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EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTION FOR A CLASS OF BOUNDARY VALUE PROBLEMS WITH FRACTIONAL q-DIFFERENCES

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Abstract. In this paper, we use the fixed point theorem in partially ordered sets, to establish the existence and uniqueness of positive solution to the nonlinear q-fractional boundary value problem

$$(D_q^\alpha u)(x) = -f(x,u(x)), \quad 0 < x < 1,$$

$$(D_q^i u)(0) = 0, \quad i = 0,\dots,n-2, \quad D_q^2 u(1) = \beta \sum_{i=1}^{m} D_q^2 u(\eta_i),$$

where $q \in (0,1), m \geq 1$ and $n \geq 3$ are integers, $n-1 < \alpha \leq n$ and $\beta, \eta_i \in (0,1)$ for $i = 1,\dots,m$, $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function. Also, we give an illustrative example in order to indicate the validity of the assumptions.

1. Introduction

Recently, fractional differential calculus have attracted a lot of attention by many researchers of different fields, such as; physics, chemistry, biology, economics, control theory, and biophysics, etc. [17, 18, 21, 23]. In particular, the existence of solutions to fractional boundary value problems is recently under strong research, see [5, 6, 7] and references therein. The q-difference calculus or quantum calculus is an old subject that was initially developed by Jackson [14, 15]. It is rich in history and in applications as the reader can confirm in the work by Ernst [9]. Fractional q-difference equations have gained attention of several researchers. For some earlier work on the topic, we refer to [2] and Agarwal [1], whereas some new work on existence solutions of fractional q-difference equations can be found in [10, 11, 12, 20]. Ferreira [10], studied the existence of positive solutions to nonlinear q-difference boundary value problem:

$$(D_q^\alpha u)(x) = -f(x,u(x)), \quad 0 < x < 1, 1 < \alpha \leq 2,$$
$$u(0) = u(1) = 0.$$

In other paper, Ferreira [11], considered the existence of positive solutions to nonlinear q-difference boundary value problem:

$$(D_q^\alpha u)(x) = -f(x,u(x)), \quad 0 < x < 1, 2 < \alpha \leq 3,$$
$$u(0) = (D_q u)(0) = 0,$$
$$(D_q u)(1) = \beta \geq 0.$$

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In both papers, he applied a fixed point theorem in cones. M.El-Shahed and Farah M.Al-Askar [8] studied the existence of positive solutions to nonlinear q-difference equation:

$$\begin{aligned}&({}_CD_q^{\alpha}u)(x)+a(x)f(u(x))=0, \quad 0\le x\le 1,\ 2<\alpha\le 3,\\&u(0)=(D_q^2u)(0)=0,\\&\gamma D_qu(1)+\beta D_q^2u(1)=0,\end{aligned}$$

where $\gamma,\beta\ge 0$ and ${}_CD_q^{\alpha}$ is the fractional q-derivative of Caputo type. Inspire of [19], we consider the existence and uniqueness of positive solution for nonlinear q-difference boundary value problem of the form

$$\begin{aligned}&(D_q^{\alpha}u)(x)=-f(x,u(x)), \quad 0\le x\le 1,\ n-1<\alpha\le n,\ n\ge 3,\\&(D_q^{i}u)(0)=0, \quad i=0,\dots,n-2,\\&D_q^2u(1)=\beta\sum_{i=1}^{m}D_q^2u(\eta_i), \quad \beta,\eta_i\in(0,1)\ for\ i=1,\dots,m.\end{aligned} \tag{1.1}$$

by using the fixed point theorem in partially ordered sets that was introduced by Amini-Harandi [3].

2. Preliminaries

In this section, we present some definitions and results which will be needed later, for more information see [1, 10, 11, 16, 24]. Let $q\in(0,1)$, define

$$[a]_q=\frac{1-q^a}{1-q}, \quad a\in\mathbb{R}.$$

The q-analogue of the power function $(a-b)^n$ with $n\in\mathbb{N}_0:=\{0,1,2,\dots\}$ is

$$(a-b)^0=1, \quad (a-b)^n=\prod_{k=0}^{n-1}(a-bq^k), \quad n\in\mathbb{N}, a,b\in\mathbb{R}.$$

Moreover, if $\alpha\in\mathbb{R}$, then

$$(a-b)^{(\alpha)}=a^{\alpha}\prod_{n=0}^{\infty}\frac{a-bq^n}{a-bq^{\alpha+n}}.$$

The q-gamma function is defined by

$$\Gamma_q(x)=\frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x\in\mathbb{R}\backslash\{0,-1,-2,\dots\},$$

and satisfies $\Gamma_q(x+1)=[x]_q\Gamma_q(x)$. The q-derivative of a function f is defined by

$$(D_qf)(x)=\frac{f(x)-f(qx)}{(1-q)x}, \quad (x\neq 0),\ (D_qf)(0)=\lim_{x\to 0}(D_qf)(x),$$

and q-derivatives of higher order by

$$(D_q^0f)(x)=f(x), \quad (D_q^nf)(x)=D_q(D_q^{n-1}f)(x), \quad n\in\mathbb{N}.$$

The q-integral of a function f defined on [0,b] is given by

$$(I_qf)(x)=\int_0^x f(t)d_qt=x(1-q)\sum_{n=0}^{\infty}f(xq^n)q^n, 0\le |q|<1, x\in[0,b].$$

If $a\in[0,b]$ and f is defined on the interval [0,b], its integral from a to b is defined by

$$\int_a^b f(t)d_qt=\int_0^b f(t)d_qt-\int_0^a f(t)d_qt.$$

Similar for derivatives, it can be defined an operator I_q^n , namely,

$$(I_q^0f)(x)=f(x),\quad (I_q^nf)(x)=I_q(I_q^{n-1}f)(x),\quad n\in\mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_qI_qf)(x)=f(x),$$

and if f is continuous at $x=0$, then

$$(I_qD_qf)(x)=f(x)-f(0).$$

Here three formulas are presented that will be used later, ($_iD_q$ denotes the derivative with respect to variable i)

$$[a(t-s)]^{(\alpha)}=a^{\alpha}(t-s)^{(\alpha)}. \tag{2.1}$$

$$_tD_q(t-s)^{(\alpha)}=[\alpha]_q(t-s)^{(\alpha-1)}. \tag{2.2}$$

$$_sD_q(t-s)^{(\alpha)}=-[\alpha]_q(t-qs)^{(\alpha-1)}. \tag{2.3}$$

$$_xD_q\int_0^x f(x,t)d_qt=\int_0^x {}_xD_qf(x,t)d_qt+f(qx,x). \tag{2.4}$$

Lemma 2.1 ([10])**.** *If $\alpha>0$ and $a\le b\le t$, then $(t-a)^{(\alpha)}\ge(t-b)^{(\alpha)}$.*

Definition 2.2. Let $\alpha\ge0$ and f be a function defined on [0,1]. The fractional q-integral of the Riemann-Liouville type is $(I_q^0f)(x)=f(x)$ and

$$(I_q^{\alpha}f)(x)=\frac{1}{\Gamma_q(\alpha)}\int_0^x(x-qt)^{(\alpha-1)}f(t)d_qt,\quad \alpha>0, x\in[0,1].$$

The formula for q-integration by parts is

$$\int_a^b u(x)(D_qv)(x)d_q(x)=[u(x)v(x)]_a^b-\int_a^b v(qx)(D_qu)(x)d_qx.$$

The following lemma is stated in [24].

Lemma 2.3. *For $\alpha\in\mathbb{R}^+,\lambda\in(-1,\infty)$, the following is valid*

$$I_q^{\alpha}((x-a)^{(\lambda)})=\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)}(x-a)^{(\alpha+\lambda)}.$$

Particularly, for $\lambda=0$, using a q-integration by parts and the relation (2.3) we have

$$(I_q1)(x)=\frac{1}{\Gamma_q(\alpha)}\int_0^x(x-qt)^{(\alpha-1)}d_qt=\frac{1}{\Gamma_q(\alpha)}\int_0^x\frac{D_q((x-t)^{(\alpha)})}{-[\alpha]_q}d_qt$$

$$= \frac{-1}{\Gamma_q(\alpha+1)} \int_0^x D_q((x-t)^{(\alpha)}) d_q t = \frac{1}{\Gamma_q(\alpha+1)} (x-a)^{(\alpha)}.$$

Definition 2.4 ([24])**.** The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined by $(D_q^0 f)(x) = f(x)$ and

$$(D_q^\alpha f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Now, we list some properties that can be found in [1, 24].

Lemma 2.5. *Let $\alpha, \beta \geq 0$ and f be a function defined on [0,1]. Then, the next formulas hold:*

1. $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x)$.
2. $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

For the forthcoming analysis, we state the following theorem that was proved in [10].

Theorem 2.6. *Let $\alpha \geq 0$ and $n \in \mathbb{N}$. Then, the following equality holds:*

$$(I_q^\alpha D_q^p f)(x) = (D_q^p I_q^\alpha f)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} (D_q^k f)(0).$$

In the following we state a fixed point theorem in partially ordered set, which was introduced in [3].

Let S denote the class of those functions $\gamma : [0,\infty) \to [0,1)$ which satisfies the condition $\gamma(t_n) \to 1$ implies $t_n \to 0$.

Theorem 2.7. *Let $(M, \leq)$ be a partially ordered set and suppose that there exists a metric d in M such that (M, d) is a complete metric space. Let $T : M \to M$ be an increasing mapping such that there exists an element $x_0 \in M$ with $x_0 \leq Tx_0$. Suppose that there exists $\gamma \in S$ such that for each $x, y \in M$ with $y \leq x$*

$$d(Tx, Ty) \leq \gamma(d(x, y)) d(x, y).$$

Then T has a fixed point if either T is continuous or M satisfies the following conditions

(A) *if an increasing sequence $\{x_n\} \to x$ in M; then $x_n \leq x, \quad \forall n$.*
(B) *Moreover, the fixed point is unique, if for each $x, y \in M$, there exists $z \in M$ which is comparable to x and y.*

Remark 2.8. The basic space used in this paper is $C[0,1]$, which is a Banach space with the norm $||u|| = \max_{0 \leq x \leq 1} |u(x)|$, and can be equipped with a partial order given by

$$u, v \in C[0,1], \quad u \leq v \Leftrightarrow u(x) \leq v(x).$$

In [22], it is proved that $(C[0,1], \leq)$, satisfied codition (A) of Theorem 2.7 and, for $u, v \in C[0,1]$ the function $\max\{u, v\} \in C[0,1]$, so $(C[0,1], \leq)$ satisfies condition (B) of Theorem 2.7.

3. Fractional boundary value problem

In this section, we investigate sufficient conditions for the existence and uniqueness solution for the boundary value problem (1.1). First, we state the following lemma;

Lemma 2.1. *Let $\eta_i \in (0,1)$ and $\beta \neq \frac{1}{\sum_{i=1}^{m} \eta_i^{\alpha-3}}$. If $f \in C[0,1]$, then the boundary value problem*

$$(D_q^\alpha u)(x) = -f(x), \quad 0 < x < 1,\ n-1 < \alpha \leq n,\ n \geq 3,$$

$$(D_q^i u)(0) = 0, \quad i = 0, \dots, n-2, \quad D_q^2 u(1) = \beta \sum_{i=1}^{m} D_q^2 u(\eta_i),$$

has a unique solution

$$u(x) = \int_0^1 G(x, qt) f(t) d_q(t) + \frac{\beta x^{\alpha-1}}{\Gamma_q(\alpha)(1 - \beta \sum_{i=1}^{m} \eta_i^{\alpha-3})} \int_0^1 H(\eta_i, qt) f(t) d_q(t).$$

where

$$G(x,t) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} x^{\alpha-1}(1-t)^{(\alpha-3)} - (x-t)^{(\alpha-1)}, & 0 \leq t \leq x \leq 1, \\ x^{\alpha-1}(1-t)^{(\alpha-3)}, & 0 \leq x \leq t \leq 1. \end{cases}$$

and

$$H(\eta_i, t) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} \sum_{i=1}^{m} \eta_i^{\alpha-3}(1-t)^{(\alpha-3)} - \sum_{i=1}^{m} (\eta_i - t)^{(\alpha-3)}, & 0 \leq t \leq \eta_i \leq 1, \\ \sum_{i=1}^{m} \eta_i^{\alpha-3}(1-t)^{(\alpha-3)}, & 0 \leq \eta_i \leq t \leq 1. \end{cases}$$

Proof. By Lemma 2.5 and Theorem 2.6 for $p = n$, we have

$$\begin{aligned} (D_q^\alpha u)(x) = -f(x) \Leftrightarrow (I_q^\alpha D_q^n I_q^{n-\alpha} u)(x) = -I_q^\alpha f(x) \Leftrightarrow \\ u(x) = c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + \cdots + c_{n-1} x^{\alpha-n+1} \\ + c_n x^{\alpha-n} - \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t) d_q(t) \end{aligned}$$

for some constant $c_i \in \mathbb{R}$, for $1 \leq i \leq n$.

Applying the boundary condition $(D_q^i u)(0) = 0,\ i = 0, \dots, n-2$, we have $c_2 = c_3 = \cdots = c_n = 0$. So,

$$u(x) = c_1 x^{\alpha-1} - \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t) d_q(t),$$

$$D_q u(x) = [\alpha-1] c_1 x^{\alpha-2} - \frac{1}{\Gamma_q(\alpha)} \int_0^x [\alpha-1](x-qt)^{(\alpha-2)} f(t) d_q(t),$$

and

$$D_q^2 u(x) = [\alpha-1][\alpha-2] c_1 x^{\alpha-3} - \frac{1}{\Gamma_q(\alpha)} \int_0^x [\alpha-1][\alpha-2](x-qt)^{(\alpha-3)} f(t) d_q(t).$$

Now, using the boundary condition $D_q^2 u(1) = \beta \sum_{i=1}^{m} D_q^2 u(\eta_i)$, we have

$$D_q^2 u(1) = [\alpha-1][\alpha-2] c_1 - \frac{1}{\Gamma_q(\alpha)} \int_0^1 [\alpha-1][\alpha-2](1-qt)^{(\alpha-3)} f(t) d_q(t)$$

$$= \beta\Big(\sum_{i=1}^{m}[\alpha-1][\alpha-2]c_1\eta_i^{\alpha-3}\Big)$$
$$-\beta\Big(\sum_{i=1}^{m}\frac{1}{\Gamma_q(\alpha)}\int_0^{\eta_i}[\alpha-1][\alpha-2](\eta_i-qt)^{(\alpha-3)}f(t)d_q(t)\Big).$$

So

$$c_1 = \frac{1}{\Gamma_q(\alpha)(1-\beta\sum_{i=1}^{m}\eta_i^{\alpha-3})}\Big(\int_0^1(1-qt)^{(\alpha-3)}f(t)d_q(t)$$
$$-\beta\sum_{i=1}^{m}\int_0^{\eta_i}\beta(\eta_i-qt)^{(\alpha-3)}f(t)d_q(t)\Big).$$

Therefore the unique solution of the problem (1.1) is

$$\begin{aligned}
u(x) &= \frac{-1}{\Gamma_q(\alpha)}\int_0^x(x-qt)^{(\alpha-1)}f(t)d_q(t)\\
&\quad+\frac{x^{\alpha-1}}{\Gamma_q(\alpha)(1-\beta\sum_{i=1}^{m}\eta_i^{\alpha-3})}\Big(\int_0^1(1-qt)^{(\alpha-3)}f(t)d_q(t)\\
&\quad-\beta\sum_{i=1}^{m}\int_0^{\eta_i}(\eta_i-qt)^{(\alpha-3)}f(t)d_q(t)\Big)\\
&= \frac{-1}{\Gamma_q(\alpha)}\int_0^x(x-qt)^{(\alpha-1)}f(t)d_q(t)\\
&\quad+\Big(\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}+\frac{\beta x^{\alpha-1}\sum_{i=1}^{m}\eta_i^{\alpha-3}}{\Gamma_q(\alpha)(1-\beta\sum_{i=1}^{m}\eta_i^{\alpha-3})}\Big)\int_0^1(1-qt)^{(\alpha-3)}f(t)d_q(t)\\
&\quad-\frac{\beta x^{\alpha-1}}{\Gamma_q(\alpha)(1-\beta\sum_{i=1}^{m}\eta_i^{\alpha-3})}\sum_{i=1}^{m}\int_0^{\eta_i}(\eta_i-qt)^{(\alpha-3)}f(t)d_q(t)\\
&= \frac{1}{\Gamma_q(\alpha)}\int_0^x(x^{\alpha-1}(1-qt)^{(\alpha-3)}-(x-qt)^{(\alpha-1)})f(t)d_q(t)\\
&\quad+\frac{1}{\Gamma_q(\alpha)}\int_x^1 x^{\alpha-1}(1-qt)^{(\alpha-3)}f(t)d_q(t)\\
&\quad+\frac{\beta x^{\alpha-1}\sum_{i=1}^{m}\eta_i^{\alpha-3}}{\Gamma_q(\alpha)(1-\beta\sum_{i=1}^{m}\eta_i^{\alpha-3})}\int_0^1(1-qt)^{(\alpha-3)}f(t)d_q(t)\\
&\quad-\frac{\beta x^{\alpha-1}}{\Gamma_q(\alpha)(1-\beta\sum_{i=1}^{m}\eta_i^{\alpha-3})}\sum_{i=1}^{m}\int_0^{\eta_i}(\eta_i-qt)^{(\alpha-3)}f(t)d_q(t)\\
&= \int_0^1 G(x,t)f(t)d_q(t)+\frac{\beta x^{\alpha-1}}{\Gamma_q(\alpha)(1-\beta\sum_{i=1}^{m}\eta_i^{\alpha-3})}\int_0^1 H(\eta_i,t)f(t)d_q(t).
\end{aligned}$$

□

Lemma 2.2. *G is a continuous function and $G(x,t) > 0$.*

Proof. The continuity of G is clear. on the other hand, put

$$g_1(x,t) = x^{\alpha-1}(1-t)^{(\alpha-3)} - (x-t)^{(\alpha-1)},$$

$$g_2(x,t) = x^{\alpha-1}(1-t)^{(\alpha-3)},$$

obviously, $g_2(x,t) \geq 0$. Now

$$\begin{aligned} g_1(x,qt) &= x^{\alpha-1}(1-qt)^{(\alpha-3)} - (x-qt)^{(\alpha-1)} \\ &\geq x^{\alpha-1}\Big((1-qt)^{(\alpha-3)} - (1-q\frac{t}{x})^{(\alpha-1)}\Big) \\ &\geq x^{\alpha-1}\Big((1-qt)^{(\alpha-3)} - (1-qt)^{(\alpha-1)}\Big) \geq 0. \end{aligned}$$

Therefore, $G(x,qt) \geq 0$. □

Let $\mathscr{A}$ denote the class of those functions $\varphi : [0,\infty) \to [0,\infty)$ which satisfies the following conditions:

(a) φ is increasing,
(b) for each $x > 0$, $\varphi(x) < x$,
(c) $\gamma(x) = \frac{\varphi(x)}{x} \in S$.

For example, $\varphi(t) = \mu t$ where $0 \leq \mu < 1$, $\varphi(t) = \frac{t}{1+t}$ are in $\mathscr{A}$.

The main result of this paper is the following;

Theorem 2.3. *The boundary value problem (1.1) has a unique positive solution $u(x)$, if the following conditions are satisfied:*

(i) $f : [0,1] \times [0,\infty) \to [0,\infty)$ *is continuous and nondecreasing with respect to the second variable, there exists $\varphi \in \mathscr{A}$ such that*

$$f(t,u) - f(t,v) \leq \varphi(u-v), \quad \forall u \geq v.$$

(ii) *put*

$$K = \int_0^1 G(1,qt)d_qt, \quad L = \int_0^1 H(\eta_i,qt)d_qt,$$

and let

$$K + \frac{\beta}{(1-\beta\sum_{i=1}^{m}\eta_i^{\alpha-3})}L \leq 1.$$

where G and H are defined as in Lemma 2.1.

Proof. Consider the cone

$$M = \{u \in C[0,1]; u(x) \geq 0\}.$$

M can be equipped with a partial order given by

$$u,v \in C[0,1], u \leq v \Leftrightarrow u(x) \leq v(x).$$

so $(M,\leq)$ is a partially ordered set. Also, Since M is a closed subset of $C[0,1]$, M is a complete metric space with the distance

$$d(u,v) = \sup_{x\in[0,1]} |u(x) - v(x)|.$$

Now, we consider the operator T as follows;

$$Tu(x)=\int_0^1 G(x,qt)f(t,u(t))d_q(t)+\frac{\beta x^{\alpha-1}}{(1-\beta\sum_{i=1}^m \eta_i^{\alpha-3})}\int_0^1 H(\eta_i,qt)f(t,u(t))d_q(t).$$

By Lemma 2.2 and condition (i), we have $T(M)\subset M$. We now show that all the condition of Theorem 2.7 are satisfied. Firstly, by condition (i), we show that T is an increasing mapping, so for $u,v\in M$,$u\geq v$,

$$\begin{aligned}Tu(x)&=\int_0^1 G(x,qt)f(t,u(t))d_q(t)\\&\quad+\frac{\beta x^{\alpha-1}}{(1-\beta\sum_{i=1}^m \eta_i^{\alpha-3})}\int_0^1 H(\eta_i,qt)f(t,u(t))d_q(t)\\&\geq\int_0^1 G(x,qt)f(t,v(t))d_q(t)\\&\quad+\frac{\beta x^{\alpha-1}}{(1-\beta\sum_{i=1}^m \eta_i^{\alpha-3})}\int_0^1 H(\eta_i,qt)f(t,v(t))d_q(t)\\&=Tv(x).\end{aligned}$$

On the other hand, for $u\geq v$ and by condition (i) we have

$$\begin{aligned}d(Tu,Tv)&=\sup_{0\leq x\leq 1}|(Tu)(x)-(Tv)(x)|=\sup_{0\leq x\leq 1}((Tu)(x)-(Tv)(x))\\&\leq\sup_{0\leq x\leq 1}\int_0^1 G(x,qt)(f(t,u(t))-f(t,v(t)))d_q(t)\\&\quad+\frac{\beta}{(1-\beta\sum_{i=1}^m \eta_i^{\alpha-3})}\int_0^1 H(\eta_i,qt)(f(t,u(t))-f(t,v(t)))d_q(t)\\&\leq\sup_{0\leq x\leq 1}\int_0^1 G(x,qt)\varphi(u(t)-v(t))d_q(t)\\&\quad+\frac{\beta}{(1-\beta\sum_{i=1}^m \eta_i^{\alpha-3})}\int_0^1 H(\eta_i,qt)\varphi(u(t)-v(t))d_q(t).\end{aligned}$$

Now, since φ is nondecreasing and condition (ii), we have

$$\begin{aligned}d(Tu,Tv)&\leq\varphi(d(u,v))\Big(\sup_{0\leq x\leq 1}\int_0^1 G(x,qt)d_q(t)+\frac{\beta}{(1-\beta\sum_{i=1}^m \eta_i^{\alpha-3})}\int_0^1 H(\eta_i,qt)d_q(t)\Big)\\&=\varphi(d(u,v))\Big(K+\frac{\beta}{(1-\beta\sum_{i=1}^m \eta_i^{\alpha-3})}L\Big)\\&\leq\varphi(d(u,v))=\frac{\varphi(d(u,v))}{d(u,v)}.d(u,v).\end{aligned}$$

So

$$d(Tu,Tv)\leq\gamma(d(u,v)).d(u,v).$$

As $G(x,qt)\geq 0$ and $f\geq 0$, $(T0)(x)=\int_0^1 G(x,qt)f(t,0)d_q(t)\geq 0$, so by Theorem 2.7, problem (1.1), has at least one nonnegative solution. Moreover, by Remark

(2.8) $(M, \leq)$ satisfies condition (B) of Theorem 2.7 and this implies the uniqueness of the solution and the proof is done. $\square$

Example 2.4. Let $\alpha = 3.5, q = 0.5$ and $\beta = \frac{1}{4}, \eta = 0.6$, consider the following problem:

$$(D_{0.5}^{3.5}u)(x) + \frac{u(x)}{1+u(x)} = 0, \tag{2.1}$$

subject to the boundary conditions

$$u(0) = 0, D_{0.5}u(0) = 0, D_{0.5}^2u(0) = 0, \quad D_{0.5}^2u(1) = \frac{1}{4}D_{0.5}^2u(0.6). \tag{2.2}$$

Now, we consider the conditions of Theorem 2.3. $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and since $\frac{\partial f}{\partial u} = \frac{1}{(1+u(x))} > 0$. So f is increasing with respect to the second variable. Also, for $u \geq v$ and $x \in [0,1]$ we have

$$f(x,u) - f(x,v) = \frac{u-v}{(1+u)(1+v)} \leq \frac{u-v}{1+u-v} = \phi(u-v),$$

where $\phi(t) = \frac{t}{1+t}$., so condition (i) is satisfied. Moreover, by definition of function G, Lemma 2.3, and the fact that $\alpha - 3 \geq 0$, we have

$$K = \int_0^1 [(1-qt)^{(\alpha-3)} - (1-qt)^{(\alpha-1)}]d_qt = \frac{\Gamma_q(\alpha-2)}{\Gamma_q(\alpha)\Gamma_q(\alpha-1)} - \frac{1}{\Gamma_q(\alpha+1)} \approx 0.11,$$

and

$$H(\eta, qt) \leq \sqrt{(0.6)}\frac{\Gamma_q(\alpha-2)}{\Gamma_q(\alpha)\Gamma_q(\alpha-1)} \approx 0.30.$$

So

$$K + \frac{\beta}{(1-\beta\eta^{\alpha-3})}L \approx 0.20,$$

therefore, condition (ii) of Theorem 2.3 is also satisfied and problem (2.1)-(2.2) has a unique positive solution.

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