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A NON-INTERIOR POINT HOMOTOPY-SMOOTHING METHODS FOR SOLVING EQUILIBRIUM PROBLEMS

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Abstract. In this paper, the homotopy-smoothing methods for solving the equilibrium problem (shortly, (EP)) is proposed without the initial point in the interior of the feasible set assumption and the equivalence between (EP) and the Robinson's normal equation is given and then the smooth homotopy equation is suggested by Robinson's normal equation of (EP) and the twice continuously differentiable approximation of the metric projection operator. The existence and global convergence of a smooth homotopy path from almost any starting point in $\mathbb{R}^n$ to a solution of (EP) is derived under some mild conditions.

1. Introduction

Throughout this paper, let K be a nonempty closed and convex subset of the n-dimensional Euclidean space $\mathbb{R}^n$ and $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a function such that $F(y, y) = 0$ for all $y \in K$ and, for each $x \in K$, $y \mapsto F(x, y)$ is convex and continuously differentiable on $\mathbb{R}^n$ and its partial derivative $F'_y(x, y)|_{y=x}$ is twice continuously differentiable on $\mathbb{R}^n$, where $F'_y(x, y)|_{y=x}$ means the derivative value of F with respect to the second variable y at $y = x$.

Now, we consider the following *equilibrium problem* (EP):

Find $x \in K$ such that

$$F(x, y) \geq 0$$

for all $y \in K$. Denote the solutions set of (EP) by Θ.

It is well known that the equilibrium problem were first introduced by Blum and Oettli [6], which provided a very general formulation of many problems as follows:

(1) **Optimization problems**: $\min_{y\in K} g(y)$, where $g : K \to \mathbb{R}$ is a mapping. In this case, one can define $F(x, y) = g(y) - g(x)$ for all $x, y \in K$;
(2) **Variational inequalities**: Find $x \in K$ such that

$$\langle T(x), y - x\rangle \geq 0$$

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for all $y \in K$, where $T : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued mapping. In this case, one can define $F(x, y) = \langle T(x), y - x\rangle$ for all $x, y \in K$;

(3) **Nonlinear complementarities**: Find $x \in K$ such that

$$\langle T(x), x\rangle = 0, \quad \langle T(x), y\rangle \geq 0$$

for all $y \in K$, where K is a closed convex cone of $\mathbb{R}^n$ and $T : K \to \mathbb{R}^n$ is a mapping. In this case, one can define $F(x, y) = \langle T(x), y\rangle$ for all $x, y \in K$.

The equilibrium problems have been intensively studied over the past decades on various models of the equilibrium problems and the properties of their solutions such as nonemptiness, boundeness and stability of their solution set and well-posedness (see, for example, [2, 8, 10, 18, 19, 22] and the references therein). One of the most important and interesting problems in the theory of equilibrium is to develop efficient and implementable algorithms for solving equilibrium and its generalizations, for details, we refer to ([5, 7, 11, 12, 29, 30, 34, 41]) and the references therein.

Homotopy method, which is a kind of the fixed point method, had been developed in 1970s (see, for example, [3, 13, 21, 24, 33, 35] and the references therein). A remarkable advantage of the homotopy method is that the algorithm generated by it possesses the global convergence under weaker conditions. Watson [36, 37, 38] utilized the homotopy-type methods to solve the linear complementarity problems and nonlinear complementarity problems, respectively. In order to avoid the requirement that Jacobian matrix is regular as Watson [38], Ding and Yin [14] presented a new homotopy method for the nonlinear complementarity problems. Chen and Ye [9] proposed a homotopy-smoothing method for solving the variational inequality problem and showed that the method converges globally and superlinearly under mild conditions and, moreover, the method found a solution of the problem in finite steps under some special cases.

Later on, Lin and Li [26] and Xu et al. [39, 40] studied nonmonotone variational inequalities by the combined homotopy method (see, for example, [20]) together with its equivalent KKT system and derived existence and convergence of homotopy pathway without the monotonicity. Nevertheless, the combined homotopy method to solve the KKT system increases the number of variables which make the problem more complicated.

Recently, Fan and Yu [16] applied a smoothing homotopy method to solve the nonsmooth equation reformulation of bounded box constrained variational inequality problem and established the existence and convergence of the homotopy pathway without monotonicity. Further, they studied the variational inequality problem on unbounded convex set in [17], constructed the homotopy equation by the smooth approximation to its Robinson's normal equation (see [31]) and proved that, for the starting point chosen almost everywhere in $\mathbb{R}^n$, the existence and convergence of a smooth homotopy pathway to some of its solution. In order to relax the existent assumption of the initial interior point in Xu et al. [39], Shang and Yu [32] introduced a new homotopy method for solving variational inequalities in unbounded sets based on a smooth perturbation of its constraints and proved the existence of solution path and a globally convergent of the proposed algorithm under some assumption. To the best our knowledge, there are little results concerning the homotopy method for equilibrium problems.

Inspired and motivated by the above works, the aim of this paper is to propose a homotopy-smoothing method for solving the equilibrium problem (EP) based on Robinson's normal equation. Under some suitable assumptions, the existence and global convergence of the smooth homotopy path from almost any starting point in $\mathbb{R}^n$ are proven.

2. Main results

Let $x \in K$ and denote by $f_x(y) = F(x, y)$ and $T(x) = F'_y(x, y)|_{y=x}$, respectively.

Now, we consider the following *optimization problem* (shortly, $(P^{EP}; x)$) corresponding with (EP):

$$\inf_{y \in K} f_x(y). \tag{2.1}$$

It is easy to see that $x \in K$ is a solution to (EP) if and only if it is an optimal solution to $(P^{EP}; x)$.

The following results are well-known:

Lemma 2.1. *$x \in K$ is an optimal solution to $(P^{EP}; x)$ if and only if it is a solution of the following variational inequality problem (shortly, (VI)):*

$$\langle T(x), y - x\rangle \geq 0 \tag{2.2}$$

for all $y \in K$.

Proof. For the sake of completeness, the readers could refer to [25]. □

It is worth noting that (VI) is equivalent to the following *nonsmooth equation* (see [15, 23])

$$y - \Pi_K(y - T(y)) = 0, \tag{2.3}$$

where $\Pi_K(y)$ is the metric projection from y onto K. The (VI) is also equivalent to the Robinson's normal equation (shortly, (RNE)) ([31]):

$$R(x) = T(\Pi_K(x)) + x - \Pi_K(x) = 0. \tag{2.4}$$

The following lemma shows the equivalence between (EP) and (RNE):

Lemma 2.2. (1) *If $x \in \mathbb{R}^n$ is a solution of (RNE), then $x^* = \Pi_K(x)$ is a solution of (EP);*

(2) *If x^* is a solution to (EP), then $x = x^* - F'_y(x^*, y)|_{y=x^*}$ is a solution of (RNE).*

Proof. It follows from Lemma 2.1 that (EP) is equivalent to (VI). Again, from [17, Sect. 2, pp. 212] and [42, Sect. 2], we know that, if $x \in \mathbb{R}^n$ is a solution of (RNE), then $x^* = \Pi_K(x)$ is a solution of (VI).

Conversely, if x^* is a solution of (VI), then $x = x^* - T(x^*)$ is a solution of (RNE). Note that $T(x^*) = F'_y(x^*, y)|_{y=x^*}$. Therefore, the desired results are obtained. This completes the proof. □

For this reason, we now try to find a solution of (RNE) by a homotopy-smoothing method. Nevertheless, $R(x)$ is nonsmooth since the metric projection $\Pi_K(x)$. Motivated by [42], we introduce a C^2-smooth operator $s(x,l)$ to approximate the metric projection $\Pi_K(x)$ such that the proposed homotopy-smoothing method is globally convergent.

Assumption 2.3. Let $x \in \mathbb{R}^n$ and $l \in [0,1]$, assume that

(1) There exists a constant $c > 0$ such that
$$\|s(x,l) - \Pi_K(x)\| \leq cl;$$
(2) There exists a constant $\gamma > 0$ such that
$$\langle \Pi_K(x) - x, s(x,l) - \Pi_K(x)\rangle \leq \gamma l;$$
(3) $s(x,l) \in K$;
(4) $s(x,l)$ is twice continuously differentiable on $\mathbb{R}^n \times (0,1]$;
(5) $s(x,0) = \Pi_K(x)$.

If Assumption 2.3 holds, then, by (1), we have
$$\|s(x,l) - \Pi_K(x)\| \to 0 \text{ as } l \to 0.$$
Again, from (5), it follows that $s(x,l)$ is the smooth approximation of $\Pi_K(x)$.
Now, we define a mapping $R : \mathbb{R}^n \times (0,1] \to \mathbb{R}^n$ by

$$R(x,l) = T(s(x,l)) + x - s(x,l) \tag{2.5}$$

for all $(x,l) \in \mathbb{R}^n \times (0,1]$. Then $R(x,l)$ is twice continuously differentiable on $\mathbb{R}^n \times (0,1]$ and so, for each $x \in \mathbb{R}^n$, $R(x,l)$ converges to $R(x)$ as $l \to 0$. For the relations between two constants c and γ in Assumption 2.3, one can refer to [42, Subsections 2.1-2.3].

In order to solve the (RNE), let $x^0 \in \mathbb{R}^n$ be given and $l \in (0,1]$, we construct the following smooth homotopy equation (shortly, (SHE)):

$$H(x,l) = H(x^0,x,l) := (1-l)[T(s(x,l)) + x - s(x,l)] + l(x - x^0) = 0. \tag{2.6}$$

Now, we recall some well-known results and definitions from differential topology which are needed in our main results.

Definition 2.4 ([28])**.** Let $U \subseteq \mathbb{R}^n$ be an open set and $\phi : U \to \mathbb{R}^p$ be a C^α-mapping, where $\alpha > \max\{0, n-p\}$. A vector $w \in \mathbb{R}^p$ is called a *regular value* for ϕ if
$$\text{Range}\left[\frac{\partial \phi(x)}{\partial x}\right] = \mathbb{R}^p$$
for all $x \in \phi^{-1}(w)$.

Lemma 2.5 (Inverse Image Theorem [28])**.** *Let $\phi : U \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a C^α-mapping, where $\alpha > \max\{0, n-p\}$. If 0 is a regular value of ϕ, then $\phi^{-1}(0)$ consists of some $(n-p)$-dimensional C^α manifolds.*

Lemma 2.6 (Classification Theorem of One-Dimensional Smooth Manifolds [28]). *A one-dimensional smooth manifold is homeomorphic to a unite circle or a unit interval.*

Lemma 2.7 (Parameterized Sard Theorem [3]). *Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open sets and $\phi : V \times U \to \mathbb{R}^k$ be a C^α-mapping, where $\alpha > \max\{0, m-k\}$. If $0 \in \mathbb{R}^k$ is a regular value of ϕ, then, for almost all $a \in V$, 0 is a regular value of $\phi_a = \phi(a, \cdot)$.*

Theorem 2.8. *Assume that Assumption 2.3 is true and there exist two nonempty bounded subsets B_0 and B_1 of K such that, for each $x \in K \setminus B_1$, there exists $z \in B_0$ such that*

$$\langle T(x), x - z \rangle > 0,$$

where $T(x) = F'_y(x, y)|_{y=x}$. Then the following results hold:

(1) *(EP) is solvable, i.e., $\Theta \neq \emptyset$;*

(2) *For almost all $x^0 \in \mathbb{R}^n$, (SHE) determines a smooth curve $\Gamma \subseteq \mathbb{R}^n \times (0,1]$ starting from $(x^0, 1)$ and approaching the hyperplane at $l = 0$. When $l \to 0$, the limit set $L \times \{0\} \subseteq \mathbb{R}^n \times \{0\}$ of Γ is nonempty and each point $(\hat{x}, 0) \in L \times \{0\}$ is a solution of (SHE) and so $\Pi_K(\hat{x}) \in \Theta$.*

The proof of Theorem 2.8 is complicated. For the sake of brevity, first, we prove several lemmas which lead to the final proof of this theorem.

Lemma 2.9. *If the conditions of Theorem 2.8 hold, then, for almost all $x^0 \in \mathbb{R}^n$, 0 is a regular value of $H : \mathbb{R}^n \times (0,1] \to \mathbb{R}^{n+1}$ and $H^{-1}(0)$ consists of some smooth curves, where $H^{-1}(0) = \{(x, l) \in \mathbb{R}^n \times (0,1] : H(x, l) = 0\}$. Among them, a smooth curve Γ starts from $(x^0, 1)$.*

Proof. Since, for each $x \in K$, $y \mapsto F(x, y)$ is convex and triple continuously differentiable on $\mathbb{R}^n$, $T(x)$ is twice continuously differentiable on $\mathbb{R}^n$ and so is $H(x^0, x, l)$. Then, for all $x^0 \in \mathbb{R}^n$ and $l \in (0,1]$, by the definition of $H(x^0, x, l)$ as a mapping of the variables x^0, x, l, one has

$$H'_{x^0}(x^0, x, l) = -l\mathbf{I},$$

where $\mathbf{I}$ is the $n \times n$ unit matrix and the Jacobi matrix of $H(x^0, x, l)$, denoted by $JH(x^0, x, l)$,

$$\begin{aligned} JH(x^0, x, l) &= (H'_{x^0}(x^0, x, l), H'_x(x^0, x, l), H'_l(x^0, x, l)) \\ &= (-l\mathbf{I}, H'_x(x^0, x, l), H'_l(x^0, x, l)) \end{aligned}$$

is of full row rank, namely, 0 is a regular value of $H(x^0, x, l)$. It follows from Lemmas 2.5 and 2.7 that, for almost all $x^0 \in \mathbb{R}^n$, 0 is a regular value of $H(x, l)$ and the inverse $H^{-1}(0)$ consists of some one-dimensional C^α manifolds, where $\alpha > 1$. Note that $H'_x(x, 1) = \mathbf{I}$ is nonsingular. Hence x^0 is the unique solution to the smooth equation $H(x, 1) = 0$ since $H(x^0, 1) = 0$. From this, there must exist a smooth curve Γ in $H^{-1}(0)$ starting from $(x^0, 1)$. This completes the proof. □

Lemma 2.10. *Let $(x^0, l_0) \in \mathbb{R}^n \times (0,1]$ be given and the conditions of Theorem 2.8 hold. If 0 is a regular value of $H : \mathbb{R}^n \times (0,1] \to \mathbb{R}^{n+1}$, then Γ is a bounded curve in $\mathbb{R}^n \times (0,1]$.*

Proof. Suppose to the contrary that Γ is an unbounded curve in $\mathbb{R}^n \times (0,1]$. Then there exists a sequence $\{(x^m, l_m)\} \subseteq \Gamma$ such that $\|(x^m, l_m)\| \to \infty$ as $m \to \infty$ and $\inf_m l_m > 0$. Further, from $\{l_m\} \subseteq (0,1]$, it follows that

$$\|x^m\| \to \infty \tag{2.7}$$

as $m \to \infty$.

Suppose that the sequence $s(x^m, l_m)$ is bounded. Since, for each $x \in K$, $y \mapsto F(x,y)$ is triple continuously differentiable, it follows that $\{T(s(x^m, l_m))\}$ is bounded too. By (SHE), we have

$$(1-l_m)[T(s(x^m,l_m)) + x^m - s(x^m,l_m)] + l_m(x^m - x^0) = 0,$$

i.e.,

$$x^m = (l_m - 1)[T(s(x^m,l_m)) - s(x^m,l_m)] + l_m x^0. \tag{2.8}$$

This yields that the sequence $\{x^m\}$ is bounded, which contradicts (2.7). Consequently, the sequence $s(x^m, l_m)$ is unbounded. This together with Assumption 2.3 (1) shows that $\{\Pi_K(x^m)\}$ is unbounded. Without loss of generality, let $\|\Pi_K(x^m)\| \to \infty$ as $m \to \infty$. Then $\|s(x^m, l_m)\| \to \infty$ as $m \to \infty$. This implies that there exists a positive integer number M_0 such that, for any $m \geq M_0$, $s(x^m, l_m) \in K \setminus B_1$. So, there exists $z \in B_0$ such that

$$\langle T(s(x^m,l_m)), s(x^m,l_m) - z\rangle > 0. \tag{2.9}$$

By the properties of the metric projection operator Π_K, we have

$$\langle x^m - \Pi_K(x^m), z - \Pi_K(x^m)\rangle \leq 0. \tag{2.10}$$

Now, we assert that $l_m < 1$ for sufficiently large m. If not and let $l_m = 1$ in (2.8), then we derive that $x^m = x^0$, which contradicts (2.7). Then there exists a positive integer number M_1 such that $1 - l_m > 0$. It follows from (2.8) that

$$T(s(x^m,l_m)) = \frac{l_m x^0 - x^m + (1-l_m)s(x^m,l_m)}{1-l_m}. \tag{2.11}$$

Set $M = \max\{M_0, M_1\}$. For any $m \geq M$, one has

$$\begin{aligned}
&(1-l_m)\langle T(s(x^m,l_m)), s(x^m,l_m) - z\rangle \\
&= \left\langle l_m x^0 - x^m + (1-l_m)s(x^m,l_m), s(x^m,l_m) - z\right\rangle \\
&= \left\langle l_m(x^0 - x^m) + (1-l_m)(s(x^m,l_m) - x^m), s(x^m,l_m) - z\right\rangle \\
&= l_m \left\langle x^0 - \Pi_K(x^m), s(x^m,l_m) - \Pi_K(x^m)\right\rangle \\
&\quad + l_m \left\langle x^0 - \Pi_K(x^m), \Pi_K(x^m) - z\right\rangle \\
&\quad + \langle \Pi_K(x^m) - x^m, s(x^m,l_m) - \Pi_K(x^m)\rangle + \langle \Pi_K(x^m) - x^m, \Pi_K(x^m) - z\rangle \\
&\quad + (1-l_m)\cdot[\langle s(x^m,l_m) - \Pi_K(x^m), s(x^m,l_m) - \Pi_K(x^m)\rangle \\
&\quad + \langle s(x^m,l_m) - \Pi_K(x^m), \Pi_K(x^m) - z\rangle] \\
&\leq c l_m^2 \|x^0 - \Pi_K(x^m)\| + l_m \left\langle x^0 - \Pi_K(x^m), \Pi_K(x^m) - z\right\rangle + \gamma l_m \\
&\quad + (1-l_m)[c^2 l_m^2 + c l_m \|\Pi_K(x^m) - z\|].
\end{aligned}$$

Then we have

$$\begin{aligned}
&\lim_{m\to\infty}\frac{(1-l_m)\left\langle T(s(x^m,l_m)),s(x^m,l_m)-z\right\rangle}{\|\Pi_K(x^m)\|^2}\\
&\leq \lim_{m\to\infty}\Big\{\frac{cl_m^2\|x^0-\Pi_K(x^m)\|+\gamma l_m+(1-l_m)[c^2l_m^2+cl_m\|\Pi_K(x^m)-z\|]}{\|\Pi_K(x^m)\|^2}\\
&\qquad+\frac{l_m\left\langle x^0-\Pi_K(x^m),\Pi_K(x^m)-z\right\rangle}{\|\Pi_K(x^m)\|^2}\Big\}\\
&=-\lim_{m\to\infty} l_m\\
&\leq-\inf_{m\geq 1} l_m\\
&<0.
\end{aligned}$$

Hence there exists a positive integer number $\bar{M}$ with $\bar{M}\geq M$ such that, for any $m\geq\bar{M}$,

$$\frac{(1-l_m)\left\langle T(s(x^m,l_m)),s(x^m,l_m)-z\right\rangle}{\|\Pi_K(x^m)\|^2}\leq 0.$$

Again, from $1-l_m>0$, one has

$$\left\langle T(s(x^m,l_m)),s(x^m,l_m)-z\right\rangle\leq 0$$

for all $m\geq\bar{M}$, which contradicts (2.9). Therefore, $\Gamma\subseteq\mathbb{R}^n\times(0,1]$ is a bounded curve. This completes the proof. □

We are now able to prove Theorem 2.8.

Proof. It follows from Lemmas 2.9 and 2.10 that, for almost all $x^0\in\mathbb{R}^n$, 0 is a regular value of $H:\mathbb{R}^n\times(0,1]\to\mathbb{R}^{n+1}$ and $H^{-1}(0)$ is a smooth curve $\Gamma\subseteq\mathbb{R}^n\times(0,1]$ starting from $(x^0,1)$. By the proof of Lemma 2.9, $H'_x(x,1)|_{x=x^0}=\mathbf{I}$. This together with Lemma 2.6 yields that Γ is homeomorphic to $(0,1]$. The limit points of Γ must belong to $\mathbb{R}^n\times[0,1]$. Let $(\bar{x},\bar{l})$ be a limit point of Γ. Again, from the proof of Lemma 2.9, x^0 is the unique solution of the smooth equation $H(x,1)=0$. Then $(\bar{x},\bar{l})\in\mathbb{R}^n\times\{0\}$. Therefore, $(\bar{x},0)$ is the limit point of Γ and so $\bar{x}$ is a solution of (RNE). By Lemma 2.2, $\Pi_K(\bar{x})$ is a solution of (EP). This completes the proof. □

Remark 2.11. (1) The global convergence of the homotopy path generated by (SHE) is established under some quite mild assumptions and provides a unified framework for the existing optimization problem, variational inequalities and complimentarities by the homotopy method;

(2) The conditions of Theorem 2.8 are different from some known results in the literature (see, for example, [17, 42, Theorem 1]). Since the condition "there exist two nonempty bounded subsets B_0 and B_1 of K such that, for each $x\in K\setminus B_1$, there exists $z\in B_0$ such that

$$\langle T(x),x-z\rangle>0,$$

where $T(x)=F'_y(x,y)|_{y=x}$" implies that the variational inequalities, which is to find $x^*\in K$ such that

$$\langle T(x),y-x\rangle\geq 0$$

for all $y \in K$, has no solution at infinity (see, for example, [42, Definition 1]) and implies that Assumption 1 of [17] holds;

(3) The proof of Theorem 2.8 is different from that of Theorem 1 in [17]. Since we found that "$VIP(F, X)$ has a solution at infinity" can not be derived in the proof of [42, Theorem 1] without the assumption "$\inf_k t_k > 0$";

(4) Theorem 2.8 also can be applied to bilevel programming problem, bilevel variational inequalities and bilevel equilibrium problem and so on (see, for example, [1, 27]).

Now, we give some examples to illustrate Theorem 2.8.

Example 2.12. Let $K = [-2, 1]$ and $F(x, y) = x^2(y - x)$ for all $x, y \in \mathbb{R}$. Then

$$T(x) = F'_y(x, y)|_{y=x} = x^2$$

for all $x \in K$ and so there exist two nonempty bounded subsets $B_1 = \{-2, 0\}$ and $B_0 = \{-2\}$ such that, for each $x \in K \setminus B_1$, there exists $z = -2 \in B_0$ such that

$$\langle T(x), x + 2\rangle = x^2(x + 2) > 0.$$

For almost all $x^0 \in \mathbb{R}$, the homotopy paths defined by (SHE) converge to $\{-6, 0\} \times \{0\}$. Then $\Theta = \{-2, 0\}$.

The following example shows that the closedness of K in Theorem 2.8 is indispensable:

Example 2.13. Let $K = (1, 2]$ and $F(x, y) = x^2(y - x)$ for all $x, y \in \mathbb{R}$. It is easy to see that $\Theta = \emptyset$.

Now, we denote the arc-length of the smooth homotopy path Γ by ξ. By Theorem 2.8, there exist continuously differentiable functions $x(\xi)$ and $l(\xi)$ such that

$$H(x(\xi), l(\xi)) = (1 - l(\xi))[T(s(x(\xi), l(\xi))) + x(\xi) - s(x(\xi), l(\xi))] + l(\xi)(x(\xi) - x^0)$$

and $H(x(\xi), l(\xi)) = 0$, where $x(0) = x^0$ and $l(0) = l_0 = 1$.

Corollary 2.14. *The smooth homotopy path Γ is determined by the initial value problem to the following ordinary differential equation:*

$$\begin{cases} JH(x(\xi), l(\xi))\begin{pmatrix} \dot{x} \\ \dot{l} \end{pmatrix} = 0, \\ x(0) = x^0, \\ l(0) = l_0 = 1. \end{cases}$$

Moreover, for the x component of the solution $(x(\bar{\xi}), l(\bar{\xi}))$ of (2) with $\bar{\xi}$ and $l(\bar{\xi}) = 0$, $\Pi_K(x)$ is a solution of (EP).

Proof. The conclusion follows from Theorem 2.8. □

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