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SYSTEM OF VARIATIONAL INEQUALITIES WITH CONSTRAINTS OF MIXED EQUILIBRIA, VARIATIONAL INEQUALITIES, AND CONVEX MINIMIZATION AND FIXED POINT PROBLEMS

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Abstract. We introduce and analyze an iterative algorithm by hybrid steepest-descent viscosity method for finding a solution of the general system of variational inequalities with constraints of several problems: finitely many generalized mixed equilibria, finitely many variational inequalities, the minimization problem for a convex and continuously Fréchet differentiable functional and the fixed point problem of infinitely many nonexpansive mappings in a real Hilbert space. We prove strong convergence theorem for the iterative algorithm under suitable conditions. The iterative algorithm is based on Korpelevich's extragradient method, hybrid steepest-descent method, viscosity approximation method, averaged mapping approach to the GPA and strongly positive bounded linear operator technique. On the other hand, we also derive the weak convergence of the proposed algorithm under the new mild assumptions.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H and P_C be the metric projection of H onto C. Let $S : C \to H$ be a nonlinear mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping $S : C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Sx - Sy\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

In particular, if $L = 1$ then S is called a nonexpansive mapping; if $L \in [0, 1)$ then S is called a contraction. A mapping V is called strongly positive on H if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Vx, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

Let $A : C \to H$ be a nonlinear mapping on C. The classical variational inequality problem (VIP) is to find a point $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

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The solution set of VIP (1.1) is denoted by VI(C, A). We observe that the VIP (1.1) was first discussed by Lions [11]. In 1976, Korpelevich [10] proposed an iterative algorithm for solving the VIP (1.1) in Euclidean space $\mathbf{R}^n$:

$$\begin{cases} y_n = P_C(x_n - \tau A x_n), \\ x_{n+1} = P_C(x_n - \tau A y_n), \quad \forall n \geq 0, \end{cases}$$

with $\tau > 0$ a given number, which is known as the extragradient method.

Let $\varphi : C \to \mathbf{R}$ be a real-valued function, $A : H \to H$ be a nonlinear mapping and $\Theta : C \times C \to \mathbf{R}$ be a bifunction. Peng and Yao [14] introduced the following generalized mixed equilibrium problem (GMEP) of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

We denote the set of solutions of GMEP (1.2) by GMEP(Θ, φ, A). The GMEP (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games and others. In particular, it covers the generalized equilibrium problem [17], the mixed equilibrium problem [7] and the equilibrium problem [8, 5] as special cases.

It was assumed in [14] that $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (A1)-(A4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (B1) or (B2), where

(A1) $\Theta(x, x) = 0$ for all $x \in C$;

(A2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for any $x, y \in C$;

(A3) Θ is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Theta(tz + (1 - t)x, y) \leq \Theta(x, y);$$

(A4) $\Theta(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x \rangle < 0;$$

(B2) C is a bounded set.

Given a positive number $r > 0$. Let $T_r^{(\Theta,\varphi)} : H \to C$ be the solution set of the auxiliary mixed equilibrium problem, that is, for each $x \in H$,

$$T_r^{(\Theta,\varphi)}(x) := \{y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y \rangle \geq 0, \forall z \in C\}.$$

Let $F_1, F_2 : C \to H$ be two mappings. Consider the following general system of variational inequalities (GSVI) of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \nu_1 F_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \nu_2 F_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.3}$$

where $\nu_1 > 0$ and $\nu_2 > 0$ are two constants. It was considered and studied in [6, 4, 21]. In 2008, Ceng, Wang and Yao [6] transformed the GSVI (1.3) into a fixed point problem; that is, for given $\bar{x}, \bar{y} \in C$, $(\bar{x}, \bar{y})$ is a solution of the GSVI (1.3) if and only if $\bar{x}$ is a fixed point of the mapping $G : C \to C$ defined by

$$Gx = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)x, \quad \forall x \in C,$$

where $\bar{y} = P_C(I - \nu_2 F_2)\bar{x}$.

We remark that if the mapping $F_j : C \to H$ is ζ_j-inverse-strongly monotone for $j = 1, 2$, then the mapping G is nonexpansive provided $\nu_j \in (0, 2\zeta_j]$ for $j = 1, 2$. We denote by GSVI(G) the fixed point set of the mapping G.

Let $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N} \in (0, 1]$, $n \geq 1$. Given the nonexpansive self-mappings $S_1, S_2, \ldots, S_N$ on C, for each $n \geq 1$, the mappings $U_{n,1}, U_{n,2}, \ldots, U_{n,N}$ are defined by

$$\begin{cases} U_{n,1} = \lambda_{n,1} S_1 + (1 - \lambda_{n,1})I, \\ U_{n,2} = \lambda_{n,2} S_n U_{n,1} + (1 - \lambda_{n,2})I, \\ U_{n,n-1} = \lambda_{n-1} S_{n-1} U_{n,n} + (1 - \lambda_{n-1})I, \\ \cdots \\ U_{n,N-1} = \lambda_{n,N-1} S_{N-1} U_{n,N-2} + (1 - \lambda_{n,N-1})I, \\ W_n := U_{n,N} = \lambda_{n,N} S_N U_{n,N-1} + (1 - \lambda_{n,N})I. \end{cases} \tag{1.4}$$

The W_n is called the W-mapping generated by $S_1, \ldots, S_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying assumptions (A1)-(A4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function with restriction (B1) or (B2). Let the mapping $A : H \to H$ be δ-inverse strongly monotone, and $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings on H such that $\Omega := \cap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta, \varphi, A) \neq \emptyset$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$ and $Q : H \to H$ an l-Lipschitzian mapping with constant $l \geq 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$, $\{\gamma_n\}$ is a sequence in $(0, 2\delta]$ and $\{\lambda_{n,i}\}_{i=1}^N$ is a sequence in $[a, b]$ with $0 < a \leq b < 1$. In 2012, combining the hybrid steepest-descent method in [20] and viscosity approximation method, Ceng, Guu and Yao [3] introduced the following hybrid iterative algorithm for finding a common element of the solution set of GMEP (1.2) and the fixed point set of finitely many nonexpansive mappings $\{S_i\}_{i=1}^N$:

$$\begin{cases} x_1 \in H \text{ chosen arbitrarily}, \\ \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n \rangle \\ \quad + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n \gamma Q x_n + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \mu F)W_n u_n, \quad \forall n \geq 1. \end{cases} \tag{1.5}$$

Assume that the following conditions are satisfied: (i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$; (ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$; (iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\delta$ and $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$; and (iv) $\lim_{n\to\infty}(\lambda_{n+1,i} - \lambda_{n,i}) = 0$ for $i = 1, 2, \ldots, N$. It was proven in [3, Theorem 3.1] that both $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* = P_\Omega(I - \mu F + \gamma Q)x^*$, which is the unique solution in Ω to the VIP

$$\langle (\mu F - \gamma Q)x^*, x^* - x \rangle \leq 0, \quad \forall x \in \Omega. \tag{1.6}$$

Let $f : C \to \mathbf{R}$ be a convex and continuously Fréchet differentiable functional. Consider the convex minimization problem (CMP) of minimizing f over the constraint set C

$$\text{minimize}\{f(x) : x \in C\}. \tag{1.7}$$

We denote by Γ the set of minimizers of CMP (1.7). The gradient-projection algorithm (GPA) generates a sequence $\{x_n\}$ determined by the gradient ∇f and the metric projection P_C:

$$x_{n+1} := P_C(x_n - \lambda \nabla f(x_n)), \quad \forall n \geq 0, \tag{1.8}$$

or more generally,

$$x_{n+1} := P_C(x_n - \lambda_n \nabla f(x_n)), \quad \forall n \geq 0, \tag{1.9}$$

where, in both (1.8) and (1.9), the initial guess x_0 is taken from C arbitrarily, the parameters λ or λ_n are positive real numbers.

On the other hand, let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let M, N be two integers. Let Θ_k be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)-(A4) and $\varphi_k : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function, where $k \in \{1, 2, \dots, M\}$. Let $B_k : H \to H$ and $A_i : C \to H$ be μ_k-inverse strongly monotone and η_i-inverse-strongly monotone, respectively, where $k \in \{1, 2, \dots, M\}$, $i \in \{1, 2, \dots, N\}$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $Q : H \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := \cap_{k=1}^{M} \text{GMEP}(\Theta_k, \varphi_k, B_k) \cap \cap_{i=1}^{N} \text{VI}(C, A_i) \cap \Gamma \neq \emptyset$ and that either (B1) or (B2) holds. Very recently, Ceng and Al-Homidan [2] proposed the following iterative algorithm by hybrid steepest-descent viscosity method

$$\begin{cases} x_1 \in H \text{ chosen arbitrarily,} \\ u_n = T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I - r_{M,n}B_M)T_{r_{M-1,n}}^{(\Theta_{M-1},\varphi_{M-1})}(I - r_{M-1,n}B_{M-1}) \cdots \\ \qquad T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I - r_{1,n}B_1)x_n, \\ v_n = P_C(I - \lambda_{N,n}A_N)P_C(I - \lambda_{N-1,n}A_{N-1}) \cdots \\ \qquad P_C(I - \lambda_{2,n}A_2)P_C(I - \lambda_{1,n}A_1)u_n, \\ x_{n+1} = s_n\gamma Q x_n + \beta_n x_n + ((1 - \beta_n)I - s_n\mu F)T_n v_n, \quad \forall n \geq 1, \end{cases} \tag{1.10}$$

where $P_C(I - \lambda_n \nabla f) = s_n I + (1 - s_n)T_n$ (here T_n is nonexpansive, $s_n = \frac{2-\lambda_n L}{4} \in (0, \frac{1}{2})$ for each $\lambda_n \in (0, \frac{2}{L})$). Assume that the following conditions hold: (i) $s_n \in (0, \frac{1}{2})$ for each $\lambda_n \in (0, \frac{2}{L})$, and $\lim_{n\to\infty} s_n = 0$ ($\Leftrightarrow$ $\lim_{n\to\infty} \lambda_n = \frac{2}{L}$); (ii) $\{\beta_n\} \subset (0, 1)$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$; (iii) $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\lim_{n\to\infty} |\lambda_{i,n+1} - \lambda_{i,n}| = 0$ for all $i \in \{1, 2, \dots, N\}$; and (iv) $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$ and $\lim_{n\to\infty} |r_{k,n+1} - r_{k,n}| = 0$ for all $k \in \{1, 2, \dots, M\}$. It was proven in [2, Theorem 21] that $\{x_n\}$ converges strongly as $\lambda_n \to \frac{2}{L}$ ($\Leftrightarrow$ $s_n \to 0$) to a point $q \in \Omega$, which is a unique solution in Ω to the VIP

$$\langle (\mu F - \gamma V)q, p - q \rangle \geq 0, \quad \forall p \in \Omega.$$

Equivalently, $q = P_\Omega(I - \mu F + \gamma V)q$.

Motivated and inspired by the above facts, we first introduce and analyze an iterative algorithm by hybrid steepest-descent viscosity method for finding a solution of the GSVI (1.3) with constraints of several problems: finitely many GMEPs, finitely many VIPs, the CMP (1.7) and the fixed point problem of infinitely many nonexpansive mappings in a real Hilbert space. We prove strong convergence theorem for the iterative algorithm under suitable conditions. The iterative algorithm is based on Korpelevich's extragradient method, hybrid steepest-descent method in [20], viscosity approximation method, averaged mapping approach to the GPA in [18] and strongly positive bounded linear operator technique. On the other hand, we derive also its weak convergence under the new assumptions different from the strong convergence criteria. The results obtained in this paper improve and extend the corresponding results announced by many others.

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, i.e.,

$$\omega_w(x_n) := \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Recall that a mapping $A : C \to H$ is called

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C;$$

(iii) α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is obvious that if A is α-inverse-strongly monotone, then A is monotone and $\frac{1}{\alpha}$-Lipschitz continuous. Moreover, we also observe that if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\| =: d(x, C).$$

Some important properties of projections are gathered in the following proposition.

Proposition 2.1. *For given $x \in H$ and $z \in C$:*

(i) $z = P_C x \;\Leftrightarrow\; \langle x - z, y - z\rangle \leq 0,\ \forall y \in C$;

(ii) $z = P_C x \;\Leftrightarrow\; \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2,\ \forall y \in C$;

(iii) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2,\ \forall y \in H$.

Consequently, P_C is nonexpansive and monotone.

Definition 2.2. A mapping $T : H \to H$ is said to be:

(a) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H;$$

(b) firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently, if T is 1-inverse strongly monotone (1-ism),

$$\langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in H;$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive; projections are firmly nonexpansive.

It can be easily seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that a projection P_C is 1-ism. Inverse strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

Definition 2.3. A mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping, that is,

$$T \equiv (1 - \alpha)I + \alpha S$$

where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive. More precisely, when the last equality holds, we say that T is α-averaged. Thus firmly nonexpansive mappings (in particular, projections) are $\frac{1}{2}$-averaged mappings.

Proposition 2.4 (see [1])**.** *Let $T : H \to H$ be a given mapping.*

(i) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.*

(ii) *If T is ν-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.*

(iii) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > 1/2$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

Proposition 2.5 (see [1])**.** *Let $S, T, V : H \to H$ be given operators.*

(i) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is averaged and V is nonexpansive, then T is averaged.*

(ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*

(iii) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.*

(iv) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \cdots T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0, 1)$, then the composite T_1T_2 is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

(v) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^N \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 \cdots T_N).$$

The notation $\mathrm{Fix}(T)$ *denotes the set of all fixed points of the mapping* T*, that is,* $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$.

Next we list some elementary conclusions for the mixed equilibrium problems.

Proposition 2.6 (see [7])**.** *Assume that* $\Theta : C \times C \to \mathbf{R}$ *satisfies (A1)-(A4) and let* $\varphi : C \to \mathbf{R}$ *be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For* $r > 0$ *and* $x \in H$*, define a mapping* $T_r^{(\Theta,\varphi)} : H \to C$ *as follows:*

$$T_r^{(\Theta,\varphi)}(x) = \{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $x \in H$*. Then the following hold:*

(i) *for each* $x \in H$*,* $T_r^{(\Theta,\varphi)}(x)$ *is nonempty and single-valued;*
(ii) $T_r^{(\Theta,\varphi)}$ *is firmly nonexpansive, that is, for any* $x, y \in H$*,*

$$\|T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y\|^2 \leq \langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y\rangle;$$

(iii) $\mathrm{Fix}(T_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, \varphi)$*;*
(iv) $\mathrm{MEP}(\Theta, \varphi)$ *is closed and convex;*
(v) $\|T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x\|^2 \leq \frac{s-t}{s}\langle T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x, T_s^{(\Theta,\varphi)}x - x\rangle$ *for all* $s, t > 0$ *and* $x \in H$.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.7. *Let* X *be a real inner product space. Then there holds the following inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X.$$

Lemma 2.8. *Let* $A : C \to H$ *be a monotone mapping. In the context of the variational inequality problem the characterization of the projection (see Proposition 2.1 (i)) implies*

$$u \in \mathrm{VI}(C, A) \quad \Leftrightarrow \quad u = P_C(u - \lambda Au), \quad \lambda > 0.$$

Lemma 2.9 (see [9, Demiclosedness principle])**.** *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* T *be a nonexpansive self-mapping on* C*. Then* $I - T$ *is demiclosed. That is, whenever* $\{x_n\}$ *is a sequence in* C *weakly converging to some* $x \in C$ *and the sequence* $\{(I-T)x_n\}$ *strongly converges to some* y*, it follows that* $(I - T)x = y$*. Here* I *is the identity operator of* H*.*

Let $\{S_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings on H and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0, 1]$. For any $n \geq 1$, define a mapping

W_n on H as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n S_n U_{n,n+1} + (1-\lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} S_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\ \cdots \\ U_{n,k} = \lambda_k S_k U_{n,k+1} + (1-\lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} S_{k-1} U_{n,k} + (1-\lambda_{k-1})I, \\ \cdots \\ U_{n,2} = \lambda_2 S_2 U_{n,3} + (1-\lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 S_1 U_{n,2} + (1-\lambda_1)I. \end{cases} \tag{2.2}$$

Such a mapping W_n is called the W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

Lemma 2.10 (see [12, Lemma 3.2])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\cap_{n=1}^{\infty}\mathrm{Fix}(S_n) \neq \emptyset$ and let $\{\lambda_n\}$ be a sequence in $(0,b]$ for some $b \in (0,1)$. Then, for every $x \in C$ and $k \geq 1$ the limit $\lim_{n\to\infty} U_{n,k}x$ exists where $U_{n,k}$ is defined as in (2.2).*

Lemma 2.11 (see [12, Lemma 3.3])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\cap_{n=1}^{\infty}\mathrm{Fix}(S_n) \neq \emptyset$, and let $\{\lambda_n\}$ be a sequence in $(0,b]$ for some $b \in (0,1)$. Then, $\mathrm{Fix}(W) = \cap_{n=1}^{\infty}\mathrm{Fix}(S_n)$.*

The following lemma can be easily proven, and therefore, we omit the proof.

Lemma 2.12. *Let $V : H \to H$ be a $\bar{\gamma}$-strongly positive bounded linear operator with constant $\bar{\gamma} > 0$ and $Q : H \to H$ be an l-Lipschitzian mapping with $\gamma l < \bar{\gamma}$. Then for $\bar{\gamma} - \gamma l > 0$,*

$$\langle (V-\gamma Q)x - (V-\gamma Q)y, x-y\rangle \geq (\bar{\gamma} - \gamma l)\|x-y\|^2, \quad \forall x, y \in H.$$

That is, $V - \gamma Q$ is strongly monotone with constant $\bar{\gamma} - \gamma l > 0$.

Let C be a nonempty closed convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0,1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \to H$, we define the mapping $T^{\lambda} : C \to H$ by

$$T^{\lambda}x := Tx - \lambda\mu F(Tx), \quad \forall x \in C,$$

where $F : H \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on H; that is, F satisfies the conditions:

$$\|Fx - Fy\| \leq \kappa\|x-y\| \quad \text{and} \quad \langle Fx - Fy, x-y\rangle \geq \eta\|x-y\|^2$$

for all $x, y \in H$.

Lemma 2.13 (see [19, Lemma 3.1])**.** *T^{λ} is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$; that is,*

$$\|T^{\lambda}x - T^{\lambda}y\| \leq (1-\lambda\tau)\|x-y\|, \quad \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)} \in (0,1]$.

Recall that a set-valued mapping $T : D(T) \subset H \to 2^H$ is called monotone if for all $x, y \in D(T)$, $f \in Tx$ and $g \in Ty$ imply

$$\langle f - g, x - y \rangle \geq 0.$$

A set-valued mapping T is called maximal monotone if T is monotone and $(I + \lambda T)D(T) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(T)$ the graph of T. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle f - g, x - y \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let $A : C \to H$ be a monotone, k-Lipschitz-continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \ \forall u \in C\}.$$

Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$; see [15].

Lemma 2.14 (see [19]). *Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the conditions*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \beta_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that

(i) $\{\alpha_n\} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *or equivalently,*

$$\prod_{n=1}^{\infty} (1 - \alpha_n) := \lim_{n\to\infty} \prod_{k=1}^{n} (1 - \alpha_k) = 0;$$

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$, *or* $\sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.15 (see [16]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in $[0, 1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$ for each $n \geq 1$ and

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.16 ([13, p. 80]). *] Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If, in addition, $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges to zero, then $\lim_{n\to\infty} a_n = 0$.

Recall that a Banach space X is said to satisfy the Opial condition [9] if for any given sequence $\{x_n\} \subset X$ which converges weakly to an element $x \in X$, there holds the inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in X,\ y \neq x.$$

It is well known in [9] that every Hilbert space H satisfies the Opial condition.

Lemma 2.17. *Let H be a real Hilbert space. Then the following hold:*

(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all* $x, y \in H$;
(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x - y\|^2$ *for all* $x, y \in H$ *and* $\lambda, \mu \in [0, 1]$ *with* $\lambda + \mu = 1$;
(c) *If* $\{x_n\}$ *is a sequence in* H *such that* $x_n \rightharpoonup x$, *it follows that*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H.$$

3. Strong convergence criteria for the GSVI with constraints

In this section, we will introduce and analyze an iterative algorithm by hybrid steepest-descent viscosity method for finding a solution of the GSVI (1.3) with constraints of several problems: finitely many GMEPs, finitely many VIPs, the CMP (1.7) and the fixed point problem of infinitely many nonexpansive mappings in a real Hilbert space. We prove strong convergence theorem for the iterative algorithm under suitable conditions. This iterative algorithm is based on Korpelevich's extragradient method, hybrid steepest-descent method in [20], viscosity approximation method, averaged mapping approach to the GPA in [18] and strongly positive bounded linear operator technique.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let M, N be two integers. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let Θ_k be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)-(A4) and $\varphi_k : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function, where $k \in \{1, 2, \dots, M\}$. Let $B_k, A_i : H \to H$ and $F_j : C \to H$ be μ_k-inverse-strongly monotone, η_i-inverse-strongly monotone and ζ_j-inverse-strongly monotone, respectively, where $k \in \{1, 2, \dots, M\}, i \in \{1, 2, \dots, N\}$ and $j \in \{1, 2\}$. Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings on H and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $Q : H \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Let V be a $\bar{\gamma}$-strongly positive bounded linear operator with $\gamma l < \bar{\gamma}$. Assume that $\Omega := \cap_{n=1}^{\infty}\mathrm{Fix}(S_n) \cap \cap_{k=1}^{M}\mathrm{GMEP}(\Theta_k, \varphi_k, B_k) \cap \cap_{i=1}^{N}\mathrm{VI}(C, A_i) \cap \mathrm{GSVI}(G) \cap \Gamma \neq \emptyset$ and that either (B1) or (B2) holds. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1]$. For arbitrarily given $x_1 \in H$, let $\{x_n\}$ be a sequence* *generated by*

$$(3.1)\quad \begin{cases} u_n = T^{(\Theta_M,\varphi_M)}_{r_{M,n}}(I - r_{M,n}B_M)T^{(\Theta_{M-1},\varphi_{M-1})}_{r_{M-1,n}}(I - r_{M-1,n}B_{M-1})\cdots \\ \qquad T^{(\Theta_1,\varphi_1)}_{r_{1,n}}(I - r_{1,n}B_1)x_n, \\ v_n = P_C(I - \lambda_{N,n}A_N)P_C(I - \lambda_{N-1,n}A_{N-1})\cdots \\ \qquad P_C(I - \lambda_{2,n}A_2)P_C(I - \lambda_{1,n}A_1)u_n, \\ y_n = \alpha_n\gamma Qv_n + (I - \alpha_n\mu F)W_nGv_n, \\ x_{n+1} = s_n\gamma Qx_n + \beta_n x_n + ((1-\beta_n)I - s_nV)T_ny_n, \quad \forall n \ge 1, \end{cases}$$

where $P_C(I-\theta_n\nabla f) = s_nI+(1-s_n)T_n$ *(here* T_n *is nonexpansive,* $s_n = \frac{2-\theta_nL}{4} \in (0,\frac{1}{2})$ *for each* $\theta_n \in (0,\frac{2}{L})$*),* $\nu_j \in (0, 2\zeta_j)$ *for* $j = 1, 2$ *and* W_n *is the* W*-mapping defined by (2.2). Suppose that the following conditions are satisfied:*

(i) $s_n \in (0,\frac{1}{2})$ *for each* $\theta_n \in (0,\frac{2}{L})$*, and* $\lim_{n\to\infty} s_n = 0$ *(*$\Leftrightarrow$ $\lim_{n\to\infty}\theta_n = \frac{2}{L}$*);*
(ii) $\lim_{n\to\infty}\frac{\alpha_n}{s_n} = 0$, $\sum_{n=1}^{\infty} s_n = \infty$ *and* $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$*;*
(iii) $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ *and* $\lim_{n\to\infty}|\lambda_{i,n+1} - \lambda_{i,n}| = 0$ *for all* $i \in \{1, 2, \ldots, N\}$*;*
(iv) $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$ *and* $\lim_{n\to\infty}|r_{k,n+1} - r_{k,n}| = 0$ *for all* $k \in \{1, 2, \ldots, M\}$*.*

Then $\{x_n\}$ *converges strongly to a point* $x^* \in \Omega$ *provided* $\|x_n - y_n\| \to 0$ $(n \to \infty)$*, which is a unique solution in* Ω *to the VIP*

$$\langle(\gamma Q - V)x^*, p - x^*\rangle \le 0, \quad \forall p \in \Omega.$$

Proof. Since ∇f is L-Lipschitzian, it follows that ∇f is $1/L$-ism. By Proposition 2.4 (ii) we know that for $\theta > 0$, $\theta\nabla f$ is $\frac{1}{\theta L}$-ism. So by Proposition 2.4 (iii) we deduce that $I-\theta\nabla f$ is $\frac{\theta L}{2}$-averaged. Now since the projection P_C is $\frac{1}{2}$-averaged, it is easy to see from Proposition 2.5 (iv) that the composite $P_C(I - \theta\nabla f)$ is $\frac{2+\theta L}{4}$-averaged for $\theta \in (0,\frac{2}{L})$. Hence we obtain that for each $n \ge 1$, $P_C(I - \theta_n\nabla f)$ is $\frac{2+\theta_nL}{4}$-averaged for each $\theta_n \in (0,\frac{2}{L})$. Therefore, we can write

$$P_C(I - \theta_n\nabla f) = \frac{2 - \theta_nL}{4}I + \frac{2 + \theta_nL}{4}T_n = s_nI + (1 - s_n)T_n,$$

where T_n is nonexpansive and $s_n := s_n(\theta_n) = \frac{2-\theta_nL}{4} \in (0,\frac{1}{2})$ for each $\theta_n \in (0,\frac{2}{L})$. It is clear that

$$\theta_n \to \frac{2}{L} \;\Leftrightarrow\; s_n \to 0.$$

As $\lim_{n\to\infty} s_n = 0$ and $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, we may assume, without loss of generality, that $\{\beta_n\} \subset [a, \hat{a}] \subset (0, 1)$ and $\beta_n + s_n\|V\| \le 1$ for all $n \ge 1$. Since V is a $\bar{\gamma}$-strongly positive bounded linear operator on H, we know that

$$\|V\| = \sup\{\langle Vu, u\rangle : u \in H, \|u\| = 1\} \ge \bar{\gamma} > 1.$$

Taking into account that $\beta_n + s_n\|V\| \le 1$ for all $n \ge 1$, we have

$$\begin{aligned}\langle((1 - \beta_n)I - s_nV)u, u\rangle &= 1 - \beta_n - s_n\langle Vu, u\rangle \\ &\ge 1 - \beta_n - s_n\|V\| \\ &\ge 0,\end{aligned}$$

that is, $(1-\beta_n)I - s_nV$ is positive. It follows that

$$\begin{aligned} \|(1-\beta_n)I - s_nV\| &= \sup\{\langle((1-\beta_n)I - s_nV)u, u\rangle : u \in H, \|u\| = 1\} \\ &= \sup\{1-\beta_n - s_n\langle Vu, u\rangle : u \in H, \|u\| = 1\} \\ &\le 1-\beta_n - s_n\bar{\gamma}. \end{aligned} \tag{3.2}$$

Put

$$\varDelta_n^k = T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}B_k)T_{r_{k-1,n}}^{(\Theta_{k-1},\varphi_{k-1})}(I - r_{k-1,n}B_{k-1})\cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I - r_{1,n}B_1)x_n$$

for all $k \in \{1,2,\dots,M\}$ and $n \ge 1$,

$$\varLambda_n^i = P_C(I - \lambda_{i,n}B_i)P_C(I - \lambda_{i-1,n}B_{i-1})\cdots P_C(I - \lambda_{1,n}B_1)$$

for all $i \in \{1,2,\dots,N\}$, $\varDelta_n^0 = I$ and $\varLambda_n^0 = I$, where I is the identity mapping on H. Then we have that $u_n = \varDelta_n^M x_n$ and $v_n = \varLambda_n^N u_n$.

We divide the rest of the proof into several steps.

Step 1. Let us show that $\|x_n - p\| \le \max\{\|x_1 - p\|, \frac{\|(I-V)p\|}{\bar{\gamma}-1}, \frac{\|(\gamma Q-\mu F)p\|}{\tau-\gamma l}\}$ for all $n \ge 1$ and $p \in \varOmega$. Indeed, take $p \in \varOmega$ arbitrarily. Then from (2.1) and Proposition 2.6 (ii) we have

$$\begin{aligned} \|u_n - p\| &= \|T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I - r_{M,n}B_M)\varDelta_n^{M-1}x_n - T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I - r_{M,n}B_M)\varDelta_n^{M-1}p\| \\ &\le \|(I - r_{M,n}B_M)\varDelta_n^{M-1}x_n - (I - r_{M,n}B_M)\varDelta_n^{M-1}p\| \\ &\le \|\varDelta_n^{M-1}x_n - \varDelta_n^{M-1}p\| \\ &\cdots \\ &\le \|\varDelta_n^0 x_n - \varDelta_n^0 p\| \\ &= \|x_n - p\|. \end{aligned} \tag{3.3}$$

Similarly, we have

$$\begin{aligned} \|v_n - p\| &= \|P_C(I - \lambda_{N,n}A_N)\varLambda_n^{N-1}u_n - P_C(I - \lambda_{N,n}A_N)\varLambda_n^{N-1}p\| \\ &\le \|(I - \lambda_{N,n}A_N)\varLambda_n^{N-1}u_n - (I - \lambda_{N,n}A_N)\varLambda_n^{N-1}p\| \\ &\le \|\varLambda_n^{N-1}u_n - \varLambda_n^{N-1}p\| \\ &\cdots \\ &\le \|\varLambda_n^0 u_n - \varLambda_n^0 p\| \\ &= \|u_n - p\|. \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we have

$$\|v_n - p\| \le \|x_n - p\|. \tag{3.5}$$

Since $p = Gp = P_C(I - \nu_1F_1)P_C(I - \nu_2F_2)p$, F_j is ζ_j-inverse-strongly monotone for $j = 1,2$, and $0 \le \nu_j \le 2\zeta_j$ for $j = 1,2$, we deduce that, for any $n \ge 1$,

$$\begin{aligned} \|Gv_n - p\|^2 &= \|P_C(I - \nu_1F_1)P_C(I - \nu_2F_2)v_n - P_C(I - \nu_1F_1)P_C(I - \nu_2F_2)p\|^2 \\ &\le \|(I - \nu_1F_1)P_C(I - \nu_2F_2)v_n - (I - \nu_1F_1)P_C(I - \nu_2F_2)p\|^2 \\ &= \|[P_C(I - \nu_2F_2)v_n - P_C(I - \nu_2F_2)p] \end{aligned}$$

$$
\begin{aligned}
&- \nu_1[F_1P_C(I-\nu_2F_2)v_n - F_1P_C(I-\nu_2F_2)p]\|^2 \\
&\le \|P_C(I-\nu_2F_2)v_n - P_C(I-\nu_2F_2)p\|^2 \\
&\quad + \nu_1(\nu_1 - 2\zeta_1)\|F_1P_C(I-\nu_2F_2)v_n - F_1P_C(I-\nu_2F_2)p\|^2 \qquad (3.6)\\
&\le \|P_C(I-\nu_2F_2)v_n - P_C(I-\nu_2F_2)p\|^2 \\
&\le \|(I-\nu_2F_2)v_n - (I-\nu_2F_2)p\|^2 \\
&= \|(v_n-p) - \nu_2(F_2v_n - F_2p)\|^2 \\
&\le \|v_n-p\|^2 + \nu_2(\nu_2-2\zeta_2)\|F_2v_n - F_2p\|^2 \\
&\le \|v_n-p\|^2.
\end{aligned}
$$

Utilizing Lemma 2.13, from (3.1), (3.2), (3.5) and (3.6) we obtain that

$$
\begin{aligned}
\|y_n - p\| &= \|\alpha_n\gamma(Qv_n - Qp) \\
&\qquad + (I-\alpha_n\mu F)W_nGv_n - (I-\alpha_n\mu F)p + \alpha_n(\gamma Q - \mu F)p\| \\
&\le \alpha_n\gamma\|Qv_n - Qp\| + \|(I-\alpha_n\mu F)W_nGv_n - (I-\alpha_n\mu F)p\| \\
&\qquad + \alpha_n\|(\gamma Q-\mu F)p\| \\
&\le \alpha_n\gamma l\|v_n - p\| + (1-\alpha_n\tau)\|Gv_n - p\| + \alpha_n\|(\gamma Q-\mu F)p\| \\
&\le \alpha_n\gamma l\|v_n - p\| + (1-\alpha_n\tau)\|v_n - p\| + \alpha_n\|(\gamma Q-\mu F)p\| \\
&\le \alpha_n\gamma l\|x_n - p\| + (1-\alpha_n\tau)\|x_n - p\| + \alpha_n\|(\gamma Q-\mu F)p\| \\
&= (1-\alpha_n(\tau-\gamma l))\|x_n - p\| + \alpha_n\|(\gamma Q-\mu F)p\| \\
&= (1-\alpha_n(\tau-\gamma l))\|x_n - p\| + \alpha_n(\tau-\gamma l)\frac{\|(\gamma Q-\mu F)p\|}{\tau-\gamma l} \\
&\le \max\left\{\|x_n-p\|, \frac{\|(\gamma Q-\mu F)p\|}{\tau-\gamma l}\right\},
\end{aligned}
$$

and hence

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|s_n\gamma(Qx_n - Qp) + \beta_n(x_n - p) \\
&\qquad + ((1-\beta_n)I - s_nV)(T_ny_n - p) + s_n(\gamma Q - V)p\| \\
&\le s_n\gamma l\|x_n - p\| + \beta_n\|x_n - p\| \\
&\qquad + \|(1-\beta_n)I - s_nV)\|\|T_ny_n - p\| + s_n\|(\gamma Q - V)p\| \\
&\le s_n\gamma l\|x_n - p\| + \beta_n\|x_n - p\| \\
&\qquad + (1-\beta_n - s_n\bar{\gamma})\|y_n - p\| + s_n\|(\gamma Q - V)p\| \\
&\le (s_n\gamma l + \beta_n)\|x_n - p\| + (1-\beta_n - s_n\bar{\gamma})\|y_n - p\| \\
&\qquad + s_n\|(\gamma Q - V)p\| \\
&\le (s_n\gamma l + \beta_n)\|x_n - p\| \\
&\qquad + (1-\beta_n - s_n\bar{\gamma})\max\left\{\|x_n-p\|, \frac{\|(\gamma Q-\mu F)p\|}{\tau-\gamma l}\right\} \\
&\qquad + s_n\|(\gamma Q - V)p\| \\
&\le (\beta_n + s_n\gamma l)\max\left\{\|x_n-p\|, \frac{\|(\gamma Q-\mu F)p\|}{\tau-\gamma l}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ (1-\beta_n - s_n\bar{\gamma}) \max\left\{\|x_n - p\|, \frac{\|(\gamma Q - \mu F)p\|}{\tau - \gamma l}\right\} \\
&+ s_n\|(\gamma Q - V)p\| \\
&= (1 - s_n(\bar{\gamma} - \gamma l)) \max\left\{\|x_n - p\|, \frac{\|(\gamma Q - \mu F)p\|}{\tau - \gamma l}\right\} \\
&+ s_n\|(I - V)p\| \\
&= (1 - s_n(\bar{\gamma} - \gamma l)) \max\left\{\|x_n - p\|, \frac{\|(\gamma Q - \mu F)p\|}{\tau - \gamma l}\right\} \\
&+ s_n(\bar{\gamma} - 1)\frac{\|(\gamma Q - V)p\|}{\bar{\gamma} - 1} \\
&\le \max\left\{\|x_n - p\|, \frac{\|(\gamma Q - \mu F)p\|}{\tau - \gamma l}, \frac{\|(\gamma Q - V)p\|}{\bar{\gamma} - \gamma l}\right\}.
\end{aligned}
$$

By induction, we get

$$
\|x_n - p\| \le \max\left\{\|x_1 - p\|, \frac{\|(\gamma Q - \mu F)p\|}{\tau - \gamma l}, \frac{\|(\gamma Q - V)p\|}{\bar{\gamma} - \gamma l}\right\}. \quad \forall n \ge 1.
$$

Hence $\{x_n\}$ is bounded and so are the sequences $\{u_n\}, \{v_n\}, \{y_n\}$.

Step 2. Let us show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. To this end, define

$$
x_{n+1} = \beta_n x_n + (1-\beta_n)z_n, \quad \forall n \ge 1.
$$

Observe that from the definition of z_n,

$$
\begin{aligned}
z_{n+1} - z_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\
&= \frac{s_{n+1}\gamma Q x_{n+1} + ((1-\beta_{n+1})I - s_{n+1}V)T_{n+1}y_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{s_n\gamma Q x_n + ((1-\beta_n)I - s_n V)T_n y_n}{1-\beta_n} \\
&= \frac{s_{n+1}}{1-\beta_{n+1}}\gamma Q x_{n+1} - \frac{s_n}{1-\beta_n}\gamma Q x_n + T_{n+1}y_{n+1} - T_n y_n \\
&\quad + \frac{s_n}{1-\beta_n} V T_n y_n - \frac{s_{n+1}}{1-\beta_{n+1}} V T_{n+1} y_{n+1} \\
&= \frac{s_{n+1}}{1-\beta_{n+1}}(\gamma Q x_{n+1} - V T_{n+1} y_{n+1}) + \frac{s_n}{1-\beta_n}(V T_n y_n - \gamma Q x_n) \\
&\quad + T_{n+1}y_{n+1} - T_n y_n.
\end{aligned}
$$

So, it follows that

$$
\begin{aligned}
\|z_{n+1} - z_n\| \le{}& \frac{s_{n+1}}{1-\beta_{n+1}}(\|\gamma Q x_{n+1}\| + \|V T_{n+1} y_{n+1}\|) \\
&+ \frac{s_n}{1-\beta_n}(\|V T_n y_n\| + \|\gamma Q x_n\|) \\
&+ \|T_{n+1}y_{n+1} - T_n y_n\|.
\end{aligned} \tag{3.7}
$$

On the other hand, since ∇f is $\frac{1}{L}$-ism, $P_C(I-\theta_n\nabla f)$ is nonexpansive for $\theta_n \in (0, \frac{2}{L})$. So, it follows that for any given $p \in \Omega$,

$$\begin{aligned}\|P_C(I-\theta_{n+1}\nabla f)y_n\| &\leq \|P_C(I-\theta_{n+1}\nabla f)y_n - p\| + \|p\| \\ &= \|P_C(I-\theta_{n+1}\nabla f)y_n - P_C(I-\theta_{n+1}\nabla f)p\| + \|p\| \\ &\leq \|y_n - p\| + \|p\| \\ &\leq \|y_n\| + 2\|p\|.\end{aligned}$$

This together with the boundedness of $\{y_n\}$ implies that $\{P_C(I-\lambda_{n+1}\nabla f)y_n\}$ is bounded. Also, observe that

$$\begin{aligned}&\|T_{n+1}y_n - T_ny_n\| \\ &= \Big\|\frac{4P_C(I-\theta_{n+1}\nabla f)-(2-\theta_{n+1}L)I}{2+\theta_{n+1}L}y_n - \frac{4P_C(I-\theta_n\nabla f)-(2-\theta_nL)I}{2+\theta_nL}y_n\Big\| \\ &\leq \Big\|\frac{4P_C(I-\theta_{n+1}\nabla f)}{2+\theta_{n+1}L}y_n - \frac{4P_C(I-\theta_n\nabla f)}{2+\theta_nL}y_n\Big\| + \Big\|\frac{2-\theta_nL}{2+\theta_nL}y_n - \frac{2-\theta_{n+1}L}{2+\theta_{n+1}L}y_n\Big\| \\ &= \Big\|\frac{4(2+\theta_nL)P_C(I-\theta_{n+1}\nabla f)y_n - 4(2+\theta_{n+1}L)P_C(I-\theta_n\nabla f)Gy_n}{(2+\theta_{n+1}L)(2+\theta_nL)}\Big\| \\ &\quad + \frac{4L|\theta_{n+1}-\theta_n|}{(2+\theta_{n+1}L)(2+\theta_nL)}\|y_n\| \\ &= \Big\|\frac{4L(\theta_n-\theta_{n+1})P_C(I-\theta_{n+1}\nabla f)y_n + 4(2+\theta_{n+1}L)(P_C(I-\theta_{n+1}\nabla f)y_n - P_C(I-\theta_n\nabla f)y_n)}{(2+\theta_{n+1}L)(2+\theta_nL)}\Big\| \\ &\quad + \frac{4L|\theta_{n+1}-\theta_n|}{(2+\theta_{n+1}L)(2+\theta_nL)}\|y_n\| \\ &\leq \frac{4L|\theta_n-\theta_{n+1}|\|P_C(I-\theta_{n+1}\nabla f)y_n\|}{(2+\theta_{n+1}L)(2+\theta_nL)} \\ &\quad + \frac{4(2+\theta_{n+1}L)\|P_C(I-\theta_{n+1}\nabla f)y_n - P_C(I-\theta_n\nabla f)y_n\|}{(2+\theta_{n+1}L)(2+\theta_nL)} \\ &\quad + \frac{4L|\theta_{n+1}-\theta_n|}{(2+\theta_{n+1}L)(2+\theta_nL)}\|y_n\| \\ &\leq |\theta_{n+1}-\theta_n|[L\|P_C(I-\theta_{n+1}\nabla f)y_n\| + 4\|\nabla f(y_n)\| + L\|y_n\|] \\ &\leq \widetilde{M}|\theta_{n+1}-\theta_n|,\end{aligned} \tag{3.8}$$

where $\sup_{n\geq 1}\{L\|P_C(I-\theta_{n+1}\nabla f)y_n\| + 4\|\nabla f(y_n)\| + L\|y_n\|\} \leq \widetilde{M}$ for some $\widetilde{M} > 0$. So, by (3.8), we have that

$$\begin{aligned}\|T_{n+1}y_{n+1} - T_ny_n\| &\leq \|T_{n+1}y_{n+1} - T_{n+1}y_n\| + \|T_{n+1}y_n - T_ny_n\| \\ &\leq \|y_{n+1} - y_n\| + \widetilde{M}|\lambda_{n+1} - \lambda_n| \\ &\leq \|y_{n+1} - y_n\| + \frac{4\widetilde{M}}{L}(s_{n+1} + s_n).\end{aligned} \tag{3.9}$$

Note that

$$
\begin{aligned}
\|v_{n+1}-v_n\| &= \|\Lambda_{n+1}^N u_{n+1}-\Lambda_n^N u_n\| \\
&= \|P_C(I-\lambda_{N,n+1}A_N)\Lambda_{n+1}^{N-1}u_{n+1}-P_C(I-\lambda_{N,n}A_N)\Lambda_n^{N-1}u_n\| \\
&\le \|P_C(I-\lambda_{N,n+1}A_N)\Lambda_{n+1}^{N-1}u_{n+1}-P_C(I-\lambda_{N,n}A_N)\Lambda_{n+1}^{N-1}u_{n+1}\| \\
&\quad + \|P_C(I-\lambda_{N,n}A_N)\Lambda_{n+1}^{N-1}u_{n+1}-P_C(I-\lambda_{N,n}A_N)\Lambda_n^{N-1}u_n\| \\
&\le \|(I-\lambda_{N,n+1}A_N)\Lambda_{n+1}^{N-1}u_{n+1}-(I-\lambda_{N,n}A_N)\Lambda_{n+1}^{N-1}u_{n+1}\| \\
&\quad + \|(I-\lambda_{N,n}A_N)\Lambda_{n+1}^{N-1}u_{n+1}-(I-\lambda_{N,n}A_N)\Lambda_n^{N-1}u_n\| \\
&\le |\lambda_{N,n+1}-\lambda_{N,n}|\|A_N\Lambda_{n+1}^{N-1}u_{n+1}\|+\|\Lambda_{n+1}^{N-1}u_{n+1}-\Lambda_n^{N-1}u_n\| \\
&\le |\lambda_{N,n+1}-\lambda_{N,n}|\|A_N\Lambda_{n+1}^{N-1}u_{n+1}\| \\
&\quad + |\lambda_{N-1,n+1}-\lambda_{N-1,n}|\|A_{N-1}\Lambda_{n+1}^{N-2}u_{n+1}\| \\
&\quad + \|\Lambda_{n+1}^{N-2}u_{n+1}-\Lambda_n^{N-2}u_n\| \\
&\cdots \\
&\le |\lambda_{N,n+1}-\lambda_{N,n}|\|A_N\Lambda_{n+1}^{N-1}u_{n+1}\| \\
&\quad + |\lambda_{N-1,n+1}-\lambda_{N-1,n}|\|A_{N-1}\Lambda_{n+1}^{N-2}u_{n+1}\| \\
&\quad + \cdots + |\lambda_{1,n+1}-\lambda_{1,n}|\|A_1\Lambda_{n+1}^{0}u_{n+1}\|+\|\Lambda_{n+1}^0u_{n+1}-\Lambda_n^0u_n\| \\
&\le \widetilde{M}_0\sum_{i=1}^N|\lambda_{i,n+1}-\lambda_{i,n}|+\|u_{n+1}-u_n\|,
\end{aligned}
\tag{3.10}
$$

where $\sup_{n\ge1}\{\sum_{i=1}^N\|A_i\Lambda_{n+1}^{i-1}u_{n+1}\|\}\le\widetilde{M}_0$ for some $\widetilde{M}_0>0$. Also, utilizing Proposition 2.6 (ii), (v) we deduce that

$$
\begin{aligned}
\|u_{n+1}-u_n\| &= \|\Delta_{n+1}^M x_{n+1}-\Delta_n^M x_n\| \\
&= \|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}(I-r_{M,n+1}B_M)\Delta_{n+1}^{M-1}x_{n+1} \\
&\quad - T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n}B_M)\Delta_n^{M-1}x_n\| \\
&\le \|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}(I-r_{M,n+1}B_M)\Delta_{n+1}^{M-1}x_{n+1} \\
&\quad - T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n}B_M)\Delta_{n+1}^{M-1}x_{n+1}\| \\
&\quad + \|T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n}B_M)\Delta_{n+1}^{M-1}x_{n+1} \\
&\quad - T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n}B_M)\Delta_n^{M-1}x_n\| \\
&\le \|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}(I-r_{M,n+1}B_M)\Delta_{n+1}^{M-1}x_{n+1} \\
&\quad - T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n+1}B_M)\Delta_{n+1}^{M-1}x_{n+1}\| \\
&\quad + \|T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n+1}B_M)\Delta_{n+1}^{M-1}x_{n+1} \\
&\quad - T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n}B_M)\Delta_{n+1}^{M-1}x_{n+1}\| \\
&\quad + \|(I-r_{M,n}B_M)\Delta_{n+1}^{M-1}x_{n+1}-(I-r_{M,n}B_M)\Delta_n^{M-1}x_n\|
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{|r_{M,n+1}-r_{M,n}|}{r_{M,n+1}}\|T^{(\Theta_M,\varphi_M)}_{r_{M,n+1}}(I-r_{M,n+1}B_M)\varDelta^{M-1}_{n+1}x_{n+1}\\
&\quad -(I-r_{M,n+1}B_M)\varDelta^{M-1}_{n+1}x_{n+1}\|\\
&\quad +|r_{M,n+1}-r_{M,n}|\|B_M\varDelta^{M-1}_{n+1}x_{n+1}\|+\|\varDelta^{M-1}_{n+1}x_{n+1}-\varDelta^{M-1}_{n}x_{n}\|\\
&=|r_{M,n+1}-r_{M,n}|[\|B_M\varDelta^{M-1}_{n+1}x_{n+1}\|\\
&\quad +\frac{1}{r_{M,n+1}}\|T^{(\Theta_M,\varphi_M)}_{r_{M,n+1}}(I-r_{M,n+1}B_M)\varDelta^{M-1}_{n+1}x_{n+1}\\
&\quad -(I-r_{M,n+1}B_M)\varDelta^{M-1}_{n+1}x_{n+1}\|]+\|\varDelta^{M-1}_{n+1}x_{n+1}-\varDelta^{M-1}_{n}x_{n}\|\\
&\cdots \qquad (3.11)\\
&\le |r_{M,n+1}-r_{M,n}|[\|B_M\varDelta^{M-1}_{n+1}x_{n+1}\|\\
&\quad +\frac{1}{r_{M,n+1}}\|T^{(\Theta_M,\varphi_M)}_{r_{M,n+1}}(I-r_{M,n+1}B_M)\varDelta^{M-1}_{n+1}x_{n+1}\\
&\quad -(I-r_{M,n+1}B_M)\varDelta^{M-1}_{n+1}x_{n+1}\|]+\cdots\\
&\quad +|r_{1,n+1}-r_{1,n}|[\|B_1\varDelta^{0}_{n+1}x_{n+1}\|\\
&\quad +\frac{1}{r_{1,n+1}}\|T^{(\Theta_1,\varphi_1)}_{r_{1,n+1}}(I-r_{1,n+1}B_1)\varDelta^{0}_{n+1}x_{n+1}\\
&\quad -(I-r_{1,n+1}B_1)\varDelta^{0}_{n+1}x_{n+1}\|]\\
&\quad +\|\varDelta^{0}_{n+1}x_{n+1}-\varDelta^{0}_{n}x_{n}\|\\
&\le \widetilde{M}_1\sum_{k=1}^{M}|r_{k,n+1}-r_{k,n}|+\|x_{n+1}-x_n\|,
\end{aligned}
$$

where $\widetilde{M}_1>0$ is a constant such that for each $n\ge 1$

$$
\begin{aligned}
&\sum_{k=1}^{M}[\|B_k\varDelta^{k-1}_{n+1}x_{n+1}\|\\
&+\frac{1}{r_{k,n+1}}\|T^{(\Theta_k,\varphi_k)}_{r_{k,n+1}}(I-r_{k,n+1}B_k)\varDelta^{k-1}_{n+1}x_{n+1}-(I-r_{k,n+1}B_k)\varDelta^{k-1}_{n+1}x_{n+1}\|]\le\widetilde{M}_1.
\end{aligned}
$$

Simple calculation shows that

$$
\begin{aligned}
y_{n+1}-y_n&=\alpha_{n+1}\gamma Qv_{n+1}+(I-\alpha_{n+1}\mu F)W_{n+1}Gv_{n+1}\\
&\quad-\alpha_n\gamma Qv_n-(I-\alpha_n\mu F)W_nGv_n\\
&=(\alpha_{n+1}-\alpha_n)(\gamma Qv_{n+1}-\mu FW_{n+1}Gv_{n+1})+\alpha_n\gamma(Qv_{n+1}-Qv_n)\\
&\quad+(I-\alpha_n\mu F)W_{n+1}Gv_{n+1}-(I-\alpha_n\mu F)W_nGv_n.
\end{aligned}
\tag{3.12}
$$

Also, from (2.2), since $W_n,\ S_n$ and $U_{n,i}$ are all nonexpansive, we have

$$
\begin{aligned}
\|W_{n+1}Gv_{n+1}-W_nGv_n\|&\le\|W_{n+1}Gv_{n+1}-W_{n+1}Gv_n\|+\|W_{n+1}Gv_n-W_nGv_n\|\\
&\le\|v_{n+1}-v_n\|+\|W_{n+1}Gv_n-W_nGv_n\|\\
&=\|v_{n+1}-v_n\|+\|\lambda_1T_1U_{n+1,2}Gv_n-\lambda_1T_1U_{n,2}Gv_n\|\\
&\le\|v_{n+1}-v_n\|+\lambda_1\|U_{n+1,2}Gv_n-U_{n,2}Gv_n\|
\end{aligned}
$$

$$
\begin{aligned}
&= \|v_{n+1} - v_n\| + \lambda_1 \|\lambda_2 T_2 U_{n+1,3} G v_n - \lambda_2 T_2 U_{n,3} G v_n\| \\
&\leq \|v_{n+1} - v_n\| + \lambda_1 \lambda_2 \|U_{n+1,3} G v_n - U_{n,3} G v_n\| \\
&\cdots \\
&\leq \|v_{n+1} - v_n\| + \lambda_1 \lambda_2 \cdots \lambda_n \|U_{n+1,n+1} G v_n - U_{n,n+1} G v_n\| \\
&\leq \|v_{n+1} - v_n\| + \widetilde{M}_2 \prod_{i=1}^{n} \lambda_i,
\end{aligned} \tag{3.13}
$$

where $\widetilde{M}_2$ is a constant such that $\|U_{n+1,n+1}Gv_n\| + \|U_{n,n+1}Gv_n\| \leq \widetilde{M}_2$ for each $n \geq 1$. So, utilizing Lemma 2.13, from (3.10)-(3.13) and $\{\lambda_n\} \subset (0, b] \subset (0, 1)$ it follows that

$$
\begin{aligned}
\|y_{n+1} - y_n\| &\leq |\alpha_{n+1} - \alpha_n|(\gamma\|Qv_{n+1}\| + \mu\|FW_{n+1}Gv_{n+1}\|) + \alpha_n\gamma\|Qv_{n+1} - Qv_n\| \\
&\quad + \|(I - \alpha_n\mu F)W_{n+1}Gv_{n+1} - (I - \alpha_n\mu F)W_nGv_n\| \\
&\leq |\alpha_{n+1} - \alpha_n|(\gamma\|Qv_{n+1}\| + \mu\|FW_{n+1}Gv_{n+1}\|) + \alpha_n\gamma l\|v_{n+1} - v_n\| \\
&\quad + (1 - \alpha_n\tau)\|W_{n+1}Gv_{n+1} - W_nGv_n\| \\
&\leq |\alpha_{n+1} - \alpha_n|(\gamma\|Qv_{n+1}\| + \mu\|FW_{n+1}Gv_{n+1}\|) + \alpha_n\gamma l\|v_{n+1} - v_n\| \\
&\quad + (1 - \alpha_n\tau)\Big(\|v_{n+1} - v_n\| + \widetilde{M}_2 \prod_{i=1}^{n} \lambda_i\Big) \\
&\leq |\alpha_{n+1} - \alpha_n|(\gamma\|Qv_{n+1}\| + \mu\|FW_{n+1}Gv_{n+1}\|) \\
&\quad + (1 - \alpha_n(\tau - \gamma l))\|v_{n+1} - v_n\| + \widetilde{M}_2 \prod_{i=1}^{n} \lambda_i \\
&\leq |\alpha_{n+1} - \alpha_n|(\gamma\|Qv_{n+1}\| + \mu\|FW_{n+1}Gv_{n+1}\|) \\
&\quad + (1 - \alpha_n(\tau - \gamma l))\Big[\widetilde{M}_0 \sum_{i=1}^{N} |\lambda_{i,n+1} - \lambda_{i,n}| + \|u_{n+1} - u_n\|\Big] + \widetilde{M}_2 \prod_{i=1}^{n} \lambda_i \\
&\leq |\alpha_{n+1} - \alpha_n|(\gamma\|Qv_{n+1}\| + \mu\|FW_{n+1}Gv_{n+1}\|) \\
&\quad + (1 - \alpha_n(\tau - \gamma l))\Big[\widetilde{M}_0 \sum_{i=1}^{N} |\lambda_{i,n+1} - \lambda_{i,n}| \\
&\quad + \widetilde{M}_1 \sum_{k=1}^{M} |r_{k,n+1} - r_{k,n}| + \|x_{n+1} - x_n\|\Big] + \widetilde{M}_2 \prod_{i=1}^{n} \lambda_i \\
&\leq |\alpha_{n+1} - \alpha_n|(\gamma\|Qv_{n+1}\| + \mu\|FW_{n+1}Gv_{n+1}\|) + \widetilde{M}_0 \sum_{i=1}^{N} |\lambda_{i,n+1} - \lambda_{i,n}| \\
&\quad + \widetilde{M}_1 \sum_{k=1}^{M} |r_{k,n+1} - r_{k,n}| + \|x_{n+1} - x_n\| + \widetilde{M}_2 b^n \\
&\leq \|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|(\gamma\|Qv_{n+1}\| + \mu\|FW_{n+1}Gv_{n+1}\|)
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
&+ (\widetilde{M}_0 + \widetilde{M}_1 + \widetilde{M}_2)\Big(\sum_{i=1}^{N} |\lambda_{i,n+1} - \lambda_{i,n}| + \sum_{k=1}^{M} |r_{k,n+1} - r_{k,n}| + b^n \Big) \\
&\leq \|x_{n+1} - x_n\| \\
&+ \widetilde{M}_3 \Big(\sum_{i=1}^{N} |\lambda_{i,n+1} - \lambda_{i,n}| + \sum_{k=1}^{M} |r_{k,n+1} - r_{k,n}| + |\alpha_{n+1} - \alpha_n| + b^n \Big),
\end{aligned}
$$

where $\widetilde{M}_3$ is a constant such that $\widetilde{M}_0 + \widetilde{M}_1 + \widetilde{M}_2 + \gamma\|Qv_n\| + \mu\|FW_nGv_n\| \leq \widetilde{M}_3$ for each $n \geq 1$. Thus, from (3.7), (3.9) and (3.14) it follows that

$$
\begin{aligned}
\|z_{n+1} - z_n\| &\leq \frac{s_{n+1}}{1-\beta_{n+1}}(\|\gamma Q x_{n+1}\| + \|VT_{n+1}y_{n+1}\|) + \frac{s_n}{1-\beta_n}(\|VT_n y_n\| + \|\gamma Q x_n\|) \\
&+ \|T_{n+1}y_{n+1} - T_n y_n\| \\
&\leq \frac{s_{n+1}}{1-\beta_{n+1}}(\|\gamma Q x_{n+1}\| + \|VT_{n+1}y_{n+1}\|) + \frac{s_n}{1-\beta_n}(\|VT_n y_n\| + \|\gamma Q x_n\|) \\
&+ \|y_{n+1} - y_n\| + \frac{4\widetilde{M}}{L}(s_{n+1} + s_n) \\
&\leq \frac{s_{n+1}}{1-\beta_{n+1}}(\|\gamma Q x_{n+1}\| + \|VT_{n+1}y_{n+1}\|) + \frac{s_n}{1-\beta_n}(\|VT_n y_n\| + \|\gamma Q x_n\|) \\
&+ \|x_{n+1} - x_n\| \\
&+ \widetilde{M}_3 \Big(\sum_{i=1}^{N} |\lambda_{i,n+1} - \lambda_{i,n}| + \sum_{k=1}^{M} |r_{k,n+1} - r_{k,n}| + |\alpha_{n+1} - \alpha_n| + b^n \Big) \\
&+ \frac{4\widetilde{M}}{L}(s_{n+1} + s_n),
\end{aligned}
$$

which immediately implies that

$$
\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq \frac{s_{n+1}}{1-\beta_{n+1}}(\|\gamma Q x_{n+1}\| + \|VT_{n+1}y_{n+1}\|) \\
&+ \frac{s_n}{1-\beta_n}(\|VT_n y_n\| + \|\gamma Q x_n\|) \\
&+ \widetilde{M}_3 \Big(\sum_{i=1}^{N} |\lambda_{i,n+1} - \lambda_{i,n}| + \sum_{k=1}^{M} |r_{k,n+1} - r_{k,n}| + |\alpha_{n+1} - \alpha_n| + b^n \Big) \\
&+ \frac{4\widetilde{M}}{L}(s_{n+1} + s_n).
\end{aligned}
$$

Since $\lim_{n\to\infty} s_n = 0$, $\lim_{n\to\infty} \alpha_n = 0$, $\{\beta_n\} \subset [a, \hat{a}] \subset (0,1)$, $\lim_{n\to\infty} |\lambda_{i,n+1} - \lambda_{i,n}| = 0$ and $\lim_{n\to\infty} |r_{k,n+1} - r_{k,n}| = 0$ for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$, we conclude that

$$
\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0,
$$

which together with Lemma 2.15 and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, implies that

$$
\lim_{n\to\infty} \|z_n - x_n\| = 0.
$$

So, it follows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - \beta_n)\|z_n - x_n\| = 0. \tag{3.15}$$

Step 3. We prove that $\|y_n - P_C(I - \frac{2}{L}\nabla f)y_n\| \to 0,\ \|x_n - u_n\| \to 0,\ \|x_n - v_n\| \to 0,\ \|v_n - Gv_n\| \to 0$ and $\|v_n - Wv_n\| \to 0$ as $n \to \infty$.

Indeed, utilizing Lemmas 2.7 and 2.13 we obtain from (3.1) (3.5) and (3.6) that

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n\gamma(Qv_n - Qp) + (I - \alpha_n\mu F)W_nGv_n \\
&\quad - (I - \alpha_n\mu F)p + \alpha_n(\gamma Q - \mu F)p\|^2 \\
&\le \|\alpha_n\gamma(Qv_n - Qp) + (I - \alpha_n\mu F)W_nGv_n \\
&\quad - (I - \alpha_n\mu F)p\|^2 + 2\alpha_n\langle(\gamma Q - \mu F)p, y_n - p\rangle \\
&\le [\alpha_n\gamma\|Qv_n - Qp\| + \|(I - \alpha_n\mu F)W_nGv_n \\
&\quad - (I - \alpha_n\mu F)p\|]^2 + 2\alpha_n\langle(\gamma Q - \mu F)p, y_n - p\rangle \\
&\le [\alpha_n\gamma l\|v_n - p\| + (1 - \alpha_n\tau)\|Gv_n - p\|]^2 + 2\alpha_n\langle(\gamma Q - \mu F)p, y_n - p\rangle \\
&\le [\alpha_n\tau\|v_n - p\| + (1 - \alpha_n\tau)\|Gv_n - p\|]^2 + 2\alpha_n\langle(\gamma Q - \mu F)p, y_n - p\rangle \\
&\le \alpha_n\tau\|v_n - p\|^2 + (1 - \alpha_n\tau)\|Gv_n - p\|^2 + 2\alpha_n\langle(\gamma Q - \mu F)p, y_n - p\rangle \\
&\le \alpha_n\tau\|v_n - p\|^2 + (1 - \alpha_n\tau)\|v_n - p\|^2 + 2\alpha_n\langle(\gamma Q - \mu F)p, y_n - p\rangle \\
&= \|v_n - p\|^2 + 2\alpha_n\langle(\gamma Q - \mu F)p, y_n - p\rangle \\
&\le \|x_n - p\|^2 + \alpha_n\langle(\gamma Q - \mu F)p, y_n - p\rangle.
\end{aligned} \tag{3.16}$$

Note that

$$x_{n+1} = s_n\gamma Qx_n + \beta_n x_n + ((1 - \beta_n)I - s_nV)T_ny_n.$$

Hence we have

$$x_{n+1} - y_n = s_n(\gamma Qx_n - VT_ny_n) + \beta_n(x_n - y_n) + (1 - \beta_n)(T_ny_n - y_n),$$

which yields

$$\begin{aligned}
(1 - \hat{a})\|T_ny_n - y_n\| &\le (1 - \beta_n)\|T_ny_n - y_n\| \\
&= \|x_{n+1} - y_n - s_n(\gamma Qx_n - VT_ny_n) - \beta_n(x_n - y_n)\| \\
&= \|x_{n+1} - x_n - s_n(\gamma Qx_n - VT_ny_n) + (1 - \beta_n)(x_n - y_n)\| \\
&\le \|x_{n+1} - x_n\| + s_n\|\gamma Qx_n - VT_ny_n\| + \|x_n - y_n\|.
\end{aligned}$$

Since $\lim_{n\to\infty} s_n = 0$ and $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$, from the assumption $\lim_{n\to\infty} \|x_n - y_n\| = 0$ and the boundedness of $\{x_n\}, \{y_n\}$, we obtain

$$\lim_{n\to\infty} \|y_n - T_ny_n\| = 0. \tag{3.17}$$

It is clear that

$$\begin{aligned}
\|P_C(I - \theta_n\nabla f)y_n - y_n\| &= \|s_ny_n + (1 - s_n)T_ny_n - y_n\| \\
&= (1 - s_n)\|T_ny_n - y_n\| \\
&\le \|T_ny_n - y_n\|,
\end{aligned}$$

where $s_n = \frac{2-\theta_n L}{4} \in (0, \frac{1}{2})$ for each $\theta_n \in (0, \frac{2}{L})$. Hence we have

$$\begin{aligned}\Big\|P_C\Big(I-\frac{2}{L}\nabla f\Big)y_n - y_n\Big\| &\le \Big\|P_C\Big(I-\frac{2}{L}\nabla f\Big)y_n - P_C(I-\theta_n\nabla f)y_n\Big\| \\ &\quad + \|P_C(I-\theta_n\nabla f)y_n - y_n\| \\ &\le \Big\|\Big(I-\frac{2}{L}\nabla f\Big)y_n - (I-\theta_n\nabla f)y_n\Big\| \\ &\quad + \|P_C(I-\theta_n\nabla f)y_n - y_n\| \\ &\le \Big(\frac{2}{L}-\theta_n\Big)\|\nabla f(y_n)\| + \|T_n y_n - y_n\|.\end{aligned}$$

From the boundedness of $\{y_n\}$, $s_n \to 0$ ($\Leftrightarrow$ $\theta_n \to \frac{2}{L}$) and $\|T_n y_n - y_n\| \to 0$ (due to (3.17)), it follows that

$$\lim_{n\to\infty} \|y_n - P_C\Big(I-\frac{2}{L}\nabla f\Big)y_n\| = 0. \tag{3.18}$$

Also, from (2.1) it follows that for all $i \in \{1,2,\dots,N\}$ and $k \in \{1,2,\dots,M\}$

$$\begin{aligned}\|v_n - p\|^2 &= \|\Lambda_n^N u_n - p\|^2 \\ &\le \|\Lambda_n^i u_n - p\|^2 \\ &= \|P_C(I-\lambda_{i,n}A_i)\Lambda_n^{i-1}u_n - P_C(I-\lambda_{i,n}A_i)p\|^2 \\ &\le \|(I-\lambda_{i,n}A_i)\Lambda_n^{i-1}u_n - (I-\lambda_{i,n}A_i)p\|^2 \\ &\le \|\Lambda_n^{i-1}u_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n}-2\eta_i)\|A_i\Lambda_n^{i-1}u_n - A_i p\|^2 \\ &\le \|u_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n}-2\eta_i)\|A_i\Lambda_n^{i-1}u_n - A_i p\|^2 \\ &\le \|x_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n}-2\eta_i)\|A_i\Lambda_n^{i-1}u_n - A_i p\|^2,\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\|u_n - p\|^2 &= \|\Delta_n^M x_n - p\|^2 \\ &\le \|\Delta_n^k x_n - p\|^2 \\ &= \|T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}B_k)\Delta_n^{k-1}x_n - T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}B_k)p\|^2 \\ &\le \|(I-r_{k,n}B_k)\Delta_n^{k-1}x_n - (I-r_{k,n}B_k)p\|^2 \\ &\le \|\Delta_n^{k-1}x_n - p\|^2 + r_{k,n}(r_{k,n}-2\mu_k)\|B_k\Delta_n^{k-1}x_n - B_k p\|^2 \\ &\le \|x_n - p\|^2 + r_{k,n}(r_{k,n}-2\mu_k)\|B_k\Delta_n^{k-1}x_n - B_k p\|^2.\end{aligned} \tag{3.20}$$

So, from (3.16), (3.19) and (3.20) it follows that

$$\begin{aligned}\|y_n - p\|^2 &\le \|v_n - p\|^2 + 2\alpha_n\langle(\gamma Q - \mu F)p, y_n - p\rangle \\ &\le \|u_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n}-2\eta_i)\|A_i\Lambda_n^{i-1}u_n - A_i p\|^2 \\ &\quad + 2\alpha_n\langle(\gamma Q - \mu F)p, y_n - p\rangle \\ &\le \|x_n - p\|^2 + r_{k,n}(r_{k,n}-2\mu_k)\|B_k\Delta_n^{k-1}x_n - B_k p\|^2 \\ &\quad + \lambda_{i,n}(\lambda_{i,n}-2\eta_i)\|A_i\Lambda_n^{i-1}u_n - A_i p\|^2 \\ &\quad + 2\alpha_n\|(\gamma Q - \mu F)p\|\|y_n - p\|,\end{aligned}$$

which hence leads to

$$\begin{aligned}
r_{k,n}(2\mu_k - r_{k,n})\|B_k\Delta_n^{k-1}x_n - B_kp\|^2 + \lambda_{i,n}(2\eta_i - \lambda_{i,n})\|A_i\Lambda_n^{i-1}u_n - A_ip\|^2 \\
\le \|x_n - p\|^2 - \|y_n - p\|^2 + 2\alpha_n\|(\gamma Q - \mu F)p\|\|y_n - p\| \\
\le \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + 2\alpha_n\|(\gamma Q - \mu F)p\|\|y_n - p\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$ for all $i \in \{1, 2, \dots, N\}$ and $k \in \{1, 2, \dots, M\}$, by the assumption $\lim_{n\to\infty}\|x_n - y_n\| = 0$ and the the boundedness of $\{x_n\}, \{y_n\}$, we conclude immediately that

$$\lim_{n\to\infty}\|A_i\Lambda_n^{i-1}u_n - A_ip\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|B_k\Delta_n^{k-1}x_n - B_kp\| = 0, \tag{3.21}$$

for all $i \in \{1, 2, \dots, N\}$ and $k \in \{1, 2, \dots, M\}$.

Furthermore, by Proposition 2.6 (ii) we obtain that for each $k \in \{1, 2, \dots, M\}$

$$\begin{aligned}
\|\Delta_n^k x_n - p\|^2 &= \|T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}B_k)\Delta_n^{k-1}x_n - T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}B_k)p\|^2 \\
&\le \langle (I - r_{k,n}B_k)\Delta_n^{k-1}x_n - (I - r_{k,n}B_k)p, \Delta_n^k x_n - p\rangle \\
&= \frac{1}{2}(\|(I - r_{k,n}B_k)\Delta_n^{k-1}x_n - (I - r_{k,n}B_k)p\|^2 + \|\Delta_n^k x_n - p\|^2 \\
&\quad - \|(I - r_{k,n}B_k)\Delta_n^{k-1}x_n - (I - r_{k,n}B_k)p - (\Delta_n^k x_n - p)\|^2) \\
&\le \frac{1}{2}(\|\Delta_n^{k-1}x_n - p\|^2 + \|\Delta_n^k x_n - p\|^2 \\
&\quad - \|\Delta_n^{k-1}x_n - \Delta_n^k x_n - r_{k,n}(B_k\Delta_n^{k-1}x_n - B_kp)\|^2),
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\Delta_n^k x_n - p\|^2 &\le \|\Delta_n^{k-1}x_n - p\|^2 - \|\Delta_n^{k-1}x_n - \Delta_n^k x_n - r_{k,n}(B_k\Delta_n^{k-1}x_n - B_kp)\|^2 \\
&= \|\Delta_n^{k-1}x_n - p\|^2 - \|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|^2 - r_{k,n}^2\|B_k\Delta_n^{k-1}x_n - B_kp\|^2 \\
&\quad + 2r_{k,n}\langle \Delta_n^{k-1}x_n - \Delta_n^k x_n, B_k\Delta_n^{k-1}x_n - B_kp\rangle \\
&\le \|\Delta_n^{k-1}x_n - p\|^2 - \|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|^2 \qquad (3.22) \\
&\quad + 2r_{k,n}\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|\|B_k\Delta_n^{k-1}x_n - B_kp\| \\
&\le \|x_n - p\|^2 - \|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|^2 \\
&\quad + 2r_{k,n}\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|\|B_k\Delta_n^{k-1}x_n - B_kp\|.
\end{aligned}$$

Also, by Proposition 2.1 (iii), we obtain that for each $i \in \{1, 2, \dots, N\}$

$$\begin{aligned}
\|\Lambda_n^i u_n - p\|^2 &= \|P_C(I - \lambda_{i,n}A_i)\Lambda_n^{i-1}u_n - P_C(I - \lambda_{i,n}A_i)p\|^2 \\
&\le \langle (I - \lambda_{i,n}A_i)\Lambda_n^{i-1}u_n - (I - \lambda_{i,n}A_i)p, \Lambda_n^i u_n - p\rangle \\
&= \frac{1}{2}(\|(I - \lambda_{i,n}A_i)\Lambda_n^{i-1}u_n - (I - \lambda_{i,n}A_i)p\|^2 + \|\Lambda_n^i u_n - p\|^2 \\
&\quad - \|(I - \lambda_{i,n}A_i)\Lambda_n^{i-1}u_n - (I - \lambda_{i,n}A_i)p - (\Lambda_n^i u_n - p)\|^2) \\
&\le \frac{1}{2}(\|\Lambda_n^{i-1}u_n - p\|^2 + \|\Lambda_n^i u_n - p\|^2 \\
&\quad - \|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n - \lambda_{i,n}(A_i\Lambda_n^{i-1}u_n - A_ip)\|^2)
\end{aligned}$$

$$\leq \frac{1}{2}(\|u_n - p\|^2 + \|\Lambda_n^i u_n - p\|^2 - \|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n - \lambda_{i,n}(A_i\Lambda_n^{i-1}u_n - A_ip)\|^2),$$

which implies

$$\begin{aligned}\|\Lambda_n^i u_n - p\|^2 &\leq \|u_n - p\|^2 - \|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n - \lambda_{i,n}(A_i\Lambda_n^{i-1}u_n - A_ip)\|^2 \\ &= \|u_n - p\|^2 - \|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|^2 - \lambda_{i,n}^2\|A_i\Lambda_n^{i-1}u_n - A_ip\|^2 \\ &\quad + 2\lambda_{i,n}\langle \Lambda_n^{i-1}u_n - \Lambda_n^i u_n, A_i\Lambda_n^{i-1}u_n - A_ip\rangle \\ &\leq \|u_n - p\|^2 - \|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|^2 \\ &\quad + 2\lambda_{i,n}\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|\|A_i\Lambda_n^{i-1}u_n - A_ip\|.\end{aligned} \tag{3.23}$$

Thus, from (3.16), (3.22) and (3.23), we have

$$\begin{aligned}\|y_n - p\|^2 &\leq \|v_n - p\|^2 + 2\alpha_n\langle(\gamma Q - \mu F)p, y_n - p\rangle \\ &\leq \|\Lambda_n^i u_n - p\|^2 + 2\alpha_n\langle(\gamma Q - \mu F)p, y_n - p\rangle \\ &\leq \|u_n - p\|^2 - \|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|^2 \\ &\quad + 2\lambda_{i,n}\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|\|A_i\Lambda_n^{i-1}u_n - A_ip\| \\ &\quad + 2\alpha_n\langle(\gamma Q - \mu F)p, y_n - p\rangle \\ &\leq \|\Delta_n^k x_n - p\|^2 - \|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|^2 \\ &\quad + 2\lambda_{i,n}\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|\|A_i\Lambda_n^{i-1}u_n - A_ip\| \\ &\quad + 2\alpha_n\langle(\gamma Q - \mu F)p, y_n - p\rangle \\ &\leq \|x_n - p\|^2 - \|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|^2 \\ &\quad + 2r_{k,n}\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|\|B_k\Delta_n^{k-1}x_n - B_kp\| \\ &\quad - \|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|^2 + 2\lambda_{i,n}\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|\|A_i\Lambda_n^{i-1}u_n - A_ip\| \\ &\quad + 2\alpha_n\|(\gamma Q - \mu F)p\|\|y_n - p\|,\end{aligned}$$

which yields

$$\begin{aligned}&\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|^2 + \|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|^2 \\ &\leq \|x_n - p\|^2 - \|y_n - p\|^2 + 2r_{k,n}\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|\|B_k\Delta_n^{k-1}x_n - B_kp\| \\ &\quad + 2\lambda_{i,n}\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|\|A_i\Lambda_n^{i-1}u_n - A_ip\| + 2\alpha_n\|(\gamma Q - \mu F)p\|\|y_n - p\| \\ &\leq \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + 2r_{k,n}\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|\|B_k\Delta_n^{k-1}x_n - B_kp\| \\ &\quad + 2\lambda_{i,n}\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|\|A_i\Lambda_n^{i-1}u_n - A_ip\| + 2\alpha_n\|(\gamma Q - \mu F)p\|\|y_n - p\|.\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are bounded, $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$ for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$, by (3.21) and the assumption $\lim_{n\to\infty}\|x_n - y_n\| = 0$, we conclude immediately that

$$\lim_{n\to\infty}\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\| \quad \text{and} \quad \lim_{n\to\infty}\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\| = 0, \tag{3.24}$$

for all $i \in \{1, 2, \dots, N\}$ and $k \in \{1, 2, \dots, M\}$. Note that

$$\begin{aligned}\|x_n - u_n\| &= \|\varDelta_n^0 x_n - \varDelta_n^M x_n\| \\ &\le \|\varDelta_n^0 x_n - \varDelta_n^1 x_n\| + \|\varDelta_n^1 x_n - \varDelta_n^2 x_n\| + \cdots + \|\varDelta_n^{M-1} x_n - \varDelta_n^M x_n\|,\end{aligned}$$

and

$$\begin{aligned}\|u_n - v_n\| &= \|\varLambda_n^0 u_n - \varLambda_n^N u_n\| \\ &\le \|\varLambda_n^0 u_n - \varLambda_n^1 u_n\| + \|\varLambda_n^1 u_n - \varLambda_n^2 u_n\| + \cdots + \|\varLambda_n^{N-1} u_n - \varLambda_n^N u_n\|.\end{aligned}$$

Thus, from (3.24) we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|u_n - v_n\| = 0. \tag{3.25}$$

It is easy to see that as $n \to \infty$

$$\|x_n - v_n\| \le \|x_n - u_n\| + \|u_n - v_n\| \to 0.$$

On the other hand, for simplicity, we write $\tilde{p} = P_C(I - \nu_2 F_2)p$, $\tilde{v}_n = P_C(I - \nu_2 F_2)v_n$ and $w_n = Gv_n = P_C(I - \nu_1 F_1)\tilde{v}_n$ for all $n \ge 1$. Then

$$p = Gp = P_C(I - \nu_1 F_1)\tilde{p} = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)p.$$

We now show that $\lim_{n\to\infty} \|Gv_n - v_n\| = 0$, i.e., $\lim_{n\to\infty} \|w_n - v_n\| = 0$. As a matter of fact, for $p \in \varOmega$, it follows from (3.5), (3.6) and (3.16) that

$$\begin{aligned}\|y_n - p\|^2 &\le \alpha_n\tau\|v_n - p\|^2 + (1 - \alpha_n\tau)\|Gv_n - p\|^2 + 2\alpha_n\langle(\gamma Q - \mu F)p, y_n - p\rangle \\ &\le \alpha_n\tau\|v_n - p\|^2 + (1 - \alpha_n\tau)\|w_n - p\|^2 + 2\alpha_n\|(\gamma Q - \mu F)p\|\|y_n - p\| \\ &\le \alpha_n\tau\|v_n - p\|^2 + (1 - \alpha_n\tau)[\|\tilde{v}_n - \tilde{p}\|^2 + \nu_1(\nu_1 - 2\zeta_1)\|F_1\tilde{v}_n - F_1\tilde{p}\|^2] \\ &\quad + 2\alpha_n\|(\gamma Q - \mu F)p\|\|y_n - p\| \\ &\le \alpha_n\tau\|v_n - p\|^2 + (1 - \alpha_n\tau)[\|v_n - p\|^2 + \nu_2(\nu_2 - 2\zeta_2)\|F_2 v_n - F_2 p\|^2 \\ &\quad + \nu_1(\nu_1 - 2\zeta_1)\|F_1\tilde{v}_n - F_1\tilde{p}\|^2] + 2\alpha_n\|(\gamma Q - \mu F)p\|\|y_n - p\| \\ &= \|v_n - p\|^2 + (1 - \alpha_n\tau)[\nu_2(\nu_2 - 2\zeta_2)\|F_2 v_n - F_2 p\|^2 \\ &\quad + \nu_1(\nu_1 - 2\zeta_1)\|F_1\tilde{v}_n - F_1\tilde{p}\|^2] \\ &\quad + 2\alpha_n\|(\gamma Q - \mu F)p\|\|y_n - p\| \\ &\le \|x_n - p\|^2 + (1 - \alpha_n\tau)[\nu_2(\nu_2 - 2\zeta_2)\|F_2 v_n - F_2 p\|^2 \\ &\quad + \nu_1(\nu_1 - 2\zeta_1)\|F_1\tilde{v}_n - F_1\tilde{p}\|^2] \\ &\quad + 2\alpha_n\|(\gamma Q - \mu F)p\|\|y_n - p\|,\end{aligned} \tag{3.26}$$

which immediately yields

$$\begin{aligned}&(1 - \alpha_n\tau)[\nu_2(2\zeta_2 - \nu_2)\|F_2 v_n - F_2 p\|^2 + \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{v}_n - F_1\tilde{p}\|^2] \\ &\quad \le \|x_n - p\|^2 - \|y_n - p\|^2 + 2\alpha_n\|(\gamma Q - \mu F)p\|\|y_n - p\| \\ &\quad \le \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + 2\alpha_n\|(\gamma Q - \mu F)p\|\|y_n - p\|.\end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\nu_j \in (0, 2\zeta_j)$ for $j = 1, 2$ and $\{x_n\}, \{y_n\}$ are bounded, by the assumption $\lim_{n\to\infty} \|x_n - y_n\| = 0$, we get

$$\lim_{n\to\infty} \|F_2 v_n - F_2 p\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|F_1\tilde{v}_n - F_1\tilde{p}\| = 0. \tag{3.27}$$

Also, in terms of the firm nonexpansivity of P_C and the ζ_j-inverse strong monotonicity of F_j for $j = 1, 2$, we obtain from $\nu_j \in (0, 2\zeta_j), j = 1, 2$ and (3.6) that

$$\begin{aligned}
\|\tilde{v}_n - \tilde{p}\|^2 &= \|P_C(I - \nu_2 F_2)v_n - P_C(I - \nu_2 F_2)p\|^2 \\
&\le \langle (I - \nu_2 F_2)v_n - (I - \nu_2 F_2)p, \tilde{v}_n - \tilde{p}\rangle \\
&= \frac{1}{2}[\|(I - \nu_2 F_2)v_n - (I - \nu_2 F_2)p\|^2 + \|\tilde{v}_n - \tilde{p}\|^2 \\
&\quad - \|(I - \nu_2 F_2)v_n - (I - \nu_2 F_2)p - (\tilde{v}_n - \tilde{p})\|^2] \\
&\le \frac{1}{2}[\|v_n - p\|^2 + \|\tilde{v}_n - \tilde{p}\|^2 - \|(v_n - \tilde{v}_n) - \nu_2(F_2 v_n - F_2 p) - (p - \tilde{p})\|^2] \\
&= \frac{1}{2}[\|v_n - p\|^2 + \|\tilde{v}_n - \tilde{p}\|^2 - \|(v_n - \tilde{v}_n) - (p - \tilde{p})\|^2 \\
&\quad + 2\nu_2\langle (v_n - \tilde{v}_n) - (p - \tilde{p}), F_2 v_n - F_2 p\rangle - \nu_2^2\|F_2 v_n - F_2 p\|^2],
\end{aligned}$$

and

$$\begin{aligned}
\|w_n - p\|^2 &= \|P_C(I - \nu_1 F_1)\tilde{v}_n - P_C(I - \nu_1 F_1)\tilde{p}\|^2 \\
&\le \langle (I - \nu_1 F_1)\tilde{v}_n - (I - \nu_1 F_1)\tilde{p}, w_n - p\rangle \\
&= \frac{1}{2}[\|(I - \nu_1 F_1)\tilde{v}_n - (I - \nu_1 F_1)\tilde{p}\|^2 + \|w_n - p\|^2 \\
&\quad - \|(I - \nu_1 F_1)\tilde{v}_n - (I - \nu_1 F_1)\tilde{p} - (w_n - p)\|^2] \\
&\le \frac{1}{2}[\|\tilde{v}_n - \tilde{p}\|^2 + \|w_n - p\|^2 - \|(\tilde{v}_n - w_n) + (p - \tilde{p})\|^2 \\
&\quad + 2\nu_1\langle F_1\tilde{v}_n - F_1\tilde{p}, (\tilde{v}_n - w_n) + (p - \tilde{p})\rangle - \nu_1^2\|F_1\tilde{v}_n - F_1\tilde{p}\|^2] \\
&\le \frac{1}{2}[\|v_n - p\|^2 + \|w_n - p\|^2 - \|(\tilde{v}_n - w_n) + (p - \tilde{p})\|^2 \\
&\quad + 2\nu_1\langle F_1\tilde{v}_n - F_1\tilde{p}, (\tilde{v}_n - w_n) + (p - \tilde{p})\rangle].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|\tilde{v}_n - \tilde{p}\|^2 \le& \|v_n - p\|^2 - \|(v_n - \tilde{v}_n) - (p - \tilde{p})\|^2 \\
&+ 2\nu_2\langle (v_n - \tilde{v}_n) - (p - \tilde{p}), F_2 v_n - F_2 p\rangle - \nu_2^2\|F_2 v_n - F_2 p\|^2,
\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
\|w_n - p\|^2 \le& \|v_n - p\|^2 - \|(\tilde{v}_n - w_n) + (p - \tilde{p})\|^2 \\
&+ 2\nu_1\|F_1\tilde{v}_n - F_1\tilde{p}\|\|(\tilde{v}_n - w_n) + (p - \tilde{p})\|.
\end{aligned} \tag{3.29}$$

Consequently, from (3.5), (3.26) and (3.28) it follows that

$$\begin{aligned}
\|y_n - p\|^2 &\le \alpha_n\tau\|v_n - p\|^2 + (1 - \alpha_n\tau)[\|\tilde{v}_n - \tilde{p}\|^2 + \nu_1(\nu_1 - 2\zeta_1)\|F_1\tilde{v}_n - F_1\tilde{p}\|^2] \\
&\quad + 2\alpha_n\|(\gamma Q - \mu F)p\|\|y_n - p\| \\
&\le \alpha_n\tau\|v_n - p\|^2 + (1 - \alpha_n\tau)\|\tilde{v}_n - \tilde{p}\|^2 + 2\alpha_n\|(\gamma Q - \mu F)p\|\|y_n - p\| \\
&\le \alpha_n\tau\|v_n - p\|^2 + (1 - \alpha_n\tau)[\|v_n - p\|^2 - \|(v_n - \tilde{v}_n) - (p - \tilde{p})\|^2 \\
&\quad + 2\nu_2\langle (v_n - \tilde{v}_n) - (p - \tilde{p}), F_2 v_n - F_2 p\rangle - \nu_2^2\|F_2 v_n - F_2 p\|^2] \\
&\quad + 2\alpha_n\|(\gamma Q - \mu F)p\|\|y_n - p\|
\end{aligned}$$

$$\begin{aligned}
&\le \alpha_n\tau\|v_n-p\|^2+(1-\alpha_n\tau)[\|v_n-p\|^2-\|(v_n-\tilde{v}_n)-(p-\tilde{p})\|^2\\
&\quad+2\nu_2\|(v_n-\tilde{v}_n)-(p-\tilde{p})\|\|F_2v_n-F_2p\|]+2\alpha_n\|(\gamma Q-\mu F)p\|\|y_n-p\|\\
&\le \|v_n-p\|^2-(1-\alpha_n\tau)\|(v_n-\tilde{v}_n)-(p-\tilde{p})\|^2\\
&\quad+2\nu_2\|(v_n-\tilde{v}_n)-(p-\tilde{p})\|\|F_2v_n-F_2p\|\\
&\quad+2\alpha_n\|(\gamma Q-\mu F)p\|\|y_n-p\|\\
&\le \|x_n-p\|^2-(1-\alpha_n\tau)\|(v_n-\tilde{v}_n)-(p-\tilde{p})\|^2\\
&\quad+2\nu_2\|(v_n-\tilde{v}_n)-(p-\tilde{p})\|\|F_2v_n-F_2p\|\\
&\quad+2\alpha_n\|(\gamma Q-\mu F)p\|\|y_n-p\|,
\end{aligned}$$

which hence leads to

$$\begin{aligned}
(1-\alpha_n\tau)\|(v_n-\tilde{v}_n)-(p-\tilde{p})\|^2 &\le \|x_n-p\|^2-\|y_n-p\|^2\\
&\quad+2\nu_2\|(v_n-\tilde{v}_n)-(p-\tilde{p})\|\|F_2v_n-F_2p\|\\
&\quad+2\alpha_n\|(\gamma Q-\mu F)p\|\|y_n-p\|\\
&\le \|x_n-y_n\|(\|x_n-p\|+\|y_n-p\|)\\
&\quad+2\nu_2\|(v_n-\tilde{v}_n)-(p-\tilde{p})\|\|F_2v_n-F_2p\|\\
&\quad+2\alpha_n\|(\gamma Q-\mu F)p\|\|y_n-p\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n=0$, $\{x_n\},\{y_n\},\{v_n\}$ and $\{\tilde{v}_n\}$ are bounded sequences, by the assumption $\lim_{n\to\infty}\|x_n-y_n\|=0$, we conclude from (3.27) that

$$\lim_{n\to\infty}\|(v_n-\tilde{v}_n)-(p-\tilde{p})\|=0. \tag{3.30}$$

Furthermore, from (3.5), (3.26) and (3.29) it follows that

$$\begin{aligned}
\|y_n-p\|^2 &\le \alpha_n\tau\|v_n-p\|^2+(1-\alpha_n\tau)\|w_n-p\|^2+2\alpha_n\|(\gamma Q-\mu F)p\|\|y_n-p\|\\
&\le \alpha_n\tau\|v_n-p\|^2+(1-\alpha_n\tau)[\|v_n-p\|^2-\|(\tilde{v}_n-w_n)+(p-\tilde{p})\|^2\\
&\quad+2\nu_1\|F_1\tilde{v}_n-F_1\tilde{p}\|\|(\tilde{v}_n-w_n)+(p-\tilde{p})\|]\\
&\quad+2\alpha_n\|(\gamma Q-\mu F)p\|\|y_n-p\|\\
&\le \|v_n-p\|^2-(1-\alpha_n\tau)\|(\tilde{v}_n-w_n)+(p-\tilde{p})\|^2\\
&\quad+2\nu_1\|F_1\tilde{v}_n-F_1\tilde{p}\|\|(\tilde{v}_n-w_n)+(p-\tilde{p})\|\\
&\quad+2\alpha_n\|(\gamma Q-\mu F)p\|\|y_n-p\|\\
&\le \|x_n-p\|^2-(1-\alpha_n\tau)\|(\tilde{v}_n-w_n)+(p-\tilde{p})\|^2\\
&\quad+2\nu_1\|F_1\tilde{v}_n-F_1\tilde{p}\|\|(\tilde{v}_n-w_n)+(p-\tilde{p})\|\\
&\quad+2\alpha_n\|(\gamma Q-\mu F)p\|\|y_n-p\|,
\end{aligned}$$

which hence yields

$$\begin{aligned}
(1-\alpha_n\tau)\|(\tilde{v}_n-w_n)+(p-\tilde{p})\|^2 &\le \|x_n-p\|^2-\|y_n-p\|^2\\
&\quad+2\nu_1\|F_1\tilde{v}_n-F_1\tilde{p}\|\|(\tilde{v}_n-w_n)+(p-\tilde{p})\|\\
&\quad+2\alpha_n\|(\gamma Q-\mu F)p\|\|y_n-p\|\\
&\le \|x_n-y_n\|(\|x_n-p\|+\|y_n-p\|)\\
&\quad+2\nu_1\|F_1\tilde{v}_n-F_1\tilde{p}\|\|(\tilde{v}_n-w_n)+(p-\tilde{p})\|
\end{aligned}$$

$$+ 2\alpha_n\|(\gamma Q - \mu F)p\|\|y_n - p\|.$$

Since $\lim_{n\to\infty} \alpha_n = 0$ and $\{x_n\}, \{y_n\}, \{w_n\}$ and $\{\tilde{v}_n\}$ are bounded sequences, by the assumption $\lim_{n\to\infty} \|x_n - y_n\| = 0$, we conclude from (3.27) that

$$\lim_{n\to\infty} \|(\tilde{v}_n - w_n) + (p - \tilde{p})\| = 0. \tag{3.31}$$

Note that

$$\|v_n - w_n\| \le \|(v_n - \tilde{v}_n) - (p - \tilde{p})\| + \|(\tilde{v}_n - w_n) + (p - \tilde{p})\|.$$

Hence from (3.30) and (3.31) we get

$$\lim_{n\to\infty} \|v_n - Gv_n\| = \lim_{n\to\infty} \|v_n - w_n\| = 0. \tag{3.32}$$

Also, observe that

$$y_n = \alpha_n\gamma Qv_n + (I - \alpha_n\mu F)W_nGv_n.$$

Hence we get

$$y_n - W_nGv_n = \alpha_n(\gamma Qv_n - \mu FW_nGv_n).$$

So, from $\lim_{n\to\infty} \alpha_n = 0$ and the boundedness of $\{v_n\}$ we deduce that

$$\lim_{n\to\infty} \|y_n - W_nGv_n\| = 0. \tag{3.33}$$

In addition, it is readily found that

$$\begin{aligned} \|W_nv_n - v_n\| &\le \|W_nv_n - W_nGv_n\| + \|W_nGv_n - v_n\| \\ &\le \|v_n - Gv_n\| + \|W_nGv_n - v_n\| \\ &\le \|v_n - Gv_n\| + \|W_nGv_n - y_n\| + \|y_n - v_n\| \\ &\le \|v_n - Gv_n\| + \|W_nGv_n - y_n\| + \|y_n - x_n\| + \|x_n - v_n\|. \end{aligned}$$

Thus, by the assumption $\lim_{n\to\infty} \|x_n - y_n\| = 0$, from (3.25), (3.32) and (3.33) we have

$$\lim_{n\to\infty} \|W_nv_n - v_n\| = 0. \tag{3.34}$$

Taking into account that $\|v_n - Wv_n\| \le \|v_n - W_nv_n\| + \|W_nv_n - Wv_n\|$, we obtain from $\|v_n - W_nv_n\| \to 0$ and [22, Remark 3.2] that

$$\lim_{n\to\infty} \|v_n - Wv_n\| = 0. \tag{3.35}$$

Step 4. We prove that $x_n \to x^* = P_\Omega(I - (V - \gamma Q))x^*$ as $n \to \infty$.

Indeed, first of all, let us show that

$$\limsup_{n\to\infty} \langle(\gamma Q - V)x^*, x_n - x^*\rangle \le 0.$$

Since $\{x_n\}$ is bounded, we may assume, without loss of generality, that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$ and

$$\limsup_{n\to\infty} \langle(\gamma Q - V)x^*, x_n - x^*\rangle = \lim_{i\to\infty} \langle(\gamma Q - V)x^*, x_{n_i} - x^*\rangle = \langle(\gamma Q - V)x^*, w - x^*\rangle. \tag{3.36}$$

From (3.24), (3.25) and the assumption $\lim_{n\to\infty} \|x_n - y_n\| = 0$ we have that $y_{n_i} \rightharpoonup w$, $u_{n_i} \rightharpoonup w$, $v_{n_i} \rightharpoonup w$, $\Delta^k_{n_i}x_{n_i} \rightharpoonup w$ and $\Lambda^m_{n_i}u_{n_i} \rightharpoonup w$, where $k \in \{1, 2, \ldots, M\}$ and $m \in \{1, 2, \ldots, N\}$. Utilizing Lemma 2.9, we deduce from $x_{n_i} \rightharpoonup w$, $v_{n_i} \rightharpoonup w$, (3.18), (3.32) and (3.35) that $w \in \mathrm{Fix}(P_C(I - \frac{2}{L}\nabla f)) = \mathrm{VI}(C, \nabla f) = \Gamma$, $w \in$

GSVI(G) and $w \in \text{Fix}(W) = \cap_{n=1}^{\infty}\text{Fix}(S_n)$ (due to Lemma 2.11). Thus, we get $w \in \cap_{n=1}^{\infty}\text{Fix}(S_n) \cap \text{GSVI}(G) \cap \Gamma$. Next we prove that $w \in \cap_{m=1}^{N}\text{VI}(C, A_m)$. Let

$$\widetilde{T}_m v = \begin{cases} A_m v + N_C v, & v \in C, \\ \emptyset, & v \notin C, \end{cases}$$

where $m \in \{1, 2, \dots, N\}$. Let $(v, u) \in G(\widetilde{T}_m)$. Since $u - A_m v \in N_C v$ and $\Lambda_n^m u_n \in C$, we have

$$\langle v - \Lambda_n^m u_n, u - A_m v\rangle \geq 0.$$

On the other hand, from $\Lambda_n^m u_n = P_C(I - \lambda_{m,n}A_m)\Lambda_n^{m-1}u_n$ and $v \in C$, we have

$$\langle v - \Lambda_n^m u_n, \Lambda_n^m u_n - (\Lambda_n^{m-1}u_n - \lambda_{m,n}A_m\Lambda_n^{m-1}u_n)\rangle \geq 0,$$

and hence

$$\langle v - \Lambda_n^m u_n, \frac{\Lambda_n^m u_n - \Lambda_n^{m-1}u_n}{\lambda_{m,n}} + A_m\Lambda_n^{m-1}u_n\rangle \geq 0.$$

Therefore we have

$$\begin{aligned}\langle v - \Lambda_{n_i}^m u_{n_i}, u\rangle &\geq \langle v - \Lambda_{n_i}^m u_{n_i}, A_m v\rangle \\ &\geq \langle v - \Lambda_{n_i}^m u_{n_i}, A_m v\rangle \\ &\quad - \left\langle v - \Lambda_{n_i}^m u_{n_i}, \frac{\Lambda_{n_i}^m u_{n_i} - \Lambda_{n_i}^{m-1}u_{n_i}}{\lambda_{m,n_i}} + A_m\Lambda_{n_i}^{m-1}u_{n_i}\right\rangle \\ &= \langle v - \Lambda_{n_i}^m u_{n_i}, A_m v - A_m\Lambda_{n_i}^m u_{n_i}\rangle \\ &\quad + \langle v - \Lambda_{n_i}^m u_{n_i}, A_m\Lambda_{n_i}^m u_{n_i} - A_m\Lambda_{n_i}^{m-1}u_{n_i}\rangle \\ &\quad - \left\langle v - \Lambda_{n_i}^m u_{n_i}, \frac{\Lambda_{n_i}^m u_{n_i} - \Lambda_{n_i}^{m-1}u_{n_i}}{\lambda_{m,n_i}}\right\rangle \\ &\geq \langle v - \Lambda_{n_i}^m u_{n_i}, A_m\Lambda_{n_i}^m u_{n_i} - A_m\Lambda_{n_i}^{m-1}u_{n_i}\rangle \\ &\quad - \left\langle v - \Lambda_{n_i}^m u_{n_i}, \frac{\Lambda_{n_i}^m u_{n_i} - \Lambda_{n_i}^{m-1}u_{n_i}}{\lambda_{m,n_i}}\right\rangle.\end{aligned}$$

From (3.24) and since A_m is Lipschitz continuous, we obtain that $\lim_{n\to\infty}\|A_m\Lambda_n^m u_n - A_m\Lambda_n^{m-1}u_n\| = 0$. From $\Lambda_{n_i}^m u_{n_i} \rightharpoonup w$, $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $\forall i \in \{1, 2, \dots, N\}$ and (3.24), we have

$$\langle v - w, u\rangle \geq 0.$$

Since $\widetilde{T}_m$ is maximal monotone, we have $w \in \widetilde{T}_m^{-1}0$ and hence $w \in \text{VI}(C, A_m)$, $m = 1, 2, \dots, N$, which implies $w \in \cap_{m=1}^{N}\text{VI}(C, A_m)$. Next we prove that $w \in \cap_{k=1}^{M}\text{GMEP}(\Theta_k, \varphi_k, B_k)$. Since $\Delta_n^k x_n = T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}B_k)\Delta_n^{k-1}x_n, n \geq 1, k \in \{1, 2, \dots, M\}$, we have

$$\begin{aligned}\Theta_k(\Delta_n^k x_n, y) + \varphi_k(y) - \varphi_k(\Delta_n^k x_n) + \langle B_k\Delta_n^{k-1}x_n, y - \Delta_n^k x_n\rangle \\ + \frac{1}{r_{k,n}}\langle y - \Delta_n^k x_n, \Delta_n^k x_n - \Delta_n^{k-1}x_n\rangle \geq 0.\end{aligned}$$

By (A2), we have

$$\varphi_k(y) - \varphi_k(\Delta_n^k x_n) + \langle B_k\Delta_n^{k-1}x_n, y - \Delta_n^k x_n\rangle$$

$$+\frac{1}{r_{k,n}}\langle y-\Delta_n^k x_n, \Delta_n^k x_n-\Delta_n^{k-1}x_n\rangle \geq \Theta_k(y,\Delta_n^k x_n).$$

Let $z_t = ty+(1-t)w$ for all $t\in(0,1]$ and $y\in C$. This implies that $z_t\in C$. Then, we have

$$\begin{aligned}\langle z_t-\Delta_n^k x_n, B_k z_t\rangle &\geq \varphi_k(\Delta_n^k x_n)-\varphi_k(z_t)+\langle z_t-\Delta_n^k x_n, B_k z_t\rangle\\ &\quad-\langle z_t-\Delta_n^k x_n, B_k\Delta_n^{k-1}x_n\rangle\\ &\quad-\langle z_t-\Delta_n^k x_n, \frac{\Delta_n^k x_n-\Delta_n^{k-1}x_n}{r_{k,n}}\rangle+\Theta_k(z_t,\Delta_n^k x_n)\\ &=\varphi_k(\Delta_n^k x_n)-\varphi_k(z_t)+\langle z_t-\Delta_n^k x_n, B_k z_t-B_k\Delta_n^k x_n\rangle\\ &\quad+\langle z_t-\Delta_n^k x_n, B_k\Delta_n^k x_n-B_k\Delta_n^{k-1}x_n\rangle\\ &\quad-\langle z_t-\Delta_n^k x_n, \frac{\Delta_n^k x_n-\Delta_n^{k-1}x_n}{r_{k,n}}\rangle+\Theta_k(z_t,\Delta_n^k x_n).\end{aligned}\tag{3.37}$$

By (3.24), we have $\|B_k\Delta_n^k x_n-B_k\Delta_n^{k-1}x_n\|\to 0$ as $n\to\infty$. Furthermore, by the monotonicity of B_k, we obtain $\langle z_t-\Delta_n^k x_n, B_k z_t-B_k\Delta_n^k x_n\rangle\geq 0$. Then, by (A4) we obtain

$$\langle z_t-w, B_k z_t\rangle \geq \varphi_k(w)-\varphi_k(z_t)+\Theta_k(z_t,w).\tag{3.38}$$

Utilizing (A1), (A4) and (3.38), we obtain

$$\begin{aligned}0&=\Theta_k(z_t,z_t)+\varphi_k(z_t)-\varphi_k(z_t)\\ &\leq t\Theta_k(z_t,y)+(1-t)\Theta_k(z_t,w)+t\varphi_k(y)+(1-t)\varphi_k(w)-\varphi_k(z_t)\\ &\leq t[\Theta_k(z_t,y)+\varphi_k(y)-\varphi_k(z_t)]+(1-t)\langle z_t-w, B_k z_t\rangle\\ &= t[\Theta_k(z_t,y)+\varphi_k(y)-\varphi_k(z_t)]+(1-t)t\langle y-w, B_k z_t\rangle,\end{aligned}$$

and hence

$$0\leq \Theta_k(z_t,y)+\varphi_k(y)-\varphi_k(z_t)+(1-t)\langle y-w, B_k z_t\rangle.$$

Letting $t\to 0$, we have, for each $y\in C$,

$$0\leq \Theta_k(w,y)+\varphi_k(y)-\varphi_k(w)+\langle y-w, B_k w\rangle.$$

This implies that $w\in \mathrm{GMEP}(\Theta_k,\varphi_k,B_k)$ and hence $w\in\cap_{k=1}^M\mathrm{GMEP}(\Theta_k,\varphi_k,B_k)$. Consequently, $w\in\cap_{n=1}^\infty \mathrm{Fix}(S_n)\cap\cap_{k=1}^M\mathrm{GMEP}(\Theta_k,\varphi_k,B_k)\cap\cap_{i=1}^N \mathrm{VI}(C,A_i)$ $\cap\mathrm{GSVI}(G)\cap\Gamma =: \Omega$. (This shows that $\omega_w(x_n)\subset\Omega$.)

Furthermore, note that that

$$\langle (V-\gamma Q)x-(V-\gamma Q)y, x-y\rangle\geq(\bar\gamma-\gamma l)\|x-y\|^2,\quad \forall x,y\in H.$$

Hence we know that $V-\gamma Q$ is $(\bar\gamma-\gamma l)$-strongly monotone with constant $\bar\gamma-\gamma l>0$. In the meantime, it is easy to see that $V-\gamma Q$ is $(\|V\|+\gamma l)$-Lipschitzian with constant $\|V\|+\gamma l>0$. Thus, there exists a unique solution x^* in Ω to the VIP

$$\langle(\gamma Q-V)x^*, p-x^*\rangle\leq 0,\quad \forall p\in\Omega.\tag{3.39}$$

Equivalently, $x^*=P_\Omega(I-(V-\gamma Q))x^*$. So, in terms of (3.36) and (3.39) we have

$$\limsup_{n\to\infty}\langle(\gamma Q-V)x^*, x_n-x^*\rangle=\langle(\gamma Q-V)x^*, w-x^*\rangle\leq 0.\tag{3.40}$$

Finally, let us show that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. In fact, put $p = x^*$ in (3.16). Then from (3.1) we get

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|s_n\gamma(Qx_n - Qx^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - s_nV)(T_ny_n - x^*) \\
&\quad + s_n(\gamma Q - V)x^*\|^2 \\
&\le \|s_n\gamma(Qx_n - Qx^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - s_nV)(T_ny_n - x^*)\|^2 \\
&\quad + 2s_n\langle(\gamma Q - V)x^*, x_{n+1} - x^*\rangle \\
&\le [s_n\gamma\|Qx_n - Qx^*\| + \beta_n\|x_n - x^*\| \\
&\quad + \|(1-\beta_n)I - s_nV\|\|T_ny_n - x^*\|]^2 \\
&\quad + 2s_n\langle(\gamma Q - V)p, x_{n+1} - x^*\rangle \\
&\le [s_n\gamma l\|x_n - x^*\| + \beta_n\|x_n - x^*\| + (1-\beta_n - s_n\bar{\gamma})\|y_n - x^*\|]^2 \\
&\quad + 2s_n\langle(\gamma Q - V)p, x_{n+1} - x^*\rangle \\
&= \Big[(\beta_n + s_n\bar{\gamma})\frac{\beta_n + s_n\gamma l}{\beta_n + s_n\bar{\gamma}}\|x_n - x^*\| + (1-\beta_n - s_n\bar{\gamma})\|y_n - x^*\|\Big]^2 \\
&\quad + 2s_n\langle(\gamma Q - V)x^*, x_{n+1} - x^*\rangle \\
&\le (\beta_n + s_n\bar{\gamma})\frac{(\beta_n + s_n\gamma l)^2}{(\beta_n + s_n\bar{\gamma})^2}\|x_n - x^*\|^2 + (1-\beta_n - s_n\bar{\gamma})\|y_n - x^*\|^2 \\
&\quad + 2s_n\langle(\gamma Q - V)x^*, x_{n+1} - x^*\rangle \\
&\le (\beta_n + s_n\gamma l)\|x_n - x^*\|^2 + (1-\beta_n - s_n\bar{\gamma})\|y_n - x^*\|^2 \\
&\quad + 2s_n\langle(\gamma Q - V)x^*, x_{n+1} - x^*\rangle \\
&\le (\beta_n + s_n\gamma l)\|x_n - x^*\|^2 + (1-\beta_n - s_n\bar{\gamma})[\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n\langle(\gamma Q - \mu F)x^*, y_n - x^*\rangle] \\
&\quad + 2s_n\langle(\gamma Q - V)x^*, x_{n+1} - x^*\rangle \\
&= (1 - s_n(\bar{\gamma} - \gamma l))\|x_n - x^*\|^2 \\
&\quad + 2(1-\beta_n - s_n\bar{\gamma})\alpha_n\langle(\gamma Q - \mu F)x^*, y_n - x^*\rangle \\
&\quad + 2s_n\langle(\gamma Q - V)x^*, x_{n+1} - x^*\rangle \\
&\le (1 - s_n(\bar{\gamma} - \gamma l))\|x_n - x^*\|^2 + 2\alpha_n\|(\gamma Q - \mu F)x^*\|\|y_n - x^*\| \\
&\quad + 2s_n\langle(\gamma Q - V)x^*, x_{n+1} - x^*\rangle \\
&= (1 - s_n(\bar{\gamma} - \gamma l))\|x_n - x^*\|^2 \\
&\quad + s_n(\bar{\gamma} - \gamma l)\Big[\frac{2\alpha_n}{s_n}\frac{\|\gamma Q - \mu F)x^*\|\|y_n - x^*\|}{\bar{\gamma} - \gamma l} \\
&\quad + \frac{2\langle(\gamma Q - V)x^*, x_{n+1} - x^*\rangle}{\bar{\gamma} - \gamma l}\Big].
\end{aligned} \tag{3.41}$$

Since $\sum_{n=1}^{\infty} s_n = \infty$, $\lim_{n\to\infty}\frac{\alpha_n}{s_n}$ and $\limsup_{n\to\infty}\langle(\gamma Q - V)x^*, x_{n+1} - x^*\rangle \le 0$ (due to (3.40)), we deduce that $\sum_{n=1}^{\infty} s_n(\bar{\gamma} - \gamma l) = \infty$ and

$$\limsup_{n\to\infty}\Big[\frac{2\alpha_n}{s_n}\frac{\|\gamma Q - \mu F)x^*\|\|y_n - x^*\|}{\bar{\gamma} - \gamma l} + \frac{2\langle(\gamma Q - V)x^*, x_{n+1} - x^*\rangle}{\bar{\gamma} - \gamma l}\Big] \le 0.$$

Therefore, applying Lemma 2.14 to (3.41) we infer that $\lim_{n\to\infty}\|x_n-x^*\|=0$. This completes the proof. □

Remark 3.2. In Theorem 3.1, whenever $M=1$ and $N=2$, the iterative scheme (3.1) reduces to the following iterative one

$$\begin{cases} \Theta_1(u_n,y)+\varphi_1(y)-\varphi_1(u_n) \\ \quad +\langle B_1x_n,y-u_n\rangle+\frac{1}{r_{1,n}}\langle u_n-x_n,y-u_n\rangle\ge 0, \qquad \forall y\in C, \\ v_n=P_C(I-\lambda_{2,n}A_2)P_C(I-\lambda_{1,n}A_1)u_n, \\ y_n=\alpha_n\gamma Qv_n+(I-\alpha_n\mu F)W_nGv_n, \\ x_{n+1}=s_n\gamma Qx_n+\beta_nx_n+((1-\beta_n)I-s_nV)T_ny_n, \quad \forall n\ge 1. \end{cases} \tag{3.42}$$

If all conditions in Theorem 3.1 are satisfied, then $\{x_n\}$ converges strongly to a point $x^*\in\Omega:=\cap_{n=1}^{\infty}\mathrm{Fix}(S_n)\cap\mathrm{GMEP}(\Theta_1,\varphi_1,B_1)\cap\mathrm{VI}(C,A_2)\cap\mathrm{VI}(C,A_1)\cap\mathrm{GSVI}(G)\cap\Gamma$ provided $\|x_n-y_n\|\to 0$ $(n\to\infty)$, which is a unique solution in Ω to the VIP

$$\langle(\gamma Q-V)x^*,p-x^*\rangle\le 0, \quad \forall p\in\Omega.$$

Remark 3.3. In Theorem 3.1, whenever $M=N=1$, the iterative scheme (3.1) reduces to the following iterative one

$$\begin{cases} \Theta_1(u_n,y)+\varphi_1(y)-\varphi_1(u_n) \\ \quad +\langle B_1x_n,y-u_n\rangle+\frac{1}{r_{1,n}}\langle u_n-x_n,y-u_n\rangle\ge 0, \quad \forall y\in C, \\ v_n=P_C(I-\lambda_{1,n}A_1)u_n, \\ y_n=\alpha_n\gamma Qv_n+(I-\alpha_n\mu F)W_nGv_n, \\ x_{n+1}=s_n\gamma Qx_n+\beta_nx_n+((1-\beta_n)I-s_nV)T_ny_n, \quad \forall n\ge 1. \end{cases} \tag{3.43}$$

If all conditions in Theorem 3.1 are satisfied, then $\{x_n\}$ converges strongly to a point $x^*\in\Omega:=\cap_{n=1}^{\infty}\mathrm{Fix}(S_n)\cap\mathrm{GMEP}(\Theta_1,\varphi_1,B_1)\cap\mathrm{VI}(C_1,A)\cap\mathrm{GSVI}(G)\cap\Gamma$ provided $\|x_n-y_n\|\to 0$ $(n\to\infty)$, which is a unique solution in Ω to the VIP

$$\langle(\gamma Q-V)x^*,p-x^*\rangle\le 0, \quad \forall p\in\Omega.$$

4. Weak convergence criteria for the GSVI with constraints

Under mild conditions imposed on the parameter sequences, we will prove weak convergence of iterative scheme (3.1) for finding a solution of the GSVI (1.3) with constraints of several problems: finitely many GMEPs, finitely many VIPs, the CMP (1.7) and the fixed point problem of infinitely many nonexpansive mappings in a real Hilbert space. We are now in a position to present the weak convergence criteria for iterative scheme (3.1).

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let M,N be two integers. Let $f:C\to\mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let Θ_k be a bifunction from $C\times C$ to $\mathbf{R}$ satisfying (A1)-(A4) and $\varphi_k:C\to\mathbf{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function, where $k\in\{1,2,\dots,M\}$. Let $B_k,A_i:H\to H$ and $F_j:C\to H$ be μ_k-inverse-strongly monotone, η_i-inverse-strongly monotone and ζ_j-inverse-strongly monotone, respectively, where $k\in\{1,2,\dots,M\}, i\in\{1,2,\dots,N\}$ and $j\in\{1,2\}$. Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings on H and $\{\lambda_n\}$ be a sequence in $(0,b]$ for some $b\in(0,1)$. Let $F:H\to H$ be a κ-Lipschitzian and η-strongly* *monotone operator with positive constants* $\kappa, \eta > 0$. *Let* $Q : H \to H$ *be an* l*-Lipschitzian mapping with constant* $l \geq 0$. *Let* $0 < \mu < \frac{2\eta}{\kappa^2}$ *and* $0 \leq \gamma l < \tau$, *where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. *Let* V *be a* $\bar{\gamma}$*-strongly positive bounded linear operator with* $\gamma l < \bar{\gamma}$. *Assume that* $\Omega := \cap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \cap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, B_k) \cap \cap_{i=1}^{N} \mathrm{VI}(C, A_i) \cap \mathrm{GSVI}(G) \cap \Gamma \neq \emptyset$ *and that either (B1) or (B2) holds. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $(0, 1]$. *For arbitrarily given* $x_1 \in H$, *let* $\{x_n\}$ *be a sequence generated by iterative scheme (3.1), where* $P_C(I - \theta_n \nabla f) = s_n I + (1 - s_n)T_n$ *(here* T_n *is nonexpansive,* $s_n = \frac{2-\theta_n L}{4} \in (0, \frac{1}{2})$ *for each* $\theta_n \in (0, \frac{2}{L})$*),* $\nu_j \in (0, 2\zeta_j)$ *for* $j = 1, 2$ *and* W_n *is the* W*-mapping defined by (2.2). Suppose that the following conditions are satisfied:*

(i) $s_n \in (0, \frac{1}{2})$ *for each* $\theta_n \in (0, \frac{2}{L})$, *and* $\sum_{n=1}^{\infty} s_n < \infty$ ($\Leftrightarrow$ $\sum_{n=1}^{\infty} (\frac{2}{L} - \theta_n) < \infty$)*;*
(ii) $\sum_{n=1}^{\infty} \alpha_n < \infty$ *and* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*
(iii) $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ *for all* $i \in \{1, 2, \ldots, N\}$*;*
(iv) $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$ *for all* $k \in \{1, 2, \ldots, M\}$.

Then $\{x_n\}$ *converges weakly to a point* $w \in \Omega$ *provided* $\|x_n - y_n\| \to 0$ $(n \to \infty)$.

Proof. Repeating the same arguments as in the proof of Theorem 3.1 we can write

$$P_C(I - \theta_n \nabla f) = \frac{2 - \theta_n L}{4} I + \frac{2 + \theta_n L}{4} T_n = s_n I + (1 - s_n)T_n,$$

where T_n is nonexpansive and $s_n := s_n(\theta_n) = \frac{2-\theta_n L}{4} \in (0, \frac{1}{2})$ for each $\theta_n \in (0, \frac{2}{L})$. It is clear that

$$\sum_{n=1}^{\infty} s_n < \infty \;\Leftrightarrow\; \sum_{n=1}^{\infty} (\frac{2}{L} - \theta_n) < \infty.$$

Since $\lim_{n\to\infty} s_n = 0$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, we may assume, without loss of generality, that $\{\beta_n\} \subset [a, \hat{a}] \subset (0, 1)$ and $\beta_n + s_n\|V\| \leq 1$ for all $n \geq 1$. By the same argument as in the proof of Theorem 3.1 we can deduce that $(1 - \beta_n)I - s_n V$ is positive and

$$\|(1 - \beta_n)I - s_n V\| \leq 1 - \beta_n - s_n\bar{\gamma}. \tag{4.1}$$

Put

$$\Delta_n^k = T_{r_{k,n}}^{(\Theta_k, \varphi_k)}(I - r_{k,n}B_k) T_{r_{k-1,n}}^{(\Theta_{k-1}, \varphi_{k-1})}(I - r_{k-1,n}B_{k-1}) \cdots T_{r_{1,n}}^{(\Theta_1, \varphi_1)}(I - r_{1,n}B_1)x_n$$

for all $k \in \{1, 2, \ldots, M\}$ and $n \geq 1$,

$$\Lambda_n^i = P_C(I - \lambda_{i,n}B_i) P_C(I - \lambda_{i-1,n}B_{i-1}) \cdots P_C(I - \lambda_{1,n}B_1)$$

for all $i \in \{1, 2, \ldots, N\}$, $\Delta_n^0 = I$ and $\Lambda_n^0 = I$, where I is the identity mapping on H. Then we have that $u_n = \Delta_n^M x_n$ and $v_n = \Lambda_n^N u_n$.

We divide the rest of the proof into several steps.

Step 1. Let us show that $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in \Omega$. Indeed, take $p \in \Omega$ arbitrarily. Repeating the same arguments as those of Step 1 in the proof of Theorem 3.1, we can prove that

$$\|u_n - p\| \leq \|x_n - p\|, \tag{4.2}$$

$$\|v_n - p\| \le \|u_n - p\|, \tag{4.3}$$

$$\begin{aligned} \|Gv_n - p\|^2 &\le \|P_C(I-\nu_2F_2)v_n - P_C(I-\nu_2F_2)p\|^2 \\ &\quad + \nu_1(\nu_1 - 2\zeta_1)\|F_1P_C(I-\nu_2F_2)v_n - F_1P_C(I-\nu_2F_2)p\|^2 \\ &\le \|v_n - p\|^2 + \nu_2(\nu_2 - 2\zeta_2)\|F_2v_n - F_2p\|^2, \end{aligned} \tag{4.4}$$

and

$$\|y_n - p\| \le (1-\alpha_n(\tau-\gamma l))\|x_n - p\| + \alpha_n\|(\gamma Q - \mu F)p\|. \tag{4.5}$$

Utilizing (3.1) and (4.5) we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|s_n\gamma(Qx_n - Qp) + \beta_n(x_n - p) \\ &\quad + ((1-\beta_n)I - s_nV)(T_ny_n - p) + s_n(\gamma Q - V)p\| \\ &\le s_n\gamma l\|x_n - p\| + \beta_n\|x_n - p\| \\ &\quad + \|(1-\beta_n)I - s_nV)\|\|T_ny_n - p\| + s_n\|(\gamma Q - V)p\| \\ &\le s_n\gamma l\|x_n - p\| + \beta_n\|x_n - p\| \\ &\quad + (1-\beta_n - s_n\bar{\gamma})\|y_n - p\| + s_n\|(\gamma Q - V)p\| \\ &\le (s_n\gamma l + \beta_n)\|x_n - p\| \\ &\quad + (1-\beta_n - s_n\bar{\gamma})\|y_n - p\| + s_n\|(\gamma Q - V)p\| \\ &\le (s_n\gamma l + \beta_n)\|x_n - p\| \\ &\quad + (1-\beta_n - s_n\bar{\gamma})[(1-\alpha_n(\tau-\gamma l))\|x_n - p\| + \alpha_n\|(\gamma Q - \mu F)p\|] \\ &\quad + s_n\|(\gamma Q - V)p\| \\ &\le (s_n\gamma l + \beta_n)\|x_n - p\| \\ &\quad + (1-\beta_n - s_n\bar{\gamma})\|x_n - p\| + \alpha_n\|(\gamma Q - \mu F)p\| + s_n\|(\gamma Q - V)p\| \\ &= (1 - s_n(\bar{\gamma} - \gamma l))\|x_n - p\| + \alpha_n\|(\gamma Q - \mu F)p\| + s_n\|(\gamma Q - V)p\| \\ &\le \|x_n - p\| + \alpha_n\|(\gamma Q - \mu F)p\| + s_n\|(\gamma Q - V)p\|. \end{aligned} \tag{4.6}$$

Since $\sum_{n=1}^\infty s_n < \infty$ and $\sum_{n=1}^\infty \alpha_n < \infty$, it is known that $\sum_{n=1}^\infty (\alpha_n\|(\gamma Q - \mu F)p\| + s_n\|(\gamma Q - V)p\|) < \infty$. Hence, applying Lemma 2.16 to (4.6) we know that $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p \in \Omega$. Thus, $\{x_n\}$ is bounded and so are the sequences $\{u_n\}$, $\{v_n\}$, $\{y_n\}$.

Step 2. Let us show that that $\|y_n - P_C(I - \frac{2}{L}\nabla f)y_n\| \to 0$ as $n \to \infty$. Indeed, from (4.5) we have

$$\begin{aligned} \|y_n - p\|^2 &\le [(1-\alpha_n(\tau-\gamma l))\|x_n - p\| + \alpha_n\|(\gamma Q - \mu F)p\|]^2 \\ &= [(1-\alpha_n(\tau-\gamma l))\|x_n - p\| + \alpha_n(\tau-\gamma l))\frac{\|(\gamma Q - \mu F)p\|}{\tau - \gamma l}]^2 \\ &\le (1-\alpha_n(\tau-\gamma l))\|x_n - p\|^2 + \alpha_n(\tau-\gamma l)\frac{\|(\gamma Q - \mu F)p\|^2}{(\tau - \gamma l)^2} \\ &\le \|x_n - p\|^2 + \alpha_n\frac{\|(\gamma Q - \mu F)p\|^2}{\tau - \gamma l}, \end{aligned}$$

which together with (3.1) and Lemmas 2.7 and 2.17 (b), implies that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\beta_n(x_n-p)+(1-\beta_n)(T_ny_n-p)+s_n(\gamma Qx_n-VT_ny_n)\|^2\\
&\le \|\beta_n(x_n-p)+(1-\beta_n)(T_ny_n-p)|^2\\
&\quad + 2s_n\langle \gamma Qx_n-VT_ny_n, x_{n+1}-p\rangle\\
&= \beta_n\|x_n-p\|^2+(1-\beta_n)\|T_ny_n-p|^2-\beta_n(1-\beta_n)\|x_n-T_ny_n\|^2\\
&\quad + 2s_n\langle\|Qx_n-VT_ny_n\|\|x_{n+1}-p\|\\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p|^2-\beta_n(1-\beta_n)\|x_n-T_ny_n\|^2\\
&\quad + 2s_n\langle\|Qx_n-VT_ny_n\|\|x_{n+1}-p\|\\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)[\|x_n-p\|^2+\alpha_n\frac{\|(\gamma Q-\mu F)p\|^2}{\tau-\gamma l}]\\
&\quad -\beta_n(1-\beta_n)\|x_n-T_ny_n\|^2+2s_n\langle\|Qx_n-VT_ny_n\|\|x_{n+1}-p\|\\
&\le \|x_n-p\|^2+\alpha_n\frac{\|(\gamma Q-\mu F)p\|^2}{\tau-\gamma l}-\beta_n(1-\beta_n)\|x_n-T_ny_n\|^2\\
&\quad + 2s_n\langle\|Qx_n-VT_ny_n\|\|x_{n+1}-p\|.
\end{aligned}$$

So, it follows that

$$\begin{aligned}
a(1-\hat a)\|x_n-T_ny_n\|^2 &\le \beta_n(1-\beta_n)\|x_n-T_ny_n\|^2\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\frac{\|(\gamma Q-\mu F)p\|^2}{\tau-\gamma l}\\
&\quad + 2s_n\langle\|Qx_n-VT_ny_n\|\|x_{n+1}-p\|.
\end{aligned}$$

Since $\lim_{n\to\infty} s_n = 0$, $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\|x_n-p\|$ exists, we conclude from the boundedness of $\{x_n\},\{y_n\}$ that

$$\lim_{n\to\infty}\|x_n-T_ny_n\|=0.$$

Taking into account that $\|y_n-T_ny_n\|\le\|y_n-x_n\|+\|x_n-T_ny_n\|$, we obtain from the assumption $\lim_{n\to\infty}\|x_n-y_n\|=0$ that

$$\lim_{n\to\infty}\|y_n-T_ny_n\|=0. \tag{4.7}$$

It is clear that

$$\begin{aligned}
\|P_C(I-\theta_n\nabla f)y_n-y_n\| &= \|s_ny_n+(1-s_n)T_ny_n-y_n\|\\
&=(1-s_n)\|T_ny_n-y_n\|\\
&\le\|T_ny_n-y_n\|,
\end{aligned}$$

where $s_n=\frac{2-\theta_nL}{4}\in(0,\frac12)$ for each $\theta_n\in(0,\frac{2}{L})$. Hence we have

$$\begin{aligned}
\Big\|P_C\Big(I-\frac{2}{L}\nabla f\Big)y_n-y_n\Big\| &\le \Big\|P_C\Big(I-\frac{2}{L}\nabla f\Big)y_n-P_C(I-\theta_n\nabla f)y_n\Big\|\\
&\quad+\|P_C(I-\theta_n\nabla f)y_n-y_n\|\\
&\le\Big\|\Big(I-\frac{2}{L}\nabla f\Big)y_n-(I-\theta_n\nabla f)y_n\Big\|\\
&\quad+\|P_C(I-\theta_n\nabla f)y_n-y_n\|
\end{aligned}$$

$$\leq \Big(\frac{2}{L} - \theta_n\Big)\|\nabla f(y_n)\| + \|T_n y_n - y_n\|.$$

From the boundedness of $\{y_n\}$, $s_n \to 0$ ($\Leftrightarrow$ $\theta_n \to \frac{2}{L}$) and $\|y_n - T_n y_n\| \to 0$ (due to (4.7)), it follows that

$$\lim_{n\to\infty} \Big\|y_n - P_C\Big(I - \frac{2}{L}\nabla f\Big)y_n\Big\| = 0. \tag{4.8}$$

Step 3. We prove that $\|x_n - u_n\| \to 0$, $\|x_n - v_n\| \to 0$, $\|v_n - Gv_n\| \to 0$ and $\|v_n - Wv_n\| \to 0$ as $n \to \infty$.

Indeed, repeating the same arguments as those of (3.24), (3.25), (3.32) and (3.35) in the proof of Theorem 3.1 we get

$$\lim_{n\to\infty} \|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\| \quad \text{and} \quad \lim_{n\to\infty} \|\Delta_n^{k-1}x_n - \Delta_n^k x_n\| = 0, \qquad \forall i \in \{1, 2, \dots, N\}, k \in \{1, 2, \dots, M\}, \tag{4.9}$$

$$\lim_{n\to\infty} \|x_n - u_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|u_n - v_n\| = 0, \tag{4.10}$$

$$\lim_{n\to\infty} \|v_n - Gv_n\| = 0, \tag{4.11}$$

and

$$\lim_{n\to\infty} \|v_n - Wv_n\| = 0. \tag{4.12}$$

Step 4. We prove that $\{x_n\}$ converges weakly to a point $w \in \Omega$.

Indeed, first of all, let us show that $\omega_w(x_n) \subset \Omega$. Since $\{x_n\}$ is bounded, we may assume, without loss of generality, that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to some $w \in H$. From (4.9), (4.10) and the assumption $\lim_{n\to\infty} \|x_n - y_n\| = 0$ we have that $y_{n_i} \rightharpoonup w$, $u_{n_i} \rightharpoonup w$, $v_{n_i} \rightharpoonup w$, $\Delta_{n_i}^k x_{n_i} \rightharpoonup w$ and $\Lambda_{n_i}^m u_{n_i} \rightharpoonup w$, where $k \in \{1, 2, \dots, M\}$ and $m \in \{1, 2, \dots, N\}$. Utilizing Lemma 2.9, we deduce from $x_{n_i} \rightharpoonup w$, $v_{n_i} \rightharpoonup w$, (4.8), (4.11) and (4.12) that $w \in \mathrm{Fix}(P_C(I - \frac{2}{L}\nabla f)) = \mathrm{VI}(C, \nabla f) = \Gamma$, $w \in \mathrm{GSVI}(G)$ and $w \in \mathrm{Fix}(W) = \cap_{n=1}^\infty \mathrm{Fix}(S_n)$ (due to Lemma 2.11). Thus, we get $w \in \cap_{n=1}^\infty \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(G) \cap \Gamma$. Repeating the same arguments as in the proof of Theorem 3.1 we get $w \in \cap_{m=1}^N \mathrm{VI}(C, A_m) \cap \cap_{k=1}^M \mathrm{GMEP}(\Theta_k, \varphi_k, B_k)$. Consequently,

$$w \in \cap_{n=1}^\infty \mathrm{Fix}(S_n) \cap \cap_{k=1}^M \mathrm{GMEP}(\Theta_k, \varphi_k, B_k) \cap \cap_{i=1}^N \mathrm{VI}(C, A_i) \cap \mathrm{GSVI}(G) \cap \Gamma =: \Omega.$$

This shows that $\omega_w(x_n) \subset \Omega$.

Next let us show that $\omega_w(x_n)$ is a single-point set. As a matter of fact, let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w'$. Then we get $w' \in \Omega$. If $w \neq w'$, from the Opial condition, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - w\| &= \lim_{i\to\infty} \|x_{n_i} - w\| < \lim_{i\to\infty} \|x_{n_i} - w'\| \\ &= \lim_{n\to\infty} \|x_n - w'\| = \lim_{j\to\infty} \|x_{n_j} - w'\| \\ &< \lim_{j\to\infty} \|x_{n_j} - w\| = \lim_{n\to\infty} \|x_n - w\|,\end{aligned}$$

which attains a contradiction. So we have $w = w'$. This completes the proof. □

Remark 4.2. In Theorem 4.1, whenever $M = 1$ and $N = 2$, the iterative scheme (3.1) reduces to the iterative one (3.42). If all conditions in Theorem 4.1 are satisfied, then the sequence $\{x_n\}$ generated by (3.42) converges weakly to a point $w \in \Omega := \cap_{n=1}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{GMEP}(\Theta_1, \varphi_1, B_1) \cap \mathrm{VI}(C, A_2) \cap \mathrm{VI}(C, A_1) \cap \mathrm{GSVI}(G) \cap \Gamma$ provided $\|x_n - y_n\| \to 0$ $(n \to \infty)$

Remark 4.3. In Theorem 4.1, whenever $M = N = 1$, the iterative scheme (3.1) reduces to the iterative one (3.43). If all conditions in Theorem 4.1 are satisfied, then the sequence $\{x_n\}$ generated by (3.43) converges weakly to a point $w \in \Omega := \cap_{n=1}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{GMEP}(\Theta_1, \varphi_1, B_1) \cap \mathrm{VI}(C_1, A) \cap \mathrm{GSVI}(G) \cap \Gamma$ provided $\|x_n - y_n\| \to 0$ $(n \to \infty)$.

Remark 4.4. In the proof of Theorem 3.1, we apply Lemma 2.15 (Suzuki's Lemma) to prove $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ according to the conditions imposed on the parameter sequences $\{\lambda_{i,n}\}$ and $\{r_{k,n}\}$, that is,

$$\lim_{n\to\infty} |\lambda_{i,n+1} - \lambda_{i,n}| = 0 \quad \text{and} \quad \lim_{n\to\infty} |r_{k,n+1} - r_{k,n}| = 0, \qquad \forall i \in \{1, 2, \ldots, N\}, k \in \{1, 2, \ldots, M\}.$$

Note that in the proof of Theorem 4.1, the role of $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ is replaced by the existence of $\lim_{n\to\infty} \|x_n - p\|$ for each $p \in \Omega$. Hence our Theorem 4.1 drops the above conditions.

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