Journal of Nonlinear and Convex Analysis
Volume 16, Number 2, 2015, 339–352

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL
© Copyright 2015

FISHER QUASI-CONTRACTION OF PEROV TYPE

MARIJA CVETKOVIĆ AND VLADIMIR RAKOČEVIĆ

Abstract. Perov [27] used the concept of vector valued metric space, and obtained a Banach type fixed point theorem on such a complete generalized metric space. In this article we study fixed point results for the new extensions of Fisher's quasi contraction to cone metric space, and we give some generalized versions of the fixed point theorem of Perov. As corollaries, we generalized some results of Zima [36] and Borkowski, Bugajewski and Zima [4] for a Banach space with a non-normal cone. The theory is illustrated with some examples. It is worth mentioning that the main result in this paper could not be derived from Fisher's result by the scalarization method, and, hence, indeed improves many recent results in cone metric spaces.

1. Introduction

There exist many generalizations of the concept of metric spaces in the literature. Perov [27] used the concept of vector valued metric space, and obtained a Banach type fixed point theorem on such a complete generalized metric space. After that, fixed point results of Perov type in vector valued metric spaces were studied by many other authors (see e.g., [9, 13, 29, 30, 33] for some works in this line of research). Let us point out that Perov theorem and related results have many applications in coincidence problems, coupled fixed point problems and systems of semilinear differential inclusions.

Let (X, d) be a complete metric space. A map $T : X \mapsto X$ such that for some constant $\lambda \in (0, 1)$ and for every $x, y \in X$

$$d(Tx, Ty) \leq \lambda \cdot \max\Big\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\Big\}, \tag{1.1}$$

is called *quasi-contraction*. Let us remark that Ćirić [6] introduced and studied quasi-contraction as one of the most general contractive type map. The well known Ćirić's result is that quasi-contraction T possesses a unique fixed point.

Fisher [10] extended the definition of quasi-contraction to include all mappings T of a metric space X into itself such that for some constant $\lambda \in (0, 1)$, and for some fixed positive integers p and q,

$$d(T^p x, T^q y) \leq \lambda \cdot \max \Big\{d(T^r x, T^s y), d(T^r x, T^{r'} x), d(T^s y, T^{s'} y) : 0 \leq r, r' \leq p \quad \text{and} \quad 0 \leq s, s' \leq q\Big\}.$$

The authors are supported By Grant No. 174025 of the Ministry of Science, Technology and Development, Republic of Serbia.

2010 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. Perov's theorem, fixed point, cone metric space, quasi-contraction, Fisher's quasi-contraction.

It is proved that if T is a continuous mapping on the complete metric space Xinto itself satisfying the inequality, then T has a unique fixed point. Further, it is shown that the condition that T be continuous is unnecessary if q (or p) $= 1$.

L. G. Huang and X. Zhang [15] (see also [35]) used the concept of cone metric spaces as a generalization of metric spaces. They have replaced the real numbers (as the co-domain of a "metric") by an ordered Banach space. The authors described the convergence in cone metric spaces and introduced their completeness. Then they proved some fixed point theorems for contractive mappings on cone metric spaces. Recently, in [1,18] and [31] some common fixed point theorems were proved for maps on cone metric spaces. However, in [1, 15, 18, 19] and [34] the authors usually use the normality property of cones in their results.

D. Ilić and V. Rakočević [19] introduced quasi-contractive mappings in cone metric spaces, and proved a fixed point theorem for such mappings, when the underlying cone is normal. Z. Kadelburg, S. Radenović and V. Rakočević [22], without using the normality condition, proved related results, but only in the case when contractive constant $\lambda \in (0, 1/2)$. Later, Haghi, Rezapour and Shahzad [32] and also Gajić and V. Rakočević [12] proved same results without the additional assumption and for $\lambda \in (0, 1)$ by providing a new technical proof; see ([11, 14, 16]) for the more related results.

In this article we study fixed point results for the new extensions of Fisher's quasi contraction to cone metric space, and we give some generalized versions of the fixed point theorem of Perov. As corollaries we have generalized some results of Zima [36] and Borkowski, Bugajewski and Zima [4] for a Banach space with a non-normal cone. The theory is illustrated with some examples. It is important to mention that the main result in this paper could not be derived from Fisher's result by the scalarization method, and indeed improves many recent results in cone metric spaces.

Consistent with [15] (see, e.g., [1–3,7,12,17,21,31,34] for more details and recent results), the following definitions and results will be needed in the sequel.

Let E be a real Banach space. A subset P of E is called a cone if:

(i) P is closed, nonempty and $P \neq \{0\}$;
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;
(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subseteq E$, we define the partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$ (interior of P).

The cone P in a real Banach space E is called normal if

$$\inf\{\|x + y\| : x, y \in P \text{ and } \|x\| = \|y\| = 1\} > 0, \tag{1.2}$$

or, equivalently, if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K \|y\| . \tag{1.3}$$

The least positive number satisfying (2.2) is called the normal constant of P. It is clear that $K \geq 1$.

Definition 1.1. Let X be a nonempty set, and let P be a cone on a real ordered Banach space E. Suppose that the mapping $d : X \times X \mapsto E$ satisfies:

(d1) $0 \leq d(x,y)$ for all $x,y \in X$ and $d(x,y)=0$ if and only if $x=y$;
(d2) $d(x,y)=d(y,x)$ for all $x,y \in X$;
(d3) $d(x,y) \leq d(x,z)+d(z,y)$ for all $x,y,z \in X$.
Then d is called a cone metric on X and (X,d) is called a cone metric space.

It is known that the class of cone metric spaces is bigger than the class of metric spaces.

Example 1.2. Let $E=l^1$, $P=\Big\{\{x_n\}_{n\geq 1} \in E : x_n \geq 0, \text{ for all } n\Big\}$, (X,ρ) be a metric space and $d : X \times X \mapsto E$ defined by $d(x,y)=\left\{\frac{\rho(x,y)}{2^n}\right\}_{n\geq 1}$. Then (X,d) is a cone metric space.

Example 1.3. Let $X=\mathbb{R}$, $E=\mathbb{R}^n$ and $P=\{(x_1,\dots,x_n) \in \mathbb{R}^n : x_i \geq 0\}$. It is easy to see that $d : X \times X \mapsto E$ defined by $d(x,y)=(|x-y|, k_1|x-y|, \dots, k_{n-1}|x-y|)$ is a cone metric on X, where $k_i \geq 0$ for all $i \in \{1,\dots,n-1\}$.

Example 1.4 ([7])**.** Let $E=C^1[0,1]$ with $\|x\|=\|x\|_\infty+\|x'\|_\infty$ on $P=\{x \in E : x(t) \geq 0, t \in [0,1]\}$. This cone is not normal. Consider, for example,

$$x_n(t)=\frac{1-\sin nt}{n+2} \quad \text{and} \quad y_n(t)=\frac{1+\sin nt}{n+2}.$$

Since, $\|x_n\|=\|y_n\|=1$ and $\|x_n+y_n\|=\frac{2}{n+2} \to 0$, it follows by (1.1) that P is non-normal.

Let X be a nonempty set and $n \in \mathbb{N}$.

Definition 1.5. A mapping $d : X \times X \mapsto \mathbb{R}^n$ is called a *vector-valued metric* on X if the following statements are satisfied for all $x,y,z \in X$.

(d1) $d(x,y) \geq 0_n$ and $d(x,y)=0_n$ if and only if $x=y$ where $0_n=(0,\dots,0) \in \mathbb{R}^n$;
(d2) $d(x,y)=d(y,x)$;
(d3) $d(x,y) \leq d(x,z)+d(z,y)$.

If $x=(x_1,\dots,x_n), y=(y_1,\dots,y_n) \in \mathbb{R}^n$, then $x \leq y$ means that $x_i \leq y_i$, $i=1,\dots,n$. This partial order determines normal cone $P=\{x=(x_1,\dots,x_n) \in \mathbb{R}^n : x_i \geq 0, i=1,2,\dots,n\}$ on $\mathbb{R}^n$, with the normal constant $K=1$. A nonempty set X with a vector-valued metric d is called a *generalized metric space*.

Throughout this paper we denote by $\mathcal{M}_{n,n}$ the set of all $n \times n$ matrices, by $\mathcal{M}_{n,n}(\mathbb{R}^+)$ the set of all $n \times n$ matrices with nonnegative elements. It is well known that if $A \in \mathcal{M}_{n,n}$, then $A(P) \subseteq P$ if and only if $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$. We write Θ for the zero $n \times n$ matrix and I_n for the identity $n \times n$ matrix. For the sake of simplicity we will identify row and column vector in $\mathbb{R}^m$.

A matrix $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ is said to be convergent to zero if $A^n \to \Theta$ as $n \to \infty$.

Theorem 1.6 (Perov [27,28])**.** *Let (X,d) be a complete generalized metric space, $f : X \mapsto X$ and $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ be a matrix convergent to zero, such that*

$$d(f(x),f(y)) \leq A(d(x,y)), \quad x,y \in X.$$

Then:

(i) *f has a unique fixed point $x^* \in X$;*

(ii) *the sequence of successive approximations* $x_n = f(x_{n-1}), n \in \mathbb{N}$ *converges to* x^* *for all* $x_0 \in X$;

(iii) $d(x_n, x^*) \leq A^n(I_n - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}$;

(iv) *if* $g : X \mapsto X$ *satisfies the condition* $d(f(x), g(x)) \leq c$ *for all* $x \in X$ *and some* $c \in \mathbb{R}^n$, *then by considering the sequence* $y_n = g^n(x_0), n \in \mathbb{N}$, *one has*

$$d(y_n, x^*) \leq (I_n - A)^{-1}(c) + A^n(I_n - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}.$$

For completeness of the paper and convenience of the reader, in Section 2 we collect some basic definitions and facts which are applied in subsequent sections.

2. Preliminaries

In the following we suppose that E is a Banach space, P is a cone in E with $\text{int}P \neq \emptyset$ and $\leq$ is the partial order on E with respect to P.

Let $\{x_n\}$ be a sequence in X, and $x \in X$. If for every c in E with $0 \ll c$, there is n_0 such that for all $n > n_0$, $d(x_n, x) \ll c$, then it is said that $\{x_n\}$ converges to x, and we denote this by $\lim_{n\to\infty} x_n = x$, or $x_n \to x$, $n \to \infty$. If for every c in E with $0 \ll c$, there is n_0 such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Let us recall [15] that if P is a normal cone, even in the case $\text{int}P = \emptyset$, then $x_n \in X$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$, $n \to \infty$. Further, $x_n \in X$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$, $n, m \to \infty$.

Let (X, d) be a cone metric space. Then the following properties are often used (particulary when dealing with cone metric spaces in which the cone need not be normal):

(p_1) If $u \leq v$ and $v \ll w$ then $u \ll w$;

(p_2) If $0 \leq u \ll c$ for each $c \in \text{int}\,P$ then $u = 0$;

(p_3) If $a \leq b + c$ for each $c \in \text{int}\,P$ then $a \leq b$;

(p_4) If $0 \leq x \leq y$, and $a \geq 0$, then $0 \leq ax \leq ay$;

(p_5) If $0 \leq x_n \leq y_n$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, then $0 \leq x \leq y$;

(p_6) If $0 \leq d(x_n, x) \leq b_n$ and $b_n \to 0$, then $x_n \to x$;

(p_7) If E is a real Banach space with a cone P and if $a \leq \lambda a$, where $a \in P$ and $0 < \lambda < 1$, then $a = 0$;

(p_8) If $c \in \text{int}\,P$, $0 \leq a_n$ and $a_n \to 0$, then there exists n_0 such that for all $n > n_0$ we have $a_n \ll c$.

From (p_8) it follows that the sequence $\{x_n\}$ converges to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$ and $\{x_n\}$ is a Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$. In the situation with a non-normal cone we have only one part of Lemmas 1 and 4 from [15]. Also, in this case the fact that $d(x_n, y_n) \to d(x, y)$ if $x_n \to x$ and $y_n \to y$ is not applicable.

We write $\mathscr{B}(E)$ for the set of all bounded linear operators on E and $\mathscr{L}(E)$ for the set of all linear operators on E. $\mathscr{B}(E)$ is a Banach algebra, and if $A \in \mathscr{B}(E)$ let

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \inf_n \|A^n\|^{1/n},$$

be the spectral radius of A. Let us remark that if $r(A) < 1$, then the series $\sum_{i=0}^{\infty} A^n$ is absolutely convergent, $I - A$ is invertible in $\mathscr{B}(E)$ and

$$\sum_{i=0}^{\infty} A^n = (I - A)^{-1}.$$

Furthermore, if $\|A\| < 1$, then $I - A$ is invertible and

$$\|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

3. Main results

In this section we prove our main results. We start with some auxiliary results.

Lemma 3.1. *Let (X, d) be a cone metric space. Suppose that $\{x_n\}$ is a sequence in X and that $\{b_n\}$ is a sequence in E and $b_n \to 0, n \to \infty$. If there exists $n_0 \in \mathbb{N}$ such that $0 \leq d(x_n, x_m) \leq b_n$ for each $n \geq n_0$ and each $m \geq n_0$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. For every $c \gg 0$ there exists $n_1 \in \mathbb{N}$ such that $b_n \ll c, n > n_1$. It follows that $0 \leq d(x_n, x_m) \ll c, m > n > \max\{n_0, n_1\}$, i.e., $\{x_n\}$ is a Cauchy sequence. □

Lemma 3.2. *Let E be Banach space, $P \subseteq E$ cone in E and $A : E \mapsto E$ a linear operator. The following conditions are equivalent:*

(i) *A is increasing, i.e., $x \leq y$ implies $A(x) \leq A(y)$.*

(ii) *A is positive, i.e., $A(P) \subset P$.*

Proof. If A is monotonically increasing and $p \in P$, then, by definitions, it follows $p \geq 0$ and $A(p) \geq A(0) = 0$. Thus, $A(p) \in P$, and $A(P) \subseteq P$.

To prove the other implication, let us assume that $A(P) \subseteq P$ and $x, y \in E$ are such that $x \leq y$. Now $y - x \in P$, and so $A(y - x) \in P$. Thus $A(x) \leq A(y)$. □

Definition 3.3. Let (X, d) be a cone metric space. A map $f : X \mapsto X$ such that for some $A \in \mathscr{B}(E)$, $r(A) < 1$ and for some fixed positive integers p and q, and for every $x, y \in X$, there exists

$$u \in \mathrm{F}_f^{p,q}(x, y) \equiv \Big\{ d(f^r x, f^s y), d(f^r x, f^{r'} x), d(f^s y, f^{s'} y) : 0 \leq r, r' \leq p \quad \text{and} \quad 0 \leq s, s' \leq q \Big\}.$$

such that

$$d(f^p x, f^q y) \leq A(u),$$

is called $(p, q)-$quasi-contraction (Fisher's quasi-contraction, F quasi-contraction) of Perov type.

The results in the next theorem are applied to the cone metric spaces in the case when cone is not necessary normal, and Banach space should not be finite dimensional. This extends the results of Perov for matrices, and also as a corollary we generalize Theorem 1 of Zima [36].

Theorem 3.4. *Let (X,d) be a complete cone metric space and P be a cone with $int P \neq \emptyset$. Suppose the mapping $f : X \mapsto X$ is a $(p,q)-$quasi-contraction of Perov type, $A(P) \subseteq P$ and let f be continuous. Then f has a unique fixed point in X and for any $x \in X$, the iterative sequence $\{f^n x\}$ converges to the fixed point.*

Proof. We will assume that $p \geq q$.

Let $x \in X$ be arbitrary. We shall show that $\{f^n x\}$ is a Cauchy sequence. First we prove that

$$d(f^n x, f^p x) \leq (I-A)^{-1}A(\omega(x)), \qquad n \geq p, \tag{3.1}$$

where

$$\omega(x) = \sum_{0 \leq i < p} d(f^i x, f^p x).$$

Clearly, (3.1) is true for $n = p$. Suppose that (3.1) is true for each $m \leq n_0 - 1$, and let us prove (3.1) for $m = n_0 \geq p+1$.

Because f is $(p,q)-$quasi-contraction, there exist $i, j \in \mathbb{N}$, such that

$$d(f^{n_0} x, f^p x) \leq A(d(f^i x, f^j x)). \tag{3.2}$$

(1) If $i, j \leq p$, then

$$\begin{aligned} d(f^{n_0} x, f^p x) &\leq A(d(f^i x, f^p x) + d(f^p x, f^j x)) \\ &\leq A(\omega(x)) \leq (I-A)^{-1}A(\omega(x)). \end{aligned}$$

Remark that we have used that $i \neq j$ in this inference, but if $i = j$ (3.1) evidently holds.

(2) If $p < i < n_0$, $j \leq p$ then (3.1) and (3.2) imply

$$\begin{aligned} d(f^{n_0} x, f^p x) &\leq A(d(f^i x, f^p x)) + A(d(f^p x, f^j x)) \\ &\leq A(I-A)^{-1}A(\omega(x)) + A(\omega(x)) \\ &= (I-A)^{-1}A(\omega(x)). \end{aligned}$$

(3) For $p < i < n_0$, $p < j < n_0$, we have

$$d(f^{n_0} x, f^p x) \leq A^k(d(f^{i_0} x, f^{j_0} x)),$$

where $i_0 < p$ or $j_0 < p$ and $1 < k$.

Assume that at least $i_0 < p$.

$$\begin{aligned} d(f^{n_0} x, f^p x) &\leq A^k(d(f^{i_0} x, f^p x)) + A^k(d(f^p x, f^{j_0} x)) \\ &\leq A^k(\omega(x)) + A^k(I-A)^{-1}A(\omega(x)) \\ &\leq (I-A)^{-1}A(\omega(x)), \end{aligned}$$

since $j_0 \leq j < n_0$, so the inequality (3.1) holds in this case.

(4) For $i = n_0$, $j \leq p$, the triangle inequality, $A(P) \subseteq P$ and (3.2) imply

$$\begin{aligned} d(f^{n_0} x, f^p x) &\leq A(d(f^{n_0} x, f^p x)) + A(d(f^p x, f^j x)) \\ &\leq A(d(f^{n_0} x, f^p x)) + A(\omega(x)), \end{aligned}$$

so (3.1) is satisfied.

(5) Finally, consider $i = n_0$ and $p < j \leq n_0$.

If $j = n_0$, it follows $d(f^{n_0} x, f^p x) \leq A(0)$ i.e. $d(f^{n_0} x, f^p x) = 0$.

Otherwise,

$$d(f^{n_0}x, f^px) \leq A(d(f^jx, f^{n_0}x)), \tag{3.3}$$

and there exist $i_0 \leq j_0 \leq n_0$, $i_0 < p$ and some $1 < k_0$ such that

$$d(f^jx, f^{n_0}x) \leq A^{k_0}(d(f^{i_0}x, f^{j_0}x)).$$

If $j_0 \leq p$, then (3.1) follows by the last inequality and (3.3). Notice that if $j_0 < n_0$, then

$$\begin{aligned} d(f^{n_0}x, f^px) &\leq A^{1+k_0}(d(f^{i_0}x, f^{j_0}x)) \\ &\leq A^{1+k_0}(d(f^{i_0}x, f^px)) + A^{1+k_0}(d(f^px, f^{j_0}x)) \\ &\leq A^{1+k_0}(\omega(x)) + A^{1+k_0}(I-A)^{-1}A(\omega(x)) \\ &= A^{1+k_0}(I-A)^{-1}(I-A+A)(\omega(x)) \\ &\leq (I-A)^{-1}A(\omega(x)). \end{aligned} \tag{3.4}$$

But if $j_0 = n_0$, then

$$d(f^{n_0}x, f^px) \leq A^{1+k_0}(d(f^{i_0}x, f^px)) + A^{1+k_0}(d(f^px, f^{n_0}x)). \tag{3.5}$$

Then, for some $k_1 \geq 1$ and $i_1 \leq j_1 \leq n_0$, $i_1 < p$, $d(f^px, f^{n_0}x) \leq A^{k_1}(d(f^{i_1}x, f^{j_1}x))$, so by (3.5) we get

$$d(f^{n_0}x, f^px) \leq A^{1+k_0}(d(f^{i_0}x, f^px)) + A^{1+k_0+k_1}(d(f^{i_1}x, f^{j_1}x)). \tag{3.6}$$

Again, if $j_1 < n_0$, as in (3.4), we have (3.1). Otherwise,

$$\begin{aligned} d(f^{n_0}x, f^px) \leq\ & A^{1+k_0}(d(f^{i_0}x, f^px)) + A^{1+k_0+k_1}(d(f^{i_1}x, f^px)) \\ & + A^{1+k_0+2k_1}(d(f^{i_1}x, f^{n_0}x)). \end{aligned}$$

Hence, for arbitrary $n \in \mathbb{N}$,

$$\begin{aligned} d(f^{n_0}x, f^px) &\leq A^{1+k_0}(d(f^{i_0}x, f^px)) \\ &\quad + \sum_{m=1}^{n-1} A^{1+k_0+mk_1}(d(f^{i_1}x, f^px)) + A^{1+k_0+nk_1}(d(f^{i_1}x, f^{n_0}x)) \\ &\leq \sum_{m=0}^{n-1} A^{1+k_0+mk_1}A(\omega(x)) + A^{1+k_0+nk_1}(d(f^{i_1}x, f^{n_0}x)) \\ &\leq (I-A)^{-1}A^{1+k_0}(\omega(x)) + A^{1+k_0+nk_1}(d(f^{i_1}x, f^{n_0}x)) \\ &\leq (I-A)^{-1}A(\omega(x)) + A^{1+k_0+nk_1}(d(f^{i_1}x, f^{n_0}x)). \end{aligned}$$

However, $A^{1+k_0+nk_1}(d(f^{i_1}x, f^{n_0}x)) \to 0$, $n \to \infty$. For each $c \gg 0$ there exists $n_c \in \mathbb{N}$ such that $A^{1+k_0+nk_1}(d(f^{i_1}x, f^{n_0}x)) \ll c$ for $n > n_c$, so

$$d(f^{n_0}x, f^px) \leq (I-A)^{-1}A(\omega(x)) + c, \quad c \gg 0,$$

i.e. $d(f^{n_0}x, f^px) \leq (I-A)^{-1}A(\omega(x))$.

Thus, by induction, we have obtained (3.1) for every $n \in \mathbb{N}$. Now, let us prove that, for each n,

$$d(f^nx, f^jx) \leq (I-A)^{-1}(\omega(x)), \quad j = 0, 1, 2, \ldots, p. \tag{3.7}$$

This follows by (3.1), since

$$\begin{aligned} d(f^n x, f^j x) &\leq d(f^n x, f^p x) + d(f^p x, f^j x) \\ &\leq (I-A)^{-1} A(\omega(x)) + \omega(x) \\ &= (I-A)^{-1}(\omega(x)). \end{aligned}$$

Because f is a $(p,q)-$quasi-contraction, and according to the (3.7) we get that for $n > m \geq p$, $m = kp + r$, $0 \leq r < p$, $k \geq 1$

$$d(f^n x, f^m x) \leq A^k(d(f^i x, f^j x)) \leq A^k(I-A)^{-1}(\omega(x)),$$

where $0 \leq i \leq j \leq n$ and $i \leq p$.

Remark that $A^k(I-A)^{-1}(\omega(x)) \to 0$, $k \to \infty\,(m \to \infty)$, implies that $\{f^n x\}$ is a Cauchy sequence in X. We will prove that $z = \lim\limits_{n\to\infty} f^n x \in X$ is a unique fixed point of f.

Since f is a continuous, it follows that $fz = z$. The uniqueness of z follows from the definition of $(p,q)-$quasi-contraction. □

In the particular case of Theorem 2 when $q = 1$ (or $p = 1$), the condition that f be continuous is unnecessary (see [10]). We then have

Theorem 3.5. *Let (X,d) be a complete cone metric space and P be a cone,* $\operatorname{int}(P) \neq \emptyset$. *Suppose the mapping $f : X \mapsto X$ is a $(p,1)-$quasi-contraction of Perov type, $A(P) \subseteq P$. Then f has a unique fixed point in X and for any $x \in X$, the iterative sequence $\{f^n x\}$ converges to the fixed point.*

Proof. Let x be an arbitrary point in X. Then, as in the proof of Theorem 3.4, the sequence $\{f^n x\}$ is a Cauchy sequence in the complete cone metric space X and so has a limit z in X. For $n > p$, we now have

$$\begin{aligned} d(z, fz) &\leq d(z, f^n x) + d(f^n x, fz) \\ &= d(z, f^n x) + d(f^p f^{n-p} x, fz) \\ &\leq d(z, f^n x) + A(u_n), \end{aligned}$$

where

$$u_n \in \left\{ d(f^r f^{n-p}x, fz), d(f^r f^{n-p}x, z), d(f^r f^{n-p}x, f^{r'} f^{n-p}x), d(z, fz) : 0 \leq r, r' \leq p \right\}.$$

But,

$$d(f^r f^{n-p}x, fz) \leq d(f^r f^{n-p}x, z) + d(z, fz),$$

so, since $\lim\limits_{n\to\infty} f^n x = z$, for each $c \gg 0$ we may choose n_0 for which $d(f^n x, z), d(f^n x, f^m x) \ll c$, $n, m \geq n_0$. Choose $n > n_0 + p$, then

$$d(z, fz) \quad \leq \quad c + A(d(z, fz)) + A(c) \quad \text{for any } c \gg 0.$$

By observing $c = \frac{c}{n}$, $n \in \mathbb{N}$, we get $d(z, fz) \leq A(d(z, fz))$ i.e. $fz = z$ because $(I-A)^{-1}(P) \subseteq P$. Uniqueness obviously follows. □

When $p = q = 1$, we have the following corollary:

Corollary 3.6. *Let (X,d) be a complete cone metric space and P be a cone with $\operatorname{int} P \neq \emptyset$. Suppose the mapping $f : X \mapsto X$ is a $(1,1)-$quasi-contraction of Perov type (Ćirić's quasi-contraction), that is for some constant $A \in \mathscr{B}(E)$, $r(A) < 1$ and $A(P) \subseteq P$ and for every $x, y \in X$, there exists*

$$u \in C(f,x,y) \equiv \{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\},$$

such that

$$d(fx,fy) \leq A(u).$$

Then f has a unique fixed point in X and for any $x \in X$, the iterative sequence $\{f^n x\}$ converges to the fixed point.

Remark 3.7. Let us remark that Du [8] has investigated the equivalence of vectorial versions of fixed point theorems in generalized cone metric spaces and scalar versions of fixed point theorems in (general) metric spaces (in usual sense). He has shown that the Banach contraction principles in general metric spaces and in TVS-cone metric spaces are equivalent. His theorems also extend some results in Huang and Zhang [15], Rezapour and Hamlbarani [31] and others.

Du [8] has used the nonlinear scalarization function ξ_e and function d_ξ as follows: Let $d_\xi = \xi_e \circ d$, where ξ_e is defined by

$$\xi_e(u) = \inf\{r \in \mathbb{R} : u \in re - P\},$$

for each $u \in E$ for some $e \in \operatorname{int}P$. Then d_ξ is a metric on X by Theorem 2.1 of [8]. If T is a Fisher's quasi-contraction with $\lambda \in (0,1)$, and for some fixed positive integers p and q, then applying Lemma 1.1 of [8], we have

$$(3.8)\quad d_\xi(T^p x, T^q y) \leq \lambda \cdot \max \Big\{ d_\xi(T^r x, T^s y), d_\xi(T^r x, T^{r'} x), d_\xi(T^s y, T^{s'} y) : 0 \leq r, r' \leq p \quad \text{and} \quad 0 \leq s, s' \leq q \Big\}.$$

Hence, Theorem 4.2 of [11] directly follows from Fisher's result by Theorem 2 of [10]. However, if T is a Fisher's quasi - contraction restricted with a linear bounded mapping, we cannot conclude that there exists some $\lambda \in (0,1)$ such that (3.8) is satisfied, and so Theorem 3.4 could not be derived from Fisher's result. Therefore, Theorem 3.4 indeed improves the corresponding result of [10]. Similar observations are valid for the Ćirić's quasi-contraction of Perov type and for the Banach's contraction of Perov type [5]. For some more recent results see [20, 23, 24].

Remark 3.8. Let us remark that the initial assumption $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$, in Perov theorem, is unnecessary. Based on the previous comments, we obtain the next result, were we do not suppose that $A(P) \subseteq P$.

Theorem 3.9. *Let (X,d) be a complete cone metric space and P be a normal cone with a normal constant K. Let the mapping $f : X \mapsto X$ be a continuous $(p,q)-$quasi-contraction of Perov type, $K\|A\| < 1$ Then f has a unique fixed point in X and for any $x \in X$, the iterative sequence $\{f^n x\}$ converges to the fixed point.*

Proof. Without loss of generality, we may assume that $p \geq q$. Let $x \in X$ be arbitrary, and

$$\eta(x) = \sum_{0\leq i<j\leq p} \|d(f^i x, f^j x)\|.$$

We shall prove that

$$\|d(f^n x, f^p x)\| \leq \frac{K\|A\|}{1-K\|A\|} \cdot \eta(x), \ n \geq p. \tag{3.9}$$

Obviously, the inequality (3.9) holds for $n = p$. Suppose that (3.9) holds for every $n \leq n_0 - 1$, $n_0 > p$. If $n = n_0$, then

$$d(f^{n_0} x, f^p x) \leq A(d(f^i x, f^j x)), \tag{3.10}$$

where $i, j \in \{0, \dots, n_0\}$. Few different cases will be discussed.

(1) If $0 \leq i, j \leq p$, then

$$\|d(f^{n_0} x, f^p x)\| \leq K\|A\| \cdot \eta(x) \leq \frac{K\|A\|}{1-K\|A\|} \cdot \eta(x).$$

(2) If $p < i < n_0$ and $j \leq p$ (analogously $i \leq p$, $p < j < n_0$), then the induction hypothesis and the triangle inequality imply

$$\begin{aligned} \|d(f^{n_0} x, f^p x)\| &\leq \frac{K^2\|A\|^2}{1-K\|A\|} \cdot \eta(x) + K\|A\| \cdot \eta(x) \\ &= \frac{K\|A\|}{1-K\|A\|} \cdot \eta(x). \end{aligned}$$

(3) In this case, consider $p < i, j < n_0$. There exist $i_0, j_0 < n_0$, $i_0 < p$ such that

$$\|d(f^i x, f^j x)\| \leq (K\|A\|)^k \|d(f^{i_0} x, f^{j_0} x)\|,$$

for some $k \geq 1$. The inequality (3.9) is satisfied if $i_0 = j_0$. If $j_0 \leq p$, then

$$\|d(f^{n_0} x, f^p x)\| \leq (K\|A\|)^{k+1} \cdot \eta(x) \leq \frac{K\|A\|}{1-K\|A\|} \cdot \eta(x).$$

Otherwise,

$$\begin{aligned} \|d(f^{n_0} x, f^p x)\| &\leq (K\|A\|)^{k+1} \big(\|d(f^{i_0} x, f^p x)\| + \|d(f^p x, f^{j_0} x)\|\big) \\ &\leq (K\|A\|)^{k+1} \big(\eta(x) + \frac{K\|A\|}{1-K\|A\|} \cdot \eta(x)\big) \\ &\leq \frac{K\|A\|}{1-K\|A\|} \cdot \eta(x), \end{aligned}$$

because $K\|A\| < 1$.

(4) Assume that $i = n_0$. By

$$\|d(f^{n_0} x, f^p x)\| \leq K\|A\| \|d(f^{n_0} x, f^p x)\| + K\|A\| \|d(f^p x, f^j x)\|,$$

if $j \leq p$, (3.9) evidently follows.

Otherwise, if $p < j < n_0$, there exist some $i_0 \leq j_0 \leq n_0$, $i_0 < p$ and $k_0 \geq 1$ for which

$$\|d(f^{n_0} x, f^j x)\| \leq (K\|A\|)^{k_0} \|d(f^{i_0} x, f^{j_0} x)\|.$$

Evidently, for $j_0 \leq p$, (3.9) is satisfied. Similarly as previously shown in the proof of Theorem 3.4, (3.9) holds if $j_0 < n_0$ by induction hypothesis.

If $j_0 = n_0$, then again as in the proof of Theorem 3.5, there are some $i_1 \leq j_1 \leq n_0$, $i_1 < p$ and $k_1 \geq 1$ that satisfy $\|d(f^p x, f^{n_0} x)\| \leq (K\|A\|)^{k_1} \|d(f^{i_1} x, f^{j_1} x)\|$. Then

$$\|d(f^{n_0} x, f^p x)\| \leq (K\|A\|)^{1+k_0} \|d(f^{i_0} x, f^p x)\| + (K\|A\|)^{1+k_0+k_1} \|d(f^{i_1} x, f^{j_1} x)\|.$$

Again, if $j_1 < n_0$, (3.9) easily follows. If $j_1 = n_0$, then after $m-1$ more steps, we get

$$\begin{aligned} \|d(f^{n_0} x . f^p x)\| \leq\ & (K\|A\|)^{1+k_0} \|d(f^{i_0} x, f^p x)\| \\ & + \sum_{l=1}^{i=m-1} (K\|A\|)^{1+k_0+lk_1} \|d(f^{i_1} x, f^p x)\| \\ & + (K\|A\|)^{1+k_0+mk_1} \|d(f^p x, f^{n_0} x)\|. \end{aligned}$$

Analogously as in the proof of Theorem 3.4, the inequality (3.9) is satisfied in this case.

Hence, (1)-(5) imply that the inequality (3.9) holds for any $n \geq p$.

Let $n \geq m > 2p$, $m = (k+1)p + r$, $k \in \mathbb{N}, 0 \leq r < p$. To estimate $d(f^n x, f^m x)$, observe $p \leq i_{n,m} \leq j_{n,m} \leq n$ and $k \geq 1$ for which

$$\|d(f^n x, f^m x)\| \leq (K\|A\|)^k \|d(f^{i_{n,m}} x, f^{j_{n,m}} x)\|.$$

Then

$$\|d(f^n x, f^m x)\| \leq \frac{2(K\|A\|)^{k+1}}{1 - K\|A\|} \cdot \eta(x),$$

by (3.9) and triangle inequality. So, $\{f^n x\}$ is a Cauchy sequence in a complete cone metric space X, thus $\lim_{n\to\infty} f^n x = z$ for some $z \in X$. Since f is a continuous, $fz = z$. Obviously, z is a unique fixed point of f in X because $K\|A\| < 1$. □

Theorem 3.10. *Let (X, d) be a complete cone metric space and P be a normal cone with a normal constant K. Suppose that the mapping $f : X \mapsto X$ is a $(p, 1)-$quasi-contraction of Perov type, $K\|A\| < 1$. Then f has a unique fixed point in X and for any $x \in X$, the iterative sequence $\{f^n x\}$ converges to the fixed point.*

Proof. Let x be an arbitrary point in X. Then, as in the proof of Theorem 3.9, the sequence $\{f^n x\}$ is a Cauchy sequence in the complete cone metric space X and so has a limit z in X. For $n > p$, we now have $d(f^n x, f(z)) \leq A(u_n)$, where

$$u_n \in \left\{ d(f^r f^{n-p} x, fz), d(f^r f^{n-p} x, z), d(f^r f^{n-p} x, f^{r'} f^{n-p} x), d(z, fz) : 0 \leq r, r' \leq p \right\}.$$

But, recall that $\lim_{n\to\infty} d(f^n x, z) = 0$ and $\lim_{n,m\to\infty} d(f^n x, f^m x) = 0$. Since

$$d(z, fz) = \lim_{n\to\infty} d(f^n x, fz) \leq A(d(z, fz)),$$

and P is a normal cone, from the norm inequality $\|d(z, fz)\| \leq K\|A\|\|d(z, fz)\|$, we get $fz = z$. Uniqueness obviously follows. □

When $p = q = 1$, we have the following corollary:

Corollary 3.11. *Let (X, d) be a complete cone metric space, P a normal cone with a normal constant K. If the mapping $f : X \mapsto X$ is a $(1,1)-$quasi-contraction of Perov type (Ćirić's quasi-contraction), that is for some operator $A \in \mathscr{B}(E)$, $K\|A\| < 1$ and for every $x, y \in X$, there exists*

$$u \in C(f, x, y) \equiv \{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\},$$

such that

$$d(fx, fy) \leq A(u),$$

then f has a unique fixed point in X and for any $x \in X$, the iterative sequence $\{f^n x\}$ converges to the fixed point of f.

Example 3.12. Let $E = C_R([0,1], \|\cdot\|_\infty)$ and $P = \{f \in E : f(t) \geq 0\}$. (X, ρ) a metric space and $d : X \times X \mapsto E$ defined by $d(x, y) = \rho(x, y)\varphi$, where $\varphi : [0,1] \mapsto R^+$ is continuous. Then (X, d) is a normal cone metric space and the normal constant of P is equal to $K = 1$.

To apply our results let us consider the solution of the equation $(I - Q)x = b$, where $b \in E$ is given, $I, Q \in \mathscr{B}(E)$ and $\|Q^2 - \alpha Q\| < 1 - \alpha$, $0 < \alpha < 1$ (see Theorem 3 of [33]). Let us take $X = E$, $\rho(x, y) = \|x - y\|$, and define $T : X \mapsto X$ by $T(x) = b + Q(x)$. Now, it is easy to see that T is continuous. If $x_n, x_0 \in X$ and $x_n \to x_0$, that is $d(x_n, x_0) \to 0$, then

$$\|d(T(x_n), T(x_0))\| = \|d(Q(x_n), Q(x_0))\| \leq \|Q\| \cdot \|d(x_n, x_0)\| \to 0.$$

Let us remark that $T(T(x)) = b + Q(b) + Q^2(x)$, so $T^2(x) - T^2(y) = Q^2(x - y)$. Thus

$$\begin{aligned} T^2(x) - T^2(y) &= Q^2(x - y) \\ &= (Q^2 - \alpha Q)(x - y) + (\alpha Q)(x - y) \\ &= (Q^2 - \alpha Q)(x - y) + (\alpha T(x - y). \end{aligned}$$

Hence,

$$\|T^2(x) - T^2(y)\| \leq \|(Q^2 - \alpha Q)\| \cdot \|x - y\| + |\alpha| \cdot \|T(x - y)\|,$$

and

$$\|T^2(x) - T^2(y)\| \leq (\|(Q^2 - \alpha Q)\| + \alpha) \max\{\|x - y\|, \|T(x - y)\|\}.$$

It follows

$$\|T^2(x) - T^2(y)\|\varphi \leq (\|(Q^2 - \alpha Q)\| + \alpha) \max\{\|x - y\|\varphi, \|T(x - y)\|\varphi\}.$$

Finally we have

$$d(T^2(x), T^2(y)) \leq A(u),$$

where $u \in \{d(x, y), d(Tx, Ty)\}$ and $A \in \mathscr{B}(E)$ is defined by $A(u) = (\|(Q^2 - \alpha Q)\| + \alpha)u$, $u \in E$.

Because $0 \leq \lambda = \|(Q^2 - \alpha Q)\| + \alpha < 1$we can apply Theorem 3.9 to conclude that there is a unique $x \in E$ such that $T(x) = x$, i.e., $(I - Q)x = b$. Moreover, for any $z \in X$, the iterative sequence $\{T^n z\}$ converges to the fixed point x of T.

Acknowledgments

The authors would like to thank the anonymous referees for the useful suggestions and remarks that contributed to the improvement of the manuscript.

References

[1] M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl. **341** (2008), 416–420.

[2] M. Abbas and B. E. Rhoades, *Fixed and periodic point results in cone metric spaces*, Applied Mathematics Letters **22** (2009), 511–515.

[3] C. D. Bari and P. Vetro, *φ-Pairs and common fixed points in cone metric space*, Rendiconti del Circolo Matematico di Palermo **57** (2008), 279–285.

[4] M. Borkowski, D. Bugajewski and M. Zima, *On some fixed-point theorems for generalized contractions and their perturbations*, J. Math. Anal. Appl. **367** (2010), 464–475.

[5] M. Cvetković and V. Rakočević, *Extensions of Perov theorem*, Carpathian J. Math., to appear.

[6] Lj. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Sc. **45** (1974), 267–273.

[7] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[8] W. Du, *A note on cone metric fixed point theory and its equivalence*, Nonlinear Anal. **72** (2010), 2259–2261.

[9] A. D. Filip, *Perov's fixed point theorem for multivalued mappings in generalized Kasahara spaces*, Stud. Univ. Babes-Bolyai Math. **56** (2011), 19–28.

[10] B. Fisher, *Quasi-contractions on metric spaces*, Proc. Amer. Math. Soc. **75** (1979), 321–325.

[11] L. Gajić, D. Ilić and V. Rakocević, *On Ćirić maps with a generalized contractive iterate at a point and Fisher's quasi-contractions in cone metric spaces*, Appl. Math. Comp. **216** (2010), 2240–2247.

[12] L. Gajić and V. Rakočević, *Quasi-contractions on a non-normal cone metric space*, Func. Anal. Appl. **46** (2012), 62–65.

[13] L. Guran, *Multivalued Perov-type theorems in generalized metric spaces*, Surveys Math. Appl. **4** (2009), 89–97.

[14] R. H. Haghi, Sh. Rezapour and N. Shahzad, *On fixed points of quasi-contraction type multifunctions*, Applied Math. Letters **25** (2012), 843–846.

[15] L. G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332** (2007), 1468–1476.

[16] H. Kunze, D. La Torre, F. Mendivil and E. R. Vrscay, *Generalized fractal transforms and self-similar objects in cone metric spaces*, Comp. Math. Appl. **64** (2012), 1761–1769.

[17] S. Janković, Z. Kadelburg nad S. Radenović, *On cone metric spaces: A survey*, Nonlinear Anal. **74** (2011), 2591–2601.

[18] D. Ilić and V. Rakočević, *Common fixed points for maps on cone metric space*, J. Math. Anal. Appl. **341** (2008), 876–882.

[19] D. Ilić and V. Rakočević, *Quasi-contraction on cone metric space*, Applied Math. Letters **22** (2009), 728–731.

[20] S. Jiang1 and Z. Li, *Extensions of Banach contraction principle to partial cone metric spaces over a non-normal solid cone*, Fixed Point Theory and Applications (2013), 2013:250.

[21] G. Jungck, S. Radenović, S. Radojević and V. Rakočević, *Common fixed point theorems for weakly compatible pairs on cone metric spaces*, Fixed Point Theory and Applications (2009), Article ID 643840, 13 pages

[22] Z. Kadelburg, S. Radenović and V. Rakočević, *Remarks on "Quasi-contraction on a cone metric space",* Applied Math. Letters **22** (2009), 1674–1679.

[23] H. Liu and S. Xu, *Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings*, Fixed Point Theory and Applications (2013), 2013:320.

[24] H. Liu and S. Xu, *Fixed point theorems of quasicontractions on cone metric spaces with Banach Aagebras*, Abst. Applied Anal. (2013), Article ID 187348, 5 pages

[25] S. B. Nadler Jr., *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969), 475–487.

[26] H. K. Pathak and N. Shahzad, *Fixed point results for generalized quasi-contraction mappings in abstract metric spaces*, Nonlinear Anal. **71** (2009), 6068–6076.

[27] A. I. Perov, *On Cauchy problem for a system of ordinary diferential equations* (in Russian), Priblizhen. Metody Reshen. Difer. Uravn. **2** (1964), 115–134.

[28] A. I. Perov and A. V. Kibenko, *On a certain general method for investigation of boundary value problems* (in Russian), Izv. Akad. Nauk SSSR Ser. Mat. **30** (1966), 249–264.

[29] A. Petruşel, *Vector-valued Metrics in Fixed Point Theory*, Babeş-Bolyai University Cluj-Napoca Faculty of Mathematics and Computer Science, May 2012.

[30] R. Precup, *The role of matrices that are convergent to zero in the study of semilinear operator systems*, Math. Comp. Model. **49** (2009), 703–708.

[31] Sh. Rezapour and R. Hamlbarani, *Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings"*, J. Math. Anal. Appl. **345** (2008), 719–724.

[32] Sh. Rezapour, R. H. Haghi nad N. Shahzad, *Some notes on fixed points of quasi-contraction maps*, Appl. Math. Lett. **23** (2010), 498–502.

[33] A. Szilárd, *A note on Perov's fixed point theorem*, Fixed Point Theory **4** (2003), 105–108.

[34] S. M. Veazpour and P. Raja, *Some Extensions of Banach's contraction principle in complete cone metric spaces,* Fixed Point Theory and Applications (2008), Article ID 768294, 11 pages

[35] P. P. Zabreiko, *K-metric and K-normed spaces: survey,* Collect. Math **48** (1997), 825–859.

[36] M. Zima, *A certain fixed point theorem and its applications to integral-functional equations*, Bull. Austral. Math. Soc. **46** (1992), 179–186.

Manuscript received November 15, 2013
revised December 3, 2014

M. Cvetković
University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niš, Serbia
E-mail address: `marijac@pmf.ni.ac.rs`

V. Rakočević
University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niš, Serbia
E-mail address: `vrakoc@ptt.rs`