Journal of Nonlinear and Convex Analysis
Volume 16, Number 2, 2015, 365–373

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

COINCIDENCE POINT THEOREMS FOR MULTI-VALUED MAPPINGS OF REICH-TYPE ON METRIC SPACES ENDOWED WITH A GRAPH

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Abstract. In this paper, we introduce the concepts of G-contraction multi-valued mappings of Reich-type on a metric space endowed with a directed graph G. Some coincidence point theorems for this type of multi-valued mapping and a surjective mapping $g : X \to X$ under some properties on X and some contractive conditions of Reich-type are established. Some examples of mappings of this type and some examples satisfying all conditions of our main theorems are also given. Our main results extend and generalize many coincidence point and fixed point theorems in partially ordered metric spaces in the literature.

1. Introduction

In 1972, Reich [15] proved fixed point theorem for single-valued Reich type operators as follow,

Theorem 1.1. *Let (X, d) be a complete metric space and let $f : X \to X$ be a Reich-type single-valued (a, b, c)-contraction, that is , there exist $a, b, c \in \mathbb{R}^+$, with $a + b + c < 1$ such that*

$$d(f(x), f(y)) \leq ad(x, y) + bd(x, f(x)) + cd(y, f(y)), \text{ for each } x, y \in X.$$

Then f is a Picard operator, that is,

(1) $F_f = \{x^*\}$,
(2) *for each $x \in X$ the sequence $\{f^n(x)\}_{n\in\mathbb{N}}$ converges in (X, d) to x^*.*

For a metric space (X, d), we let $CB(X)$, $Comp(X)$, $P_{cl}(X)$ and $P_b(X)$ to be the set of all nonempty closed bounded subsets of X, all nonempty compact subsets of X, all nonempty closed subsets of X and all bounded subsets of X, respectively. A point $x \in X$ is a fixed point of a multi-valued mapping $T : X \to 2^X$ if $x \in Tx$. The first well-known theorem for multi-valued contraction mappings was given by Nadler [13] in 1967.

Theorem 1.2. *Let (X, d) be a complete metric space and T a mapping from X into $CB(X)$. Assume that there exists $k \in [0, 1)$ such that*

$$H(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X.$$

Then there exists $z \in X$ such that $z \in Tz$.

2010 *Mathematics Subject Classification.* 47H04, 47H10.

Key words and phrases. Fixed point theorems, multi-valued mappings, partially ordered set, monotone mappings, graph-preserving.

This paper was supported by Chiang Mai University, Chiang Mai, Thailand.

Nadler's fixed point theorem for multi-valued contractive mappings has been extended in many directions (see [2], [5], [10], [12]).

Definition 1.3. If (X, d) is a metric space, then a multi-valued operator $T : X \to P_{cl}(X)$ is said to be a Reich-type multi-valued (a, b, c)-contraction if there exist $a, b, c \in \mathbb{R}^+$ with $a + b + c < 1$ such that

$$H(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty), \quad \text{for each } x, y \in X.$$

Reich [15] proved some fixed point theorem for multi-valued Reich-type (a, b, c)-contraction mappings.

From the year of 2003 many results concerning existence of fixed points of both single-valued and multi-valued mappings in metric spaces endowed with a partial ordering were established. The first result in this direction was given by Ran and Reurings [14] and they also presented its applications to linear and nonlinear matrix equations. After that many authors extended those results and investigated fixed point theorems in partially ordered metric spaces (see [14],[3],[4],[1],[9]).

In 2008, Jachymski [11] introduced the concept of G-contraction and proved some fixed point results of G-contractions in a complete metric space endowed with a graph.

Definition 1.4 ([11])**.** Let (X, d) be a metric space and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contains all loops, i.e., $\triangle = \{(x, x) : x \in X\} \subseteq E(G)$.

We say that a mapping $f : X \to X$ is a G-contraction if f preserves edges of G, i.e.,

$$x, y \in X, (x, y) \in E(G) \Rightarrow (f(x), f(y)) \in E(G) \tag{1.1}$$

and there exists $\alpha \in (0, 1)$ such that

$$x, y \in X, (x, y) \in E(G) \Rightarrow d(f(x), f(y)) \leq \alpha d(x, y).$$

Let X_f be the set $\{x \in X : (x, f(x)) \in E(G)\}$. He showed in [11] that under some certain properties on (X, d, G), a G-contraction $f : X \to X$ has a fixed point if and only if X_f is nonempty. The mapping $f : X \to X$ satisfying the condition (1.1) is also called a graph-perserving mapping.

In 2010, Beg, Butt and Radojevic [2] introduced the concept of G-contraction for a multi-valued mapping $T : X \to CB(X)$ and proved some fixed point results of this kind of mappings.

Definition 1.5 ([2])**.** Let $T : X \to CB(X)$ be a multi-valued mapping. The mapping T is said to be a G-contraction if there exists a $k \in (0, 1)$ such that

$$H(Tx, Ty) \leq kd(x, y) \text{ for all } (x, y) \in E(G)$$

and if $u \in Tx$ and $v \in Ty$ are such that

$$d(u, v) \leq kd(x, y) + \alpha, \text{ for each } \alpha > 0$$

then $(u, v) \in E(G)$.

We denote X_F to be the set $\{x \in X : (x, y) \in E(G),$ for some $y \in Tx\}$. They also showed that if X is a complete metric space with a graph G and X has the property (A), then a G-contraction mapping $T : X \to CB(X)$ has a fixed point if and only if X_F is nonempty.

In 2012, Bojor [6] extended the concept of G-contraction defined by Jachymski [11] by using some general contractive condition as follows.

Definition 1.6. Let (X, d) be a metric space. The operator $T : X \to X$ is said to be a G-Ciric-Reich-Rus operator if:

(1) $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$, $\forall x, y \in X$;
(2) there exists nonnegative numbers a, b, c with $a + b + c < 1$, such that, for each $(x, y) \in E(G)$, we have:

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty).$$

We say that G is **T-connected** if for all vertices x, y of G with $(x, y) \notin E(G)$, there exists a path in G, $(x_i)_{i=0}^N$ from x to y such that $x_0 = x$, $x_N = y$ and $(x_i, Tx_i) \in E(G)$ for all $i = 1, 2, \ldots, N-1$.

An operator $T : X \to X$ is said to be a Picard operator if T has a unique fixed point x^* and for each $x \in X$, $T^n x \to x^*$. Bojor [6] proved some fixed point results for G-Ciric-Reich-Rus operator as follows.

Theorem 1.7. *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ a G-Ciric-Reich-Rus operator. Suppose that:*

(1) *G is T-connected.*
(2) *for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x) \in E(G)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for all $n \in \mathbb{N}$.*

Then T is a Picard operator.

In 2013, Chifu, Petrusel and Bota [10] proved some fixed point theorems for a Reich type contraction with respect to the functional δ, where $\delta(A, B) = sup\{d(x, y) : x \in A, y \in B\}$.

Theorem 1.8. *Let (X, d) be a complete metric space and let G be a directed graph such that the triple (X, d, G) satisfies the following property:*

(P) *for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to x$ as $n \to \infty$, there exists a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(x_{k_n}, x) \in E(G)$.*

Let $T : X \to P_b(X)$ be a multi-valued operator. Suppose that the following assertions hold:

(i) *There exist $a, b, c \in \mathbb{R}^+$ with $b \neq 0$ and $a + b + c < 1$ such that*

$$\delta(Tx, Ty) \leq ad(x, y) + b\delta(x, Tx) + c\delta(y, Ty)$$

for all $(x, y) \in E(G)$.

(ii) *For each $x \in X$, the set*

$$\tilde{X} := \left\{ y \in Tx : (x, y) \in E(G) \text{ and } \delta(x, Tx) \leq qd(x, y) \text{ for some } q \in \left(1, \frac{1 - a - c}{b}\right) \right\}$$

is nonempty.

Then

(a) $Fix(T) = SFix(T) \neq \emptyset$.
(b) *If we additionally suppose that*

$$x^*, y^* \in Fix(T) \Rightarrow (x^*, y^*) \in E(G),$$

then $Fix(T) = SFix(T) = \{x^*\}$.

Recently, Dinvari and Frigon [8] introduced a new concept of G-contraction which is weaker than that of Beg, Butt and Radojavic [2].

Definition 1.9 ([8])**.** Let $T : X \to 2^X$ be a map with nonempty values. We say that T is a G-contraction (in the sense of Dinvari and Frigon) if there exists $\alpha \in (0, 1)$ such that

(C_G) for all $(x, y) \in E(G)$ and all $u \in Tx$, there exists $v \in Ty$ such that

$$(u, v) \in E(G) \text{ and } d(u, v) \leq \alpha d(x, y).$$

They showed that under some properties on a metric space which is weaker than Property(A), a multi-valued G-contraction with closed values has a fixed point (see [8], Theorem 2.10 and Corollary 2.11). We note that the concept of G-contraction for multi-valued mappings does not concern the concept of graph-preserving as seen for single-valued mappings. Motivated by this observation and these previous works, we are interested to introduce a concept of graph-preserving for multi-valued mappings and study their fixed point theorems in a complete metric space endowed with a directed graph.

2. Preliminaries

Let (X, d) be a metric space and $CB(X)$ the set of all nonempty closed bounded subsets of X. For $x \in X$ and $A, B \in CB(X)$, define

$$d(x, A) = \inf\{d(x, y) : y \in A\},$$

$$\delta(A, B) = \sup\{d(x, y) : x \in A, y \in B\}.$$

Denote H the Hausdorff metric induced by d, that is

$$H(A, B) = \max\{\sup_{u \in A} d(u, B), \sup_{v \in B} d(v, A)\}.$$

The following lemma which can be found in [13], is very useful for our main result.

Lemma 2.1 ([13])**.** *Let (X, d) be a metric space. If $A, B \in CB(X)$ and $a \in A$, then, for each $\epsilon > 0$ there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \epsilon.$$

Let $G = (V(G), E(G))$ be a directed graph where $V(G)$ is a set of vertices of graph and $E(G)$ is a set of its edges. Assume that G has no parallel edges.

Definition 2.2. Let x and y be vertices in a graph G, then a path in G from x to y of length $n \in \mathbb{N} \cup \{0\}$ is a sequence $\{x_i\}_{i=0}^n$ of $n + 1$ vertices such that $x_0 = x$, $x_n = y$, $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \ldots, n$. A graph G is connected if there is a path between any two vertices of G.

A partial order is a binary relation $\leq$ over the set X which satisfies the following conditions:

(1) $x \leq x$ (reflexivity);
(2) if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry);
(3) if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity);

for all $x, y \in X$. A set with a partial order $\leq$ is called a partially ordered set. We write $x < y$ if $x \leq y$ and $x \neq y$.

Definition 2.3. Let $(X, \leq)$ be a partially ordered set. For each A, $B \subset X$, we write

$$A \prec B \text{ if } a < b \text{ for any } a \in A,\ b \in B.$$

Definition 2.4. Let (X, d) be a metric space endowed with a partial order $\leq$. Let $g : X \to X$ be surjective and $T : X \to CB(X)$, T is said to be g - increasing if for any $x, y \in X$,

$$g(x) < g(y) \Rightarrow Tx \prec Ty.$$

Definition 2.5. Let X be a nonempty set, $G = (V(G), E(G))$ a graph such that $V(G) = X$ and $T : X \to CB(X)$. Then T is said to be graph-perserving if

$$(x, y) \in E(G) \Rightarrow (u, v) \in E(G) \text{ for all } u \in Tx \text{ and } v \in Ty.$$

Definition 2.6. Let X be a nonempty set, $G = (V(G), E(G))$ a graph such that $V(G) = X$ and $T : X \to CB(X)$. Then T is said to be g-graph-perserving if for any $x, y \in X$,

$$(g(x), g(y)) \in E(G) \Rightarrow (u, v) \in E(G) \text{ for all } u \in Tx \text{ and } v \in Ty.$$

3. Main results

We first introduce a new type of G-contraction.

Definition 3.1. Let (X, d) be a metric space and $G = (V(G), E(G))$ a directed graph such that $V(G) = X$. Let $g : X \to X$ be surjective and $T : X \to CB(X)$ a multi-valued mapping. T is said to be a **generalized G-contraction with respect to** g if there exist $a, b, c \in \mathbb{R}^+$ with $a + b + c < 1$ and $k \in (0, 1)$ such that

(1) $H(Tx, Ty) \leq ad(g(x), g(y)) + bd(g(x), Tx) + cd(g(y), Ty)$
for all $x, y \in X$ such that $(g(x), g(y)) \in E(G)$.
(2) If $x, y \in X$, $(g(x), g(y)) \in E(G)$ and $u \in Tx$, $v \in Ty$ are such that

$$d(u, v) \leq kd(g(x), g(y)) + \alpha \quad \text{for some } 0 < \alpha < 1,$$

then $(u, v) \in E(G)$.

Example 3.2. Let $G = (X, E(G))$ where $X = \{2, 4, 5, 6, 7, 9, 10, 11, 21\}$, $E(G) = \{(2, 4), (2, 10), (4, 6)\} \bigcup \{(5, 5), (5, 7), (5, 9), (7, 5), (7, 7), (7, 9), (9, 9), (9, 11), (11, 21)\}$ and the Euclidean metric $d(x, y) = |x - y|$ for any $x, y \in X$. Define $T : X \to CB(X)$ by

$$T(x) = \begin{cases} \{7, 9\}, & \text{if } x \in \{4, 10\} \\ \{9, 11\}, & \text{if } x = 6 \\ \{5, 7\}, & \text{if } x \in \{2, 5, 7, 9, 11, 21\}. \end{cases}$$

Let $(x, y) \in E(G)$. If $(x, y) = (2, 4)$, then $T2 = \{5, 7\}$, $T4 = \{7, 9\}$, $H(T2, T4) = 2 \leq 0.7d(2, 4) + 0.1d(2, T2) + 0.1d(4, T4)$ and $(5, 7), (5, 9), (7, 7), (7, 9) \in E(G)$.

If $(x, y) = (2, 10)$, then $T2 = \{5, 7\}$, $T10 = \{7, 9\}$, $H(T2, T10) = 2 \leq 0.7d(2, 10) + 0.1d(2, T2) + 0.1d(10, T10)$ and $(5, 7), (5, 9), (7, 7), (7, 9) \in E(G)$.

If $(x, y) = (4, 6)$, then $T4 = \{7, 9\}$, $T6 = \{9, 11\}$, $H(T4, T6) = 2 \leq 0.7d(4, 6) + 0.1d(4, T4) + 0.1d(6, T6)$ and $(7, 9), (7, 11), (9, 9), (9, 11) \in E(G)$.

If $(x, y) \in \{(5, 5), (5, 7), (5, 9), (7, 5), (7, 7), (7, 9), (9, 9), (9, 11), (11, 21)\}$, then $Tx = Ty = \{5, 7\}$ and we see that $(5, 5), (5, 7), (7, 5), (7, 7) \in E(G)$, hence T satisfies (1) and (2) of Definition 3.1. Therefore T is a generalized G-contraction with respect to g with $a = 0.7$, $b = 0.1$, $c = 0.1$, where g is an identity function.

Example 3.3. Let $G = (X, E(G))$ where $X = \{3, 4, 5, 6, 7, 10, 20\}$, $E(G) = \{(3, 3), (3, 4), (4, 3), (4, 4), (4, 5), (5, 6), (6, 7), (10, 20)\}$ and the Euclidean metric $d(x, y) = |x - y|$ for any $x, y \in X$. Define $T : X \to CB(X)$ by

$$T(x) = \begin{cases} \{3, 4\}, & \text{if } x \in \{3, 4, 5, 6, 7, 10\} \\ \{4, 5\}, & \text{if } x = 20 \end{cases}$$

Define $g : X \to X$ by $g(3) = 10, g(4) = 4, g(5) = 6, g(6) = 5, g(7) = 20, g(10) = 3$ and $g(20) = 7$. We will show T is a generalized G-contraction with respect to g. Let $(g(x), g(y)) \in E(G)$.

If $(x, y) \in \{(3, 3), (3, 4), (4, 3), (4, 4), (4, 5), (5, 6), (10, 20)\}$, then $Tx = Ty = \{3, 4\}$ and we see that $(3, 3), (3, 4), (4, 3), (4, 4) \in E(G)$, hence T satisfies (1) and (2) of Definition 3.1. If $(g(x), g(y)) = (6, 7)$, then $(x, y) = (5, 20)$, $T5 = \{3, 4\}$, $T20 = \{4, 5\}$, $H(T6, T7) = 1 \leq 0.1d(g(5), g(20)) + 0.5d(g(5), T5) + 0.2d(g(20), T20)$ and $(3, 4), (4, 4), (4, 5) \in E(G)$, hence T satisfies (1) and (2) of Definition 3.1.

If $(g(x), g(y)) = (10, 20)$, then $(x, y) = (3, 7)$, $T3 = T7 = \{3, 4\}$, $H(T3, T7) = 0$ and $(3, 3), (3, 4), (4, 3), (4, 4) \in E(G)$, hence T satisfies (1) and (2) of Definition 3.1.

Therefore T is a generalized G-contraction with respect to g with $a = 0.1$, $b = 0.5$ and $c = 0.2$.

Property A ([11])**.** For any sequence $(x_n)_{n \in \mathbb{N}}$ in X. If $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then there is a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

The following main theorem is proposed to guarantee the existence of coincidence point for multi-valued mappings generalized G-contraction with respect to g.

Theorem 3.4. *Let (X, d) be a complete metric space and $G = (V(G), E(G))$ a directed graph such that $V(G) = X$. Let $T : X \to CB(X)$ be a multi-valued mapping. Suppose that*

(1) *T is a generalized G-contraction with respect to g;*
(2) *there exists $x_0 \in X$ such that $(g(x_0), y) \in E(G)$, where $y \in Tx_0$;*
(3) *X has the property A;*

Then there exists $u \in X$ such that $g(u) \in Tu$.

Proof. Since g is a surjective, there exists $x_1 \in X$ such that $g(x_1) \in Tx_0$. By (2) we obtain $(g(x_0), g(x_1)) \in E(G)$. By (1), we get

$$H(Tx_0, Tx_1) \leq ad(g(x_0), g(x_1) + bd(g(x_0), Tx_0) + cd(g(x_1), Tx_1)$$

$$\leq ad(g(x_0), x_1) + bd(g(x_0), g(x_1) + cH(Tx_0, Tx_1).$$

Therefore

$$H(Tx_0, Tx_1) \leq \frac{a+b}{1-c} d(g(x_0), g(x_1)). \tag{3.1}$$

Let $k = \frac{a+b}{1-c}$ and $\alpha \in (0, 1)$. By Lemma 2.1, there exists $g(x_2) \in Tx_1$ such that

$$d(g(x_1), g(x_2)) \leq H(Tx_0, Tx_1) + \alpha. \tag{3.2}$$

Moreover, by (3.1) and (3.2), we get

$$\begin{aligned} d(g(x_1), g(x_2)) &\leq H(Tx_0, Tx_1) + \alpha \\ &\leq kd(g(x_0), g(x_1)) + \alpha. \end{aligned}$$

By (1), we obtain $(g(x_1), g(x_2)) \in E(G)$. Next, by assumption (1), we have

$$\begin{aligned} H(Tx_1, Tx_2) &\leq ad(g(x_1), g(x_2) + bd(g(x_1), Tx_1) + cd(g(x_2), Tx_2) \\ &\leq ad(g(x_1), x_2) + bd(g(x_1), g(x_2) + cH(Tx_1, Tx_2). \end{aligned}$$

Therefore

$$H(Tx_1, Tx_2) \leq \frac{a+b}{1-c} d(g(x_1), g(x_2)) \leq \left(\frac{a+b}{1-c}\right)^2 d(g(x_0), g(x_1)). \tag{3.3}$$

By Lemma 2.1, there exists $g(x_3) \in Tx_2$ such that

$$d(g(x_2), g(x_3)) \leq H(Tx_1, Tx_2) + \alpha^2. \tag{3.4}$$

Moreover, by (3.3) and (3.4), we get

$$\begin{aligned} d(g(x_2), g(x_3)) &\leq H(Tx_1, Tx_2) + \alpha^2 \\ &\leq \left(\frac{a+b}{1-c}\right)^2 d(g(x_0), g(x_1)) + \alpha^2 \\ &\leq kd(g(x_0), g(x_1)) + \alpha^2. \end{aligned}$$

By (1), we obtain $(g(x_2), g(x_3)) \in E(G)$. By induction, we obtain a sequence $\{g(x_n)\}$ in X with the property that for each $n \in \mathbb{N}$, $g(x_{n+1}) \in Tx_n$, $(g(x_n), g(x_{n+1})) \in E(G)$ and

$$d(g(x_n), g(x_{n+1}) \leq \left(\frac{a+b}{1-c}\right)^n d(g(x_0), g(x_1)) + \alpha^n. \tag{3.5}$$

Because $\alpha \in (0, 1)$ and using (3.5), we obtain

$$\sum_{n=0}^{\infty} d(g(x_n), g(x_{n+1}) \leq d(g(x_0), g(x_1)) \sum_{n=0}^{\infty} \left(\frac{a+b}{1-c}\right)^n + \sum_{n=0}^{\infty} \alpha^n < \infty,$$

then $(g(x_n))_{\in \mathbb{N}}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $\lim_{n\to\infty} g(x_n) = g(u)$. By assumption (3), there is a subsequence $g(x_{n_k})$ such that $(g(x_{n_k}), g(u)) \in E(G)$ for any $k \in \mathbb{N}$. Consider

$$\begin{aligned} D(g(u), Tu) &\leq d(g(u), g(x_{n_k+1})) + d(g(x_{n_k+1}), Tu) \\ &\leq d(g(u), g(x_{n_k+1})) + H(Tx_{n_k}, Tu) \\ &\leq d(g(u), g(x_{n_k+1})) + ad(g(x_{n_k}, g(u))) \end{aligned}$$

$$+ bd(g(x_{n_k}, Tx_{n_k})) + cd(g(u), Tu)$$
$$\leq d(g(u), g(x_{n_k+1})) + ad(g(x_{n_k}, g(u)))$$
$$+ bd(g(x_{n_k}), g(x_{n_k+1})) + cd(g(u), Tu).$$

Then we obtain

$$d(g(u), Tu) \leq \frac{1}{1-c} d(g(u), g(x_{n_k+1})) + \frac{a}{1-c} d(g(x_{n_k}, g(u))) + \frac{b}{1-c} d(g(x_{n_k}), g(x_{n_k+1})).$$

Since $g(x_{n_k})$ converges to $g(u)$ as $n \to \infty$, it follows that $d(g(u), Tu) = 0$. Since Tu is closed, we conclude that $g(u) \in Tu$. □

Remark 3.5. We can directly check that Example 3.3 satisfies all conditions in Theorem 3.4 and $F(T) = \{3, 4\}$.

The following corollary is an existence of a coincidence point for multi-valued mappings in partially ordered metric spaces.

Corollary 3.6. *Let (X, d) be a complete metric space endowed with a partial ordered $\leq$, $g : X \to X$ a surjection and $T : X \to CB(X)$ a multi-valued mapping. Suppose that*

(1) *T is g - increasing;*
(2) *there exist $x_0 \in X$ and $u \in Tx_0$ such that $g(x_0) < u$;*
(3) *For each sequence $\{x_k\}$ such that $g(x_k) < g(x_{k+1})$ for all $k \in \mathbb{N}$ and $g(x_k)$ converges to $g(x)$, for some $x \in X$, then $g(x_k) < g(x)$ for all $k \in \mathbb{N}$;*
(4) *there exist $a, b, c \in \mathbb{R}^+$ with $a + b + c < 1$ such that*
$$H(Tx, Ty) \leq ad(g(x), g(y)) + bd(g(x), Tx) + cd(g(y), Ty)$$
for all $x, y \in X$ such that $g(x) < g(y)$.

Then there exists $u \in X$ such that $g(u) \in Tu$.

Proof. Define $G = (V(G), E(G))$ by $V(G) = X$ and $E(G) = \{(x, y) : x < y\}$. Let $x, y \in X$ such that $(g(x), g(y)) \in E(G)$. Then $g(x) < g(y)$ so $Tx \prec Ty$. For any $u \in Tx$ and $v \in Ty$, we have $u < v$ i.e., $(u, v) \in E(G)$, and (4), we conclude that T is a generalized G-contraction with respect to g. By assumption (2), there exist x_0 and $u \in Tx_0$ such that $g(x_0) < u$, so $(g(x_0), u) \in E(G)$. Hence (2) of Theorem 3.4 is satisfied. It is easy to see that (3) of Theorem 3.4 is also satisfied. Therefore the Corollary 3.6 is obtained directly by Theorem 3.4. □

References

[1] I. Beg and A. R. Butt, *Fixed point for set valued mappings satisfying an implicit relation in partially ordered metric spaces*, Nonlinear Anal. **71** (2009), 3699–3704.

[2] I. Beg and A. R. Butt, *The contraction principle for set valued mappings on a metric space with graph*, Comput. Math. Appl. **60** (2010), 1214–1219.

[3] I. Beg and A. R. Butt, *Common fixed point for generalized set valued contractions satisfying an implicit relation in partially ordered metric spaces*, Math. Commun. **15** (2010), 65–75.

[4] I. Beg and A. Latif, *Common fixed point and coincidence point of generalized contractions in ordered metric spaces*, Fixed point theory and Applications **2012** (2012):229.

[5] M. Berinde and V. Berinde, *On a general class of multi-valued weakly Picard mappings*, J. Math Anal. Appl. **326** (2007), 772–782.
[6] F. Bojor, *Fixed point theorems for Reich type contractions on metric spaces with a graph*, Nonlinear Anal. **75** (2012), 3895–3901.
[7] E. Defez, L. Jódar and A. Law, *Jacobi matrix differential equation, polynomial solutions and their properties*, Comput. Math. Appl. **48** (2004), 789–803.
[8] T. Dinevari and M. Frigon, *Fixed point results for multivalued contractions on a metric space with a graph*, J. Math. Anal. Appl. **405** (2013), 507–517.
[9] T. G. Bhaskar and V. Laskhmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal TMA. **65** (2006), 1379–1393.
[10] C. Chifu, G. Petrusel and M. Bota, *Fixed points and strict fixed points for multivalued contractions of Reich type on metric spaces endowed with a graph*, Fixed Point Theory Appl. **2013** (2013):203.
[11] J. Jachymski, *The contraction principle for mappings on a metric with a graph*, Proc. Amer. Math. Soc. **1** (2008), 1359–1373.
[12] N. Mizoguchi and W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces*, J. Math. Anal. Appl. **141** (1989), 177–188.
[13] S. Nadler, *Multi-valued contraction mappings*, Pacific J Math. **20** (1969), 475–488.
[14] A. C. M. Ran and M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. **132** (2003), 1435–1443.
[15] S. Reich, *Fixed points of contractive functions*, Boll. Un. Mat. Ital. **5** (1972), 26–42.

Manuscript received October 25, 2013
revised March 18, 2014

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