)$ for each $y \in F(x)$ and some $z \in I_X(x)$;*

(ii) *for any $x \in X$, $(x,y] \cap I_X(x) \neq \emptyset$ for each $y \in F(x)$;*

*then $F$ has a fixed point.*

*Proof.* By Theorem 3.6, there exist $x_0$ and $y_0 \in F(x_0)$ with

$$d(y_0,x_0) \;=\; d(y_0,X) \;=\; d(y_0,I_X(x_0)).$$

Suppose $F$ satisfies condition (i) and assume that $x_0 \notin F(x_0)$. Then, by condition (i), we have $d(y_0,z) < d(y_0,x_0)$ for some $z \in I_X(x_0)$. This contradicts the choice of $x_0$. Hence $F$ has a fixed point.

Suppose $F$ satisfies condition (ii), and assume that $x_0 \notin F(x_0)$. Then, by condition (ii), $z_0 \in (x_0,y_0] \cap I_X(x_0) \neq \emptyset$. This implies that

$$\begin{aligned} d(y_0,x_0) &\leq d(y_0,z_0) \\ &= d(y_0,x_0) - d(z_0,x_0) \\ &< d(y_0,x_0), \end{aligned}$$

which is a contradiction. Hence $F$ has a fixed point. □

## 4. Fixed points for commuting mappings

Fixed point theorems for commuting mappings in CAT(0) spaces and $\mathbb{R}$-trees were given in [6, 19] where the multivalued mapping was assumed to be nonexpansive. Here we continue the development for $\mathbb{R}$-trees with results for domains that are geodesically bounded, and for multivalued mappings that are upper semicontinuous or nonexpansive.

**Theorem 4.1.** *Let $(M,d)$ be a complete $\mathbb{R}$-tree, $X$ a closed convex subset of $M$ which is geodesically bounded, $f : X \to X$ a nonexpansive mapping and $F : X \to 2^X$ an upper semicontinuous mapping with values that are nonempty closed convex subsets of $M$. If $f$ and $F$ commute, then $f$ and $F$ have a common fixed point in $X$.*

*Proof.* Since $f$ is nonexpansive, the set of fixed points $Fix(f)$ of $f$ is nonempty, closed and convex as in [9]. Let $x \in Fix(f)$. Then there is a unique $y \in F(x)$ such that $d(y,x) = d(x,F(x))$. Since

$$d(f(y),x) = d(f(y),f(x)) \leq d(y,x) = d(x,F(x))$$

and $f$ and $F$ commute, we have

$$f(y) \in f(F(x)) \subset F(f(x)) = F(x).$$

This implies that $f(y) = y$ and so

$$F(x) \cap Fix(f) \neq \emptyset \;\text{ for }\; x \in Fix(f).$$

Now by Theorem 3.4, there is a $z \in Fix(f)$ such that $z \in F(z)$. □

**Corollary 4.2.** *Let $(M, d)$ be a complete $\mathbb{R}$-tree, $X$ a closed convex subset of $M$ which is geodesically bounded, $f : X \to X$ a nonexpansive mapping and $F : X \to 2^X$ a nonexpansive mapping with values that are nonempty closed convex subsets of $M$. If $f$ and $F$ commute, then $f$ and $F$ have a common fixed point in $X$.*

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