Journal of Nonlinear and Convex Analysis
Volume 16, Number 2, 2015, 353–364

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

SUFFICIENT AND NECESSARY CONDITIONS FOR EQUILIBRIUM UNIQUENESS IN AGGREGATIVE GAMES

PIERRE VON MOUCHE AND TAKESHI YAMAZAKI

Abstract. Sufficient and necessary conditions for an aggregative game to have a unique Nash equilibrium are identified. In particular, an improvement of a result of Gaudet and Salant (1991) for Cournot oligopolies is obtained. The results are proved by exploiting the general relation between Nash equilibria and fixed points of the (virtual) aggregate cumulative best reply correspondence.

1. Introduction

The result of [3], restated in Section 2 as Theorem 2.1, is a milestone in oligopoly theory. It deals with sufficient and necessary conditions for a homogeneous Cournot oligopoly game to have a unique equilibrium and concerns a variant of a result in [4]. Contrary to the latter result, it considers the whole equilibrium set.

The proof in [3] also is much more elementary than the proof in [4] which deals with Cournot equilibria as solution of a complementarity problem to which differential topological fixed point index theory is applied. The more simple nature of the proof was realized by using the in oligopoly theory popular Selten-Szidarovzsky method (in particular the idea in [5]) which reformulates the n-dimensional equilibrium fixed point problem for the best reply correspondence as a 1-dimensional one. The 1-dimensionality makes that in general no deep theorems like Brouwer's fixed point theorem are needed for establishing equilibrium existence. This method in fact applies to correspondences with a special factorisation property and so its setting is not necessarily a game theoretic one (see [7] and references therein).

As in Theorem 2.1 conditional profit function are quasi-concave and there is a finite market satiation point, its part 1 on equilibrium existence also would follow from an equilibrium existence result à la Nikaido-Isoda. So the most interesting parts in Theorem 2.1 are parts 2 and 3 about equilibrium semi-uniqueness, i.e. that there exists at most one equilibrium.

A shortcoming of Theorem 2.1 is that assumption (2.2) below concerns a strong variant of the first Fisher-Hahn condition (see Remark 1 below) which is one of the reasons that Theorem 2.1 does not imply much simpler results that assume the usual condition. Another is that the price function can, due to the finite market satiation point, not be everywhere positive.

In our article, we provide by Theorem 4.1 a generalisation of Theorem 2.1 which can deal with aggregative games. The article is intended to be a mathematical rigorous variant of some results in chapter 3 in [8]. Also we will use the Selten-Szidarovzsky method. Theorem 4.1 not only implies Theorem 2.1 but also improves

2010 *Mathematics Subject Classification.* 47H10, 91A10.
Key words and phrases. Cumulative best reply correspondence, aggregative game, Fisher-Hahn conditions, fixed point, Cournot oligopoly, pseudo-concavity, equilibrium (semi-)uniqueness.

intrinsically on it by removing among other things the above mentioned shortcomings. Theorem 4.1 is divided into a part dealing with existence and in a part dealing with uniqueness. The smoothness conditions in the existence part are weaker than in the uniqueness part. Due to a result in the appendix, our existence proof could be provided without referring to the implicit function theorem, but the uniqueness proof does.

2. The result of Gaudet and Salant revisited

The setting for the result of [3] is a homogeneous Cournot oligopoly with firms $1,\dots,n$ with action set $\mathbb{R}_+$, strictly increasing cost functions $c_i : \mathbb{R}_+ \to \mathbb{R}$ and a decreasing price function $p : \mathbb{R}_+ \to \mathbb{R}$ such that for some positive real number v one has $p(y) > 0$ $(0 \leq y < v)$ and $p(y) = 0$ $(y \geq v)$. We refer to v as the *market satiation point* of p. So writing, for $\mathbf{a} \in \mathbb{R}^n$, $\underline{\mathbf{a}} := \sum_{l=1}^{n} a_l$, the profit function $u_i : \mathbb{R}^n_+ \to \mathbb{R}$ of firm i is given by

$$u_i(\mathbf{x}) = p(\underline{\mathbf{x}})x_i - c_i(x_i).$$

Concerning smoothness it is assumed that p is continuous and that the restriction $p\lceil[0, v]$ and every c_i is twice continuously differentiable. This defines a game in strategic form. Denote its set of (Cournot) equilibria by E. It is well-known (also see Theorem 4.1(1)) that $\underline{\mathbf{e}} < v$ for every $\mathbf{e} \in E$.

Defining for a firm i and (aggregate action of the other firms) $z \in \mathbb{R}_+$ the *reduced conditional profit* function $\tilde{u}_i^{(z)} : \mathbb{R}_+ \to \mathbb{R}$ by

$$\tilde{u}_i^{(z)}(x_i) = p(x_i + z)x_i - c_i(x_i), \tag{2.1}$$

the result in [3] essentially is:

Theorem 2.1. *Suppose for every $i \in N$:*

$$c_i{}'(x_i) > 0 \ (x_i > 0),$$

for every $y \in [0, v\,[$ there exists $\alpha_{i,y} < 0$ such that

$$p'(y) - c_i{}''(x_i) \leq \alpha_{i,y} \ (x_i \geq 0) \tag{2.2}$$

and for every $z \in [0, v\,[$

$$\tilde{u}_i^{(z)} \text{ is strictly pseudo-concave on } [0, v - z\,[. \tag{2.3}$$

Let

$$N_> := \{k \in N \mid p(0) > c'_k(0)\}$$

and for $\mathbf{x} \in \mathbf{X}$ with $\underline{\mathbf{x}} < v$

$$q_k(\mathbf{x}) := -\frac{x_k p''(\underline{\mathbf{x}}) + p'(\underline{\mathbf{x}})}{p'(\underline{\mathbf{x}}) - c_k{}''(x_k)} \ (k \in N).$$

(1) $\#E \geq 1$.
(2) $\sum_{k \in N_>} \textit{with}_{\ e_k > 0}\, q_k(\mathbf{e}) < 1$ $(\mathbf{e} \in E) \Rightarrow \#E = 1$.
(3) $E = \{\mathbf{e}\} \Rightarrow \sum_{k \in N_>} q_k(\mathbf{e}) \leq 1$. $\diamond$

Remarks. (1) Formula (2.2) concerns a strong variant of (what we call) the *first Fisher-Hahn condition* (for firm i), i.e. of

$$p'(y) - c_i''(x_i) < 0 \ (0 \leq x_i \leq y < v). \tag{2.4}$$

(2) In Theorem 2.1, every $\tilde{u}_i^{(z)}$ is strictly quasi-concave. (Indeed: for $z \geq v$, $\tilde{u}_i^{(z)} = -c_i$ is strictly decreasing, and, for $z < v$, $\tilde{u}_i^{(z)}$ is continuous, strictly quasi-concave on $[0, v - z\,[$ and strictly decreasing on $[v - z, +\infty\,[$.)

(3) The so-called *marginal revenue condition* is

$$yp''(y) + p'(y) \leq 0 \ (0 \leq y < v) \tag{2.5}$$

and the *second Fisher-Hahn condition* is

$$xp''(y) + p'(y) \leq 0 \ (0 \leq x \leq y < v). \tag{2.6}$$

If p is decreasing, then (2.5) $\Leftrightarrow$ (2.6). And (cfr. with Remark 7) we have ((2.4) $\wedge$ (2.6)) $\Rightarrow$ (2.3).

(4) If the marginal revenue condition and the first Fisher-Hahn condition hold, then, using Remark 3, $q_k(\mathbf{e}) < 0$ ($\mathbf{e} \in E$) and thus Theorem 2.1(2,3) (trivially) holds. However, as shown in [1], for this case a much more simple equilibrium uniqueness proof exists.

3. Assumptions

Consider a game in strategic form with player set $N := \{1, \ldots, n\}$, for player i a strategy set X_i and payoff function u_i. So every X_i is a non-empty set and every u_i a function $X_1 \times \cdots \times X_n \to \mathbb{R}$. We denote the set of *strategy profiles* $X_1 \times \cdots \times X_n$ also by $\mathbf{X}$. For $i \in N$, define $\mathbf{X}_{\hat{\imath}} := X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$. We sometimes identify $\mathbf{X}$ with $X_i \times \mathbf{X}_{\hat{\imath}}$ and then write $\mathbf{x} \in \mathbf{X}$ as $\mathbf{x} = (x_i; \mathbf{x}_{\hat{\imath}})$.

For $i \in N$ and $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$ the *conditional payoff function* $u_i^{(\mathbf{z})} : X_i \to \mathbb{R}$ is defined by $u_i^{(\mathbf{z})}(x_i) := u_i(x_i; \mathbf{z})$ and the *best-reply correspondence* $R_i : \mathbf{X}_{\hat{\imath}} \multimap X_i$ is defined by $R_i(\mathbf{z}) := \text{argmax } u_i^{(\mathbf{z})}$. A strategy profile $\mathbf{x}$ is called a *(Nash) equilibrium* if $x_i \in R_i(\mathbf{x}_{\hat{\imath}})$ ($i \in N$).

From now on we always assume the following Assumptions a–k.

a. For $i \in N$: $X_i = \mathbb{R}_+$ and $T_i := \sum_{l \in N \setminus \{i\}} X_l$. Let $Y := \sum_{l \in N} X_l$.

b. For each player $i \in N$ and $z \in T_i$ there exists a function $\tilde{u}_i^{(z)} : X_i \to \mathbb{R}$ such that $u_i^{(\mathbf{z})} = \tilde{u}_i^{(\underline{\mathbf{z}})}$ ($\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$). We call $\tilde{u}_i^{(z)}$ the *reduced conditional payoff function* of i.

c. Fix $v \in Y \cup \{+\infty\}$ with $v \neq 0$ such that in case $v \neq +\infty$ for every $i \in N$ and $z \in T_i$

$$\text{argmax } \tilde{u}_i^{(z)} = \begin{cases} \text{argmax } \tilde{u}_i^{(z)} \restriction [0, v - z\,[\neq \emptyset \text{ if } z < v, \\ \{0\} \text{ if } z \geq v. \end{cases}$$

Let $Y_v := \begin{cases} [0, v] \text{ if } v \neq +\infty, \\ [0, v\,[\text{ if } v = +\infty. \end{cases}$ (We always can take $v = +\infty$.)

d. Every $\tilde{u}_i^{(z)}$ is continuous and for $z < v$ differentiable on $[0, v - z\,[$.

e. For every $i \in N$ there exists a continuous function $t_i : X_i \times Y_v \to \mathbb{R}$ such that for every $z \in [0, v[$ and $x_i \in [0, v - z[$ (using also Euler's notation D for derivatives)

$$D\tilde{u}_i^{(z)}(x_i) = t_i(x_i, x_i + z).$$

f. Each function $t_i(0, \cdot)$ is decreasing.
g. For every $i \in N$ and $y \in [0, v[$, $t_i(\cdot, y)$ is strictly decreasing.
h. If $v = +\infty$, then for every $i \in N$ there exists $r_i \in X_i$ such that $t_i(x_i, x_i+z) < 0$ for every $z \in T_i$ and $x_i \in X_i$ with $x_i \geq r_i$.
i. If $v \neq +\infty$, then $t_i(x_i, v) < 0$ $(x_i \in X_i \setminus \{0\})$.
j. For every $i \in N_> := \{k \in N \mid t_k(0,0) > 0\}$ and for every $y \in Y_v$ with $t_i(0, y) \geq 0$, there exists an $x_i \in X_i$ with $t_i(x_i, y) \leq 0$.
k. For every $i \in N$ and $z \in [0, v[$, $\tilde{u}_i^{(z)}$ is strictly pseudo-concave on $[0, v - z[$.

Remarks. (5) Although $X_i = T_i = Y = \mathbb{R}_+$, we often write X_i, T_i and Y instead of $\mathbb{R}_+$. Doing this may be helpfull for the generalisation where also other strategy sets are allowed.

(6) Under Ass. a concerns one of the possible ways to define an aggregative game. Under Ass. a sufficient for Ass. b–e to hold (with $v = +\infty$) is that for every $i \in N$ the following assumption holds: there exists a smooth enough function $\pi_i : X_i \times Y \to \mathbb{R}$ such that $u_i(\mathbf{x}) = \pi_i(x_i, \underline{\mathbf{x}})$ $(\mathbf{x} \in \mathbf{X})$. (Indeed, then take $\tilde{u}_i^{(z)}(x_i) = \pi_i(x_i, x_i + z)$ and $t_i = (D_1 + D_2)\pi_i$.)

(7) Sufficient for Ass. k to hold is that for every $i \in N$, $z \in [0, v[$ and $x_i \in [0, v - z[$ we have: $t_i(x_i, x_i + z) = 0 \Rightarrow (D_1 + D_2)t_i(x_i, x_i + z) < 0$.

(8) Ass. e and i imply: $v \neq +\infty \Rightarrow t_i(0, v) \leq 0$.

(9) In Ass. c: if $v \in]z, +\infty[$, then $\operatorname{argmax} \tilde{u}_i^{(z)} = \operatorname{argmax} \tilde{u}_i^{(z)} \restriction [0, v - z]$.

4. Main result

Denoting the set of equilibria by E, our main result is the following.

Theorem 4.1. *Suppose Assumptions a–k hold.*

(1) *If* $\mathbf{e} \in E$, *then* $\underline{\mathbf{e}} \in [0, v[$. *And if* $\#E \geq 2$, *then* $\mathbf{0} \notin E$.
(2) $\#E \geq 1$.
(3) *If for every* $i \in N_>$ *also the assumptions*
  l. *there exists* $\delta_i < 0$ *and a continuously differentiable function* $\tilde{t}_i :]\delta_i, +\infty[\times]0, v[\to \mathbb{R}$ *such that* $\tilde{t}_i = t_i$ *on* $\mathbb{R}_+ \times]0, v[$,
  m. $D_1 \tilde{t}_i < 0$,

  hold, then
  I. $\sum_{k \in N_> \text{ with } e_k > 0} -\frac{D_2 t_k}{D_1 t_k}(e_k, \underline{\mathbf{e}}) < 1$ $(\mathbf{e} \in E) \Rightarrow \#E \leq 1$.
  II. $E = \{\mathbf{e}\} \Rightarrow \sum_{k \in N_>} -\frac{D_2 t_k}{D_1 t_k}(e_k, \underline{\mathbf{e}}) \leq 1$. $\diamond$

Proof. (1) First statement: we may suppose $v \neq +\infty$. By contradiction suppose $\mathbf{e} \in E$ with $\underline{\mathbf{e}} \geq v$. We have, $e_j \in \operatorname{argmax} \tilde{u}_j^{(\underline{\mathbf{e}_{\hat{j}}})}$ $(j \in N)$.

Case where there exists $j \in N$ with $\underline{\mathbf{e}_{\hat{j}}} < v$: now, by Ass. c, $e_j \in \operatorname{argmax} \tilde{u}_j^{(\underline{\mathbf{e}_{\hat{j}}})} \restriction [0, v - \underline{\mathbf{e}_{\hat{j}}}[\subseteq [0, v - \underline{\mathbf{e}_{\hat{j}}}[$ and so $\underline{\mathbf{e}} < v$, a contradiction.

Case where there is no $j \in N$ with $\underline{\mathbf{e}_{\hat{\jmath}}} < v$: now $\underline{\mathbf{e}_{\hat{\jmath}}} \geq v$ $(j \in N)$ and therefore, by Ass. c, $\mathbf{e} = \mathbf{0}$. Again a contradiction.

Second statement: by way of contradiction, suppose $\mathbf{a}, \mathbf{b} \in E$ with $0 = \mathbf{a} \neq \mathbf{b}$. Fix i with $b_i > 0$. As $\mathbf{0} \in E$, Ass. e implies $t_i(0,0) = D\tilde{u}_i^{(0)}(0) \leq 0$. By part 1, $\underline{\mathbf{b}} < v$. Now the contradiction $t_i(0,0) \leq 0 = t_i(b_i, \underline{\mathbf{b}}) < t_i(0, \underline{\mathbf{b}}) \leq t_i(0,0)$ follows: the second inequality holds as $\mathbf{b}$ is an equilibrium with $b_i \in \mathrm{Int}(X_i)$, the third follows with Ass. g and the fourth with Ass. f.

(2), (3) See Section 6. □

Remarks. (10) By Ass. f and l: $D_2 t_k(0,y) \leq 0$ $(k \in N_>)$ and $y \in \,]0, v\,[$. So Proposition 5.1(4) below implies in Theorem 4.1(3), writing $q_k(\mathbf{e}) := -\frac{D_2 t_k}{D_1 t_k}(e_k, \underline{\mathbf{e}})$, that $\sum_{k \in N} q_k(\mathbf{e}) \leq \sum_{k \in N_>} q_k(\mathbf{e}) \leq \sum_{k \in N_> \text{ with } e_k > 0} q_k(\mathbf{e})$.

5. Method

Define the *reduced best reply correspondence* $\tilde{R}_i : T_i \multimap X_i$ by

$$\tilde{R}_i(z) := \operatorname{argmax} \tilde{u}_i^{(z)}.$$

Note that $R_i(\mathbf{z}) = \tilde{R}_i(\underline{\mathbf{z}})$ $(\mathbf{z} \in \mathbf{X}_{\hat{\imath}})$.

Proposition 5.1. (1) *Every $\tilde{R}_i$ is singleton-valued. (So the correspondence $\tilde{R}_i$ can and will be interpreted as a function $T_i \to X_i$.)*

(2) *Each function $\tilde{R}_i$ is bounded.*

(3) *If $i \in N \setminus N_>$, then $\tilde{R}_i = 0$.*

(4) *For every $\mathbf{e} \in E$ and $i \in N \setminus N_>$ it holds that $e_i = 0$.* ⋄

Proof. (1) Fix $z \in T_i$. If $z \geq v$, then by Ass. c, $\tilde{R}_i(z) = \{0\}$. Now suppose $z < v$. By Ass. c, $\tilde{R}_i(z) = \operatorname{argmax} \tilde{u}_i^{(z)} \restriction]0, v-z\,[$. As, by Ass. k, $\tilde{u}_i^{(z)} \restriction]0, v-z\,[$ is strictly quasi-concave, we have $\#\tilde{R}_i(z) \leq 1$. Case $v < +\infty$: by Remark 9, $\tilde{R}_i(z) = \operatorname{argmax} \tilde{u}_i^{(z)} \restriction [0, v-z]$. Ass. d together with the Weierstrass' theorem implies $\#\tilde{R}_i(z) \geq 1$. Case $v = +\infty$: by Ass. h, for $z \in T_i$ and $x_i \geq r_i$ we have $D\tilde{u}_i^{(z)}(x_i) = t_i(x_i, x_i + z) < 0$. So $\tilde{u}_i^{(z)}$ is strictly decreasing on $[r_i, +\infty\,[$. This implies $\tilde{R}_i(z) = \operatorname{argmax} \tilde{u}_i^{(z)} \restriction [0, r_i]$. Again, the Weierstrass' theorem implies $\#\tilde{R}_i(z) \geq 1$. Thus $\#\tilde{R}_i(z) = 1$.

(2) Case $v \neq +\infty$: Ass. c implies that $z \geq v \;\Rightarrow\; \tilde{R}_i(z) = 0$ and $z < v \;\Rightarrow\; \tilde{R}_i(z) < v - z \leq v$.

Case $v = +\infty$: as, by the proof of part 1, $\tilde{R}_i(z) = \operatorname{argmax} \tilde{u}_i^{(z)} \restriction [0, r_i]$.

(3) If $z \geq v$, then $\tilde{R}_i(z) = 0$. Now suppose $z < v$. Let $a = \tilde{R}_i(z)$. By contradiction we prove that $a = 0$. So suppose $a \neq 0$. By Ass. c, $a \in \,]0, v-z\,[$. As, by Ass. d, $\tilde{u}_i^{(z)}$ is differentiable on $[0, v-z\,[$, we have, by Fermat's theorem, $D\tilde{u}_i^{(z)}(a) = 0$. By Ass. e, $t_i(a, a+z) = 0$. So, with Ass. g and f, $0 = t_i(a, a+z) < t_i(0, a+z) \leq t_i(0,0) \leq 0$, a contradiction.

(4) This follows with part 3. □

For $i \in N_>$, we define the correspondence $b_i : Y_v \multimap X_i$ by

$$b_i(y) := \{x_i \in X_i \mid t_i(x_i, y) = 0\}.$$

By Conditions g and i every b_i is at most singleton-valued. For $i \in N_>$, we define the *virtual cumulative best reply function* $\tilde{b}_i : Y_v \to \mathbb{R}$ of player i by

$$\tilde{b}_i(y) := \begin{cases} \text{the unique element of } b_i(y) \text{ if } b_i(y) \neq \emptyset, \\ 0 \text{ if } b_i(y) = \emptyset. \end{cases}$$

The adjective 'virtual' refers to the fact that b_i is not the same as (but is closely related to), what is called in [6], the *cumulative best reply correspondence* $B_i : Y_v \multimap \mathbb{R}$ defined by $B_i(y) := \{x_i \in X_i \mid y - x_i \in T_i \text{ and } x_i \in \tilde{R}_i(y - x_i)\}$. (The function $\tilde{b}_i$ corresponds to the function g_i in [3], but is in its very detail different.) For $i \in N_>$, let

$$Y_i^{(\text{ess})} := \{y \in Y_v \mid b_i(y) \neq \emptyset\} \ \supseteq \ \{y \in Y_v \mid \tilde{b}_i(y) > 0\} \ =: \ Y_i^{(\text{ess}+)}.$$

Lemma 5.2. *Suppose $i \in N_>$.*

1. *For $y \in Y_i^{(\text{ess})}$, $\tilde{b}_i(y)$ is the unique element of X_i with $t_i(\tilde{b}_i(y), y) = 0$.*
2. *For every $x_i \in X_i$ and $y \in Y_v$:*

$$\tilde{b}_i(y) \cdot \ t_i(\tilde{b}_i(y), y) = 0 \text{ and } t_i(x_i, y) = 0 \ \Rightarrow \ x_i = \tilde{b}_i(y).$$

3. $v \neq +\infty \ \Rightarrow \ \tilde{b}_i(v) = 0.$
4. *For $y \in Y_v$: $y \in Y_i^{(\text{ess}+)} \ \Leftrightarrow \ t_i(0, y) > 0$.*
5. *For $y \in Y_v$: $y \in Y_i^{(\text{ess})} \ \Leftrightarrow \ t_i(0, y) \geq 0$.* $\diamond$

Proof. (1), (2) Direct consequences of the definitions involved.

(3) By part 2 and Ass. i.

(4) '$\Rightarrow$': so $\tilde{b}_i(y) > 0$. By parts 2 and 3, $t_i(\tilde{b}_i(y), y) = 0$ and $y \neq v$. With Ass. g, $t_i(0, y) > t_i(\tilde{b}_i(y), y) = 0$.

'$\Leftarrow$': As $t_i(0, y) > 0$, Ass. j implies that there exists $x_i \in X_i \setminus \{0\}$ with $t_i(x_i, y) = 0$. So, by part 2, $\tilde{b}_i(y) = x_i > 0$.

(5) '$\Rightarrow$': if $y \in Y_i^{(\text{ess})}$, then apply part 4. If $y \in Y_i^{(\text{ess})} \setminus Y_i^{(\text{ess}+)}$, then $\tilde{b}_i(y) = 0$ and part 1 gives $t_i(0, y) = 0$.

'$\Leftarrow$': if $t_i(0, y) > 0$, then apply part 4. If $t_i(0, y) = 0$, then $0 \in b_i(y)$ and so $0 \in Y_i^{(\text{ess})}$. $\square$

Proposition 5.3. *If $\mathbf{e} \in E$ and $i \in N_>$, then $e_i = \tilde{b}_i(\underline{\mathbf{e}})$.* $\diamond$

Proof. As $\mathbf{e} \in E$ and $X_i = \mathbb{R}_+$, we have $e_i > 0 \ \Rightarrow \ t_i(e_i, \underline{\mathbf{e}}) = Du_i^{(\mathbf{e}_{\hat{\imath}})}(e_i) = D_i u_i(\mathbf{e}) = 0$, and $e_i = 0 \ \Rightarrow \ t_i(e_i, \underline{\mathbf{e}}) = D_i u_i(\mathbf{e}) \leq 0$. So, if $e_i > 0$, then $\tilde{b}_i(\underline{\mathbf{e}}) = e_i$ and if $e_i = 0$, then by Lemma 5.2(4), $\underline{\mathbf{e}} \notin Y_i^{(\text{ess}+)}$ and so $\tilde{b}_i(\underline{\mathbf{e}}) = 0 = e_i$. $\square$

Lemma 5.4. *Suppose $v = +\infty$ and $i \in N_>$.*

1. *If $y \geq r_i$, then $y - \tilde{b}_i(y) \in T_i$ and $[\tilde{b}_i(y) > 0 \ \Rightarrow \ \tilde{R}_i(y - \tilde{b}_i(y)) = \tilde{b}_i(y)]$.*
2. *There exists $\overline{y}_i > 0$ such that $\tilde{b}_i(y) \leq \frac{y}{n}$ for all $y \geq \overline{y}_i$.* $\diamond$

Proof. (1) First statement: if $\tilde{b}_i(y) = 0$, then this statement holds. Now suppose $\tilde{b}_i(y) > 0$. By Lemma 5.2(2), $t_i(\tilde{b}_i(y), y) = 0$. By Ass. h, $t_i(y, y) < 0$. So $t_i(\tilde{b}_i(y), y) > t_i(y, y)$. By Ass. g, $\tilde{b}_i(y) < y$. Thus $y - \tilde{b}_i(y) \in T_i$.

Second statement: as $\tilde{b}_i(y) > 0$, $t_i(\tilde{b}_i(y), y) = 0$ holds. As $y - \tilde{b}_i(y) \in T_i$, we obtain $D\tilde{u}_i^{(y-\tilde{b}_i(y))}(\tilde{b}_i(y)) = t_i(\tilde{b}_i(y), y) = 0$. Ass. k implies $\tilde{b}_i(y) \in \operatorname{argmax} \tilde{u}_i^{(y-\tilde{b}_i(y))} = \tilde{R}_i(y - \tilde{b}_i(y))$.

(2) By contradiction suppose there does not exist such an $\overline{y}_i$. This implies the existence of a sequence (y_m) in $\mathbb{R}_+$ with limit $+\infty$ and $\tilde{b}_i(y_m) > y_m/n$ for all m. Now, by part 1 for m large enough, $\tilde{R}_i(y_m - \tilde{b}_i(y_m)) = \tilde{b}_i(y_m) > y_m/n$. It follows that $\lim_{m\to+\infty} \tilde{R}_i(y_m - \tilde{b}_i(y_m)) = +\infty$. As, by Proposition 5.1(2), $\tilde{R}_i$ is bounded, this is absurd. □

We define the function $\tilde{b} : Y_v \to \mathbb{R}$ by

$$\tilde{b} := \sum_{k\in N_>} \tilde{b}_k$$

and refer to $\tilde{b}$ as a *aggregate virtual cumulative best reply function.* We denote the set of fixed points of $\tilde{b}$ by $\operatorname{fix}(\tilde{b})$. We understand by the *Nash-sum function* the function $\sigma : E \to \mathbb{R}$ defined by $\sigma(\mathbf{e}) := \underline{\mathbf{e}}$ and we call an element of $\sigma(E)$ *Nash-sum.*

Proposition 5.5. (1) *σ is injective.*

(2) *If $y \in \operatorname{fix}(\tilde{b})$, then with $\mathbf{e} \in \mathbf{X}$ defined by $e_i = \tilde{b}_i(y)$ $(i \in N_>)$ and $e_i = 0$ $(i \in N \setminus N_>)$, it holds that $\underline{\mathbf{e}} = y$ and $\mathbf{e} \in E$.*

(3) *$\sigma(E) = \operatorname{fix}(\tilde{b})$.* $\diamond$

Proof. (1) By contradiction suppose $\mathbf{a}, \mathbf{b} \in E$ with $\mathbf{a} \neq \mathbf{b}$ and $\underline{\mathbf{a}} = \underline{\mathbf{b}} =: y$. Fix $i \in N$ with $b_i > a_i$. As $\mathbf{a}, \mathbf{b} \in E$, we have $D_i u_i(\mathbf{b}) \geq 0 \geq D_i u_i(\mathbf{a})$, so $t_i(b_i, y) \geq t_i(a_i, y)$. But, by Ass. g, $t_i(b_i, y) < t_i(a_i, y)$.

(2) $\underline{\mathbf{e}} = \sum_{k\in N\setminus N_>} e_k + \sum_{k\in N_>} e_k = \sum_{k\in N_>} e_k = \sum_{k\in N_>} \tilde{b}_k(y) = \tilde{b}(y) = y$. Fix $i \in N$. We have to prove that $e_i = \tilde{R}_i(\underline{\mathbf{e}_{\hat{\imath}}})$. If $\underline{\mathbf{e}_{\hat{\imath}}} = v$, then, as $\underline{\mathbf{e}} \leq v$, $e_i = 0$. Also, by Ass. c, $\tilde{R}_i(v) = 0$. Now further suppose $\underline{\mathbf{e}_{\hat{\imath}}} < v$. We have $Du_i^{(\underline{\mathbf{e}_{\hat{\imath}}})}(e_i) = t_i(e_i, \underline{\mathbf{e}})$.

Case $i \in N \setminus N_>$: by Ass. f, $Du_i^{(\underline{\mathbf{e}_{\hat{\imath}}})}(e_i) = t_i(0, \underline{\mathbf{e}}) \leq t_i(0, 0) \leq 0$. By Ass. k, $u_i^{(\underline{\mathbf{e}_{\hat{\imath}}})} \upharpoonright [0, v - \underline{\mathbf{e}_{\hat{\imath}}}[$ is pseudo-concave. Therefore $0 \in \operatorname{argmax} u_i^{(\underline{\mathbf{e}_{\hat{\imath}}})} \upharpoonright [0, v - \underline{\mathbf{e}_{\hat{\imath}}}[$. So, by Ass. c, $0 \in \tilde{R}_i(\underline{\mathbf{e}_{\hat{\imath}}})$.

Case $i \in N_>$ and $e_i = 0$: as $0 = e_i = \tilde{b}_i(\underline{\mathbf{e}})$, we have $\underline{\mathbf{e}} \notin Y_i^{(\mathrm{ess}+)}$ and so, by Lemma 5.2(4), $t_i(e_i, \underline{\mathbf{e}}) = t_i(0, \underline{\mathbf{e}}) \leq 0$. As above $0 \in \tilde{R}_i(\underline{\mathbf{e}_{\hat{\imath}}})$ follows.

Case $i \in N_>$ and $e_i \neq 0$: noting that $\tilde{b}_i(y) > 0$ and therefore $t_i(\tilde{b}_i(y), y) = 0$, we have $t_i(e_i, \underline{\mathbf{e}}) = 0$. As above $e_i \in \tilde{R}_i(\underline{\mathbf{e}_{\hat{\imath}}})$ follows.

(3) '$\subseteq$': suppose $y = \underline{\mathbf{e}}$ with $\mathbf{e} \in E$. By Propositions 5.1(4) and 5.3, $y = \sum_{k\in N} e_k = \sum_{k\in N_>} e_k = \sum_{k\in N_>} \tilde{b}_k(\underline{\mathbf{e}}) = \tilde{b}(\underline{\mathbf{e}}) = \tilde{b}(y)$. '$\supseteq$': by part 2. □

Lemma 5.6. *Suppose $i \in N_>$.*

(1) *$Y_i^{(\mathrm{ess})} = \mathbb{R}_+$ or $Y_i^{(\mathrm{ess})} = [0, s_i]$ for some $s_i \in Y_v \setminus \{0\}$.*

(2) *If $Y_i^{(\mathrm{ess})} = [0, s_i]$, then $\tilde{b}_i(y) = 0$ for every $y \in Y_v$ with $y \geq s_i$.*

(3) $\tilde{b}_i : Y_v \to \mathbb{R}$ *is continuous.*

(4) $Y_i^{(\mathrm{ess}+)} = \mathbb{R}_+$ *or* $Y_i^{(\mathrm{ess}+)} = [0, w_i[$ *for some* $w_i \in]0, v[$.

(5) *If* $Y_i^{(\mathrm{ess}+)} = [0, w_i[$, *then* $w_i \in Y_i^{(\mathrm{ess})}$, $t_i(0, w_i) = 0$ *and* $\tilde{b}_i(w_i) = 0$. $\diamond$

Proof. (1) By Lemma 5.2(5), $Y_i^{(\mathrm{ess})} = \{y \in Y_v \mid t_i(0, y) \geq 0\}$. As $t_i(0, \cdot)$ is decreasing and Y_v is an interval, also $Y_i^{(\mathrm{ess})}$ is an interval. As $t_i(0, 0) > 0$, $0 \in Y_i^{(\mathrm{ess})}$ follows. The continuity of $t_i(\cdot, 0)$ and $t_i(0, 0) > 0$ imply that the interval $Y_i^{(\mathrm{ess})}$ is proper and that $Y_i^{(\mathrm{ess})}$ is closed in Y_v. This implies the desired result.

(2) By Lemma 5.2(5), $t_i(0, s_i) \geq 0$ and $t_i(0, y) < 0$ ($y \in Y_v$ with $y > s_i$). If $y = s_i = v$, then apply Lemma 5.2(3). If $y = s_i < v$, then the continuity of t_i implies $t_i(0, s_i) = 0$ and thus $\tilde{b}_i(s_i) = 0$. Finally, for $y \in Y_v$ with $y > s_i$, we have, using Ass. g, $t_i(x_i, y) < 0$ ($x_i \in X_i$) and therefore $\tilde{b}_i(y) = 0$.

(3) Consider the function $t_i : X_i \times Y_i^{(\mathrm{ess})} \to \mathbb{R}$. By Lemma 5.2(1), for $y \in Y_i^{(\mathrm{ess})}$ it holds that $\tilde{b}_i(y)$ is the unique element of X_i with $t_i(\tilde{b}_i(y), y) = 0$. With Lemma 5.2(3): $v \in Y_i^{(\mathrm{ess})} \Rightarrow t_i(0, v) = 0$. Part 1 and Ass. e,g and i imply that Theorem 7.1 in the appendix applies. This theorem guarantees that $\tilde{b}_i$ is continuous on $Y_i^{(\mathrm{ess})}$. Now with parts 1 and 4 it follows that $\tilde{b}_i$ is continuous.

(4) Analogous to part 1, noting that $v \notin Y_i^{(\mathrm{ess}+)}$ by Lemma 5.2(3).

(5) As $w_i \notin Y_i^{(\mathrm{ess}+)}$, $t_i(0, w_i) \leq 0$ holds. As $Y_i^{(\mathrm{ess}+)} \subseteq Y_i^{(\mathrm{ess})}$, part 1 implies $w_i \in Y_i^{(\mathrm{ess})}$ and so $t_i(0, w_i) \geq 0$. Thus $t_i(0, w_i) = 0$ and so $\tilde{b}_i(w_i) = 0$. □

Proposition 5.7. *The function* $\tilde{b} : Y_v \to \mathbb{R}$ *has a fixed point.* $\diamond$

Proof. If $N_> = \emptyset$, then $\tilde{b} = 0$ and $0 \in \mathrm{fix}(\tilde{b})$. Now suppose $N_> \neq \emptyset$. Lemma 5.6(3) guarantees that $\tilde{b}$ is continuous. By Lemma 5.6(4), $\tilde{b}(0) > 0$.

Case where $v \neq +\infty$: according to Lemma 5.2(1) we have for every $i \in N_>$ that $Y_i^{(\mathrm{ess})} = [0, s_i]$ where $s_i \in]0, v]$. According to Lemma 5.2(3) we have $\tilde{b}(v) = 0$. It follows that $\tilde{b}$ has a fixed point.

Case where $v = +\infty$: Lemma 5.4(2) implies the existence of $\overline{y} > 0$ such that $\tilde{b}(y) \leq y$ for every $y \geq \overline{y}$. Again, it follows that $\tilde{b}$ has a fixed point. □

For the next lemma remember that, for $i \in N_>$, $Y_i^{(\mathrm{ess}+)} = \mathbb{R}_+ = Y_v$ or $Y_i^{(\mathrm{ess}+)} = [0, w_i[\subset Y_v$ for some $w_i \in]0, v[$.

Lemma 5.8. *Suppose Assumptions l and m (in Theorem 4.1) hold. Let* $i \in N_>$.

(1) $\tilde{b}_i$ *is differentiable at every* $y_0 \in Y_i^{(\mathrm{ess}+)}$ *with* $y_0 \neq 0$ *and* $D\tilde{b}_i(y_0) = -\frac{D_2 t_i}{D_1 t_i}(\tilde{b}_i(y_0), y_0)$.

(2) *If* $Y_i^{(\mathrm{ess}+)} = [0, w_i[$, *then* $\tilde{b}_i(y_0) = 0$, *at every* $y_0 \in [w_i, v[$ *and* $\tilde{b}_i$ *is semi-differentiable at* w_i *with* $D^-\tilde{b}_i(w_i) = -\frac{D_2 t_i}{D_1 t_i}(\tilde{b}_i(w_i), w_i) \leq 0 = D^+\tilde{b}_i(w_i)$. $\diamond$

Proof. By the definition of $Y_i^{(\mathrm{ess}+)}$, $\tilde{b}_i = 0$ on $[w_i, v[$. Thus $D^+\tilde{b}_i(w_i) = 0$. The other statements can be proved with the implicit function theorem. As the proof of part 1 is a routine one, we only provide here the proof of the other statements in part 2. This proof is a little bit technical due to the fact that $(0, w_i)$ is not an interior point of the domain of t_i.

Let $W_i' :=]\delta_i, +\infty[\times]0, v[$ By Lemma 5.6(5), $w_i \in Y_i^{(\mathrm{ess})}$, $\tilde{b}_i(w_i) = 0$ and $t_i(0, w_i) = 0$. As $w_i \in]0, v[$, we have $0 = \tilde{t}_i(0, w_i)$. By Ass. l, the function $\tilde{t}_i : W_i' \to \mathbb{R}$ is continuously differentiable and, by Ass. m, $D_1\tilde{t}_i(0, w_i) \neq 0$. The implicit function theorem guarantees that there exists an open neighbourhood U_i of 0 in $\mathbb{R}$, an open neighbourhood V_i of w_i in $\mathbb{R}$ with $U_i \times V_i \subseteq W_i'$ and a unique function $\Psi_i : V_i \to \mathbb{R}$ with $\Psi_i(V_i) \subseteq U_i$ such that

$$\{(\Psi_i(y), y) \mid y \in V_i\} = \{(x_i, y) \in U_i \times V_i \mid \tilde{t}_i(x_i, y) = 0\}.$$

In addition: this function Ψ_i is continuously differentiable. So we have $\tilde{t}_i(\Psi_i(y), y) = 0$ $(y \in V_i)$. As $\tilde{b}_i$ is, by Lemma 5.6(3), continuous, there exists an open neighbourhood S_i of w_i in $\mathbb{R}$ such that $\tilde{b}_i(y) \in U_i$ $(y \in S_i)$. Now $\tilde{t}_i(\Psi_i(y), y) = 0$ $(y \in S_i \cap V_i \cap]0, w_i])$. Also $\tilde{t}_i(\tilde{b}_i(y), y) = t_i(\tilde{b}_i(y), y) = 0$ $(y \in S_i \cap V_i \cap]0, w_i])$. It follows that $\tilde{b}_i = \Psi_i$ on $S_i \cap V_i \cap]0, w_i]$ and so $\tilde{b}_i$ is left differentiable at w_i. Differentiating the identity $\tilde{t}_i(\Psi_i(y), y) = 0$ $(y \in S_i \cap V_i \cap]0, w_i])$ gives for $y = w_i$

$$D^-\tilde{b}_i(w_i) = D\Psi_i(w_i) = -\frac{D_2\tilde{t}_i}{D_1\tilde{t}_i}(\Psi_i(w_i), w_i) = -\frac{D_2\tilde{t}_i}{D_1\tilde{t}_i}(0, w_i) = -\frac{D_2 t_i}{D_1 t_i}(0, w_i).$$

By Ass. f we have $D_2 t_i(0, w_i) \leq 0$. Thus, with Ass. m, $D^-\tilde{b}_i(w_i) \leq 0$. □

Before stating the next result we note that Lemmas 5.2(3) and 5.6(4) imply: $N_> \neq \emptyset \Rightarrow \mathrm{fix}(\tilde{b}) \subseteq]0, v[$.

Proposition 5.9. *Suppose $N_> \neq \emptyset$ and $\tilde{b}$ is at every $w \in \mathrm{fix}(\tilde{b})$ semi-differentiable with $D^-\tilde{b}(w) \leq D^+\tilde{b}(w)$.*

(1) *If at each $w \in \mathrm{fix}(\tilde{b})$ it holds that $D^+\tilde{b}(w) < 1$, then $\#\mathrm{fix}(\tilde{b}) \leq 1$.*

(2) *If $\tilde{b}$ has a unique fixed point w, then $D^-\tilde{b}(w) \leq 1$.* ◇

Proof. Let $N(g)$ be the set of zeros of the function $g : Y_v \to \mathbb{R}$ defined by $g(y) := \tilde{b}(y) - y$. So $\mathrm{fix}(\tilde{b}) = N(g)$.

(1) By contradiction suppose $\#\mathrm{fix}(\tilde{b}) \geq 2$; so $\#N(g) \geq 2$ by part 1. As $D^-g \leq D^+g < 0$ on $N(g)$ and g is by Lemma 5.6(3) continuous, it follows that g has at most one zero. This is a contradiction.

(2) By contradiction suppose $D^-\tilde{b}(w) > 1$. So $g(w) = 0$ and $D^-g(w) > 0$. As $g(0) > 0$ and g is continuous, g has a zero in $]0, w[$. So $\#\mathrm{fix}(\tilde{b}) \geq 2$, a contradiction. □

6. Proof of Theorem 4.1(2,3)

(2) By Proposition 5.7, $\mathrm{fix}(\tilde{b}) \neq \emptyset$. Proposition 5.5(3) implies $E \neq \emptyset$.

(3) First note that Lemma 5.8 implies that for every $k \in N_>$ the function $\tilde{b}_k$ is at every $y \in]0, v[$ semi-differentiable with $D^-\tilde{b}_k(y) \leq D^+\tilde{b}_k(y)$. This implies that $\tilde{b}$ is at every $w \in \mathrm{fix}(\tilde{b})$ with $y \neq 0$ semi-differentiable with $D^-\tilde{b} \leq D^+\tilde{b}$.

I. Having part 1, we may suppose that $\mathbf{0} \notin E$. By Proposition 5.1(4) this implies $N_> \neq \emptyset$. We shall prove that the continuous function $\tilde{b}$ has at most one fixed point; then $\#E \leq 1$ follows from Proposition 5.5(3). Proving that $\tilde{b}$ has at most one fixed point now will be done by verifying the condition in Proposition 5.9(1). So suppose $w \in \mathrm{fix}(\tilde{b})$. It follows that

$$\begin{aligned} D^+\tilde{b}(w) &= \sum_{k \in N_>} D^+\tilde{b}_k(w) = \sum_{k \in N_> \text{ with } \tilde{b}_k(w)>0} D^+\tilde{b}_k(w) \\ &= \sum_{k \in N_> \text{ with } \tilde{b}_k(w)>0} D\tilde{b}_k(w) \\ &= \sum_{k \in N_> \text{ with } \tilde{b}_k(w)>0} -\frac{D_2 t_k}{D_1 t_k}(\tilde{b}_k(w), w). \end{aligned}$$

Here the second equality holds by Lemma 5.8(2) and the third and fourth by Lemma 5.8(1). By Proposition 5.5(3), $w \in \sigma(E)$. Fix $\mathbf{e} \in E$ such that $w = \underline{\mathbf{e}}$. Now for $k \in N_>$, by Proposition 5.3, $e_k = \tilde{b}_k(w)$. With this, as desired,

$$D^+\tilde{b}(w) = \sum_{k \in N_> \text{ with } e_k>0} -\frac{D_2 t_k}{D_1 t_k}(e_k, \underline{\mathbf{e}}) < 1.$$

II. Suppose $E = \{\mathbf{e}\}$. If $\mathbf{e} = \mathbf{0}$, then the desired result (trivially) holds. Now suppose $\mathbf{e} \neq \mathbf{0}$. By Proposition 5.5(3), $\mathrm{fix}(\tilde{b}) = \{\underline{\mathbf{e}}\}$. Also $N_> \neq \emptyset$. By Proposition 5.9(2), $D^-\tilde{b}(\underline{\mathbf{e}}) \leq 1$. Now (using Ass. f for the below inequality)

$$\begin{aligned} \sum_{k \in N_>} -\frac{D_2 t_k}{D_1 t_k}(e_k, \underline{\mathbf{e}}) &= \sum_{k \in N_>} -\frac{D_2 t_k}{D_1 t_k}(\tilde{b}_k(\underline{\mathbf{e}}), \underline{\mathbf{e}}) \\ &= \sum_{k \in N_> \text{ with } \tilde{b}_k(\underline{\mathbf{e}})>0} -\frac{D_2 t_k}{D_1 t_k}(\tilde{b}_k(\underline{\mathbf{e}}), \underline{\mathbf{e}}) \\ &\quad + \sum_{k \in N_> \text{ with } \tilde{b}_k(\underline{\mathbf{e}})=0} -\frac{D_2 t_k}{D_1 t_k}(\tilde{b}_k(\underline{\mathbf{e}}), \underline{\mathbf{e}}) \\ &\leq \sum_{k \in N_> \text{ with } \tilde{b}_k(\underline{\mathbf{e}})>0} D^-\tilde{b}_k(\underline{\mathbf{e}}) + \sum_{k \in N_> \text{ with } \tilde{b}_k(\underline{\mathbf{e}})=0} D^-\tilde{b}_k(\underline{\mathbf{e}}) \\ &= \sum_{k \in N_>} D^-\tilde{b}_k(\underline{\mathbf{e}}) = D^-\tilde{b}(\underline{\mathbf{e}}) \leq 1. \end{aligned}$$

□

7. Applications

As Theorem 4.1 deals with abstract aggregative games, one may wish to have applications to concrete games. Although there is a whole list of assumptions in this theorem, they are less demanding than they may look.

The reader is invited to check that Theorem 4.1 implies Theorem 2.1. (Take $t_i(x_i, y) = p'(y)x_i + p(y) - {c_i}'(x_i)$, where $p'(v)$ should be understood as a left derivative, $\tilde{t}_i(x_i, y) = p'(y)x_i + p(y) - \tilde{c}_i'(x_i)$ where $\tilde{c}_i(x_i) = c_i(x_i)$ $(x_i \geq 0)$ and $\tilde{c}_i(x_i) := c_i(0) + c_i'(0)x_i + \frac{1}{2}c_i''(0)x_i^2$ $(x_i < 0)$.)

Besides Cournot oligopolies there are many others aggregative games, like Bertrand oligopolies, public good games, contest games, smash-and-grab games, search games and joint production games ([1, 2, 8]). It is tempting to find out in how far Theorem 4.1 applies to these games. We only look here to transboundary pollution games with global transboundary pollution, being a special type of a public good game. In such a game the players choose an emission level (in order to produce) which causes transboundary pollution. X_i is the set of country i's possible emission levels. An emission of a country not only causes damage in this country but also abroad. Country i gains (monetary) benefits $\mathcal{P}_i(x_i)$ and faces (monetary) damage costs $\mathcal{D}_i(\sum_{l \in N} x_l)$. This leads to the net benefits function $f_i(x_1, \ldots, x_n) := \mathcal{P}_i(x_i) - \mathcal{D}_i(\sum_{l \in N} x_l)$. Now consider the case $X_i = \mathbb{R}_+$, $\mathcal{P}_i : \mathbb{R}_+ \to \mathbb{R}$ twice differentiable and strictly concave and $\mathcal{D}_i : \mathbb{R}_+ \to \mathbb{R}$ twice differentiable and convex, $\mathcal{D}'_i > 0$, $\lim_{x_i \to +\infty} \mathcal{P}'_i(x_i) = 0$ and $\lim_{y \to +\infty} \mathcal{D}'_i(y) = +\infty$. Theorem 4.1 applies to this situation (with $v = +\infty$).

Appendix

Theorem 7.1. *Suppose T is a proper interval of $\mathbb{R}$ with $0 \in T \subseteq \mathbb{R}_+$ and $f : \mathbb{R}_+ \times T \to \mathbb{R}$ is continuous. If for every $t \in T$, there exists a unique $a_\star(t) \in \mathbb{R}_+$ with $f(a_\star(t), t) = 0$ and $f(a, t) < 0$ for every $a > a_\star(t)$, then the function $a_\star : T \to \mathbb{R}$ is continuous.* $\diamond$

Proof. Fix $\bar{t} \in T$ with $\bar{t} > 0$. If we can prove that $a_\star \restriction [0, \bar{t}]$ is continuous, then it follows that $a_\star : T \to \mathbb{R}$ is continuous. As the graph of $a_\star \restriction [0, \bar{t}]$ is closed, continuity of $a_\star \restriction [0, \bar{t}]$ follows if we can show that $a_\star \restriction [0, \bar{t}]$ is bounded.

Let $t \in [0, \bar{t}]$. As $f(a_\star(t), t) = 0$ and $f(a, t) < 0$ for $a > a_\star(t)$, we can fix $a(t) > a_\star(t)$ such that $f(a(t), t) < 0$. As f is continuous at $(a(t), t)$, there exists an open ball $B_{r(t)}(a(t), t)$ in $\mathbb{R}_+ \times [0, \bar{t}]$ with radius $r(t) > 0$ around $(a(t), t)$ on which f is negative.

Let $Z := \cup_{t \in [0, \bar{t}]} B_{r(t)}(a(t), t)$ and $Z' := \cup_{t \in [0, \bar{t}]} (t - r(t), t + r(t))$. Z' is an open covering of the compact set $[0, \bar{t}]$. So there exists $t_1, \ldots, t_m \in [0, \bar{t}]$ such that $[0, \bar{t}] \subseteq \cup_{i=1}^m (t_i - r(t_i), t_i + r(t_i))$. We may suppose that $a(t_1) \geq a(t_k)$ $(1 \leq k \leq m)$.

Now fix $t \in [0, \bar{t}]$. Take $k \in \{1, \ldots, m\}$ such that $t \in (t_k - r(t_k), t_k + r(t_k))$. Then $(a(t_k), t) \in B_{r(t_k)}(a(t_k), t_k)$ and therefore $f(a(t_k), t) < 0$. This implies $a_\star(t) < a(t_k) \leq a(t_1)$. So with $z = a(t_1)$, $f(a, t) < 0$ for all $a > z$ and $t \in [0, \bar{t}]$ and therefore $a_\star \restriction [0, \bar{t}] \leq z$, thus $a_\star \restriction [0, \bar{t}]$ is bounded. □

Closing words

Thanks to Willem Pijnappel for providing Theorem 7.1.

References

[1] L. C. Corchón, *Theories of Imperfectly Competitive Markets*, Lecture Notes in Economics and Mathematical Systems, vol. 442, second edition, Springer-Verlag, Berlin,1996.

[2] R. Cornes and R. Hartley, *Well-behaved aggregative games*, Economic Discussion Paper May 24, School of Social Sciences, The University of Manchester, 2011.

[3] G. Gaudet and S. W. Salant, *Uniqueness of Cournot equilibrium: New results from old methods*, The Review of Economic Studies, **58** (1991), 399–404.

[4] C. D. Kolstad and L. Mathiesen, *Necessary and sufficient conditions for uniqueness of a Cournot equilibrium*, The Review of Economic Studies **54** (1987), 681–690.

[5] F. Szidarovszky and S. Yakowitz, *A new proof of the existence and uniqueness of the Cournot equilibrium*, International Economic Review **18** (1977), 787–789.
[6] X. Vives. *Oligopoly Pricing: Old Ideas and New Tools*, MIT Press, Cambridge, 2001.
[7] P. H. M. von Mouche, F. Quartieri, and F. Szidarovzsky, *On a fixed point problem transformation method*, in Proceedings of the 10th IC-FPTA, Cluj-Napoca, Romania, 2013, pp. 179–190.
[8] T. Yamazaki. *Aggregative Games, Lobbying Models and Endogeneous Tariffs*, Niigata University, Niigata, Japan, 2013.

Manuscript received November 1, 2013
revised May 28, 2014

P. v. Mouche
Wageningen University, The Netherlands
E-mail address: pvmouche@yahoo.fr

T. Yamazaki
Department of Economics, Niigata University, Japan
E-mail address: tyamazak@econ.niigata-u.ac.jp