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SOME CONVERGENCE THEOREMS FOR A HYBRID PAIR OF GENERALIZED NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

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Dedicated to Professor Sompong Dhompongsa on the occasion of his 65th birthday

Abstract. In 2010, Sokhuma and Kaewkhao [Fixed Point Theory Appl. 2010, Art. ID 618767, 9 pp.] introduced a modified Ishikawa iteration scheme for a pair of hybrid mappings in Banach spaces and utilized the same to prove some convergence theorems. In this paper, we study the convergence of modified Ishikawa iteration process involving a hybrid pair of generalized nonexpansive mappings in CAT(0) spaces. In process, result of Sokhuma and Kaewkhao [Fixed Point Theory Appl. 2010, Art. ID 618767, 9 pp.], Akkasriworn et al. [Int. Journal of Math. Analysis, Vol. 6, 2012, no. 19, 923-932] and Izhar Uddin et al. [accepted to be appear in Bull. Malays. Math. Sci. Soc.] are generalized and improved.

1. Introduction

A metric space (X, d) is a CAT(0) space if it is geodesically connected and every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples are the classes of pre-Hilbert spaces, R-tress (see [17]) and some others. For more details on these spaces, one can consult [4–6].

Fixed point theory in CAT(0) spaces was initiated by Kirk [20, 21] wherein he proved that every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space has a fixed point. Since then the fixed point theory for single valued as well as multivalued mappings is rapidly developing in CAT(0) spaces (e.g., [10–15, 22, 24, 25]). Here it is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leq 0$ as any CAT(k) space is a CAT(k') space for every $k' \geq k$.

In 2008, Suzuki [34] introduced a class of mappings which is larger than the class of nonexpansive mappings and named his new condition as Condition (C). Concretely speaking, a mapping T defined on a subset K of a Banach space X is said to satisfy Condition (C) if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|, \ x, y \in K.$$

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He then proved some fixed point theorems for such mappings. Motivated by Suzuki [34], García-Falsat et al. [16] defined two new classes of mappings by generalizing Condition (C) and named these two new classes of mappings as mappings satisfying Condition (E) and Condition (C_λ) respectively and thereafter utilized these two classes to prove existence results on fixed points besides investigating their asymptotic behaviour in Banach spaces.

Iterative techniques for approximating fixed points of nonexpansive single-valued mappings have been investigated by various authors (e.g., [18, 19, 27, 29, 35]) using the Mann iteration scheme or the Ishikawa iteration scheme. By now, there exists an extensive literature on the iterative fixed points for various classes of mappings. For an up to date account of literature on this theme, we refer the readers to Berinde [3].

In 2005, Sastry and Babu [30] defined Ishikawa iteration scheme for multivalued mappings in the setting of Hilbert spaces. In 2007, Panyanak [28] extended the results of Sastry and Babu to uniformly convex Banach space for multivalued nonexpansive mappings. Song and Wang [33], Shahzad, Zegeye [31], Cholamjiak and Suantai [9] and Cholamjiak et al. [8], introduced the different modified Ishikawa iteration schemes and improved the results of Sastry and Babu [30] and Panyanak [28] in many ways.

In 2010, Sokhuma and Kaewkhao [32] introduced a modified Ishikawa iterative process involving a pair of hybrid mappings in Banach spaces and utilized the same to prove their results. The purpose of this paper is to study modified Ishikawa iterative method due to Sokhuma and Kaewkhao [32] for a hybrid pair of generalized nonexpansive mappings in CAT(0) spaces.

2. Preliminaries

Let X be a Banach space and K be a nonempty subset of X. Let $CB(K)$ be the family of nonempty closed bounded subsets of K while $KC(K)$ be the family of nonempty compact convex subsets of K. A subset K of X is called proximinal if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x, k) = dist(x, K) = \inf\{\|x - y\| : y \in K\}.$$

It is well known that every closed convex subset of a uniformly convex Banach space is proximinal. We shall denote by $PB(K)$, the family of nonempty bounded proximinal subsets of K. The Hausdorff metric H on $CB(K)$ is defined as

$$H(A, B) = max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\} \text{ for } A,\ B \in CB(K).$$

A multivalued mapping $T : K \to CB(K)$ is said to be nonexpansive if

$$H(T(x), T(y)) \leq \|x - y\|, \text{ for all } x,\ y \in K.$$

We use the notation $Fix(T)$ for the set of fixed points of the mapping T while $Fix(t) \bigcap Fix(T)$ denotes the set of common fixed points of t and T, i.e. a point x is said to be a common fixed point of t and T if $x = tx \in Tx$.

In 2005, Sastry and Babu [30] defined Ishikawa iteration scheme for multivalued mappings. Sastry and Babu [30] proved that Ishikawa iteration scheme for a multivalued nonexpansive mapping T with a fixed point p converges to fixed point under certain conditions. In 2007, Panyanak [28] extended the results of Sastry and Babu to uniformly convex Banach space for multivalued nonexpansive mappings. Panyanak also modified the iteration scheme of Sastry and Babu and imposed the question of convergence of this scheme. He introduced the following modified Ishikawa iteration method. Choose $x_0 \in K$, then

$$y_n = \beta_n z_n + (1-\beta_n)x_n,\ \beta_n \in [a,b],\ 0 < a < b < 1,\ n \geq 0,$$

where $z_n \in Tx_n$ is such that $\|z_n - u_n\| = \text{dist}(u_n, Tx_n)$, and $u_n \in F(T)$ such that $\|x_n - u_n\| = \text{dist}(x_n, F(T))$, and

$$x_{n+1} = \alpha_n z_n' + (1-\alpha_n)x_n\ \alpha_n \in [a,b],$$

where $z_n' \in T(y_n)$ such that $\|z_n' - v_n\| = \text{dist}(v_n, Ty_n)$, and $v_n \in F(T)$ such that $\|y_n - v_n\| = \text{dist}(y_n, F(T))$.

In 2009, Song and Wang [33] pointed out the gap in the result of Panyanak [28]. By using the following iteration scheme, they solved/revised the gap and gave the partial answer to the question raised by Panyanak [28]. Let $\alpha_n,\ \beta_n \in [0,1]$ and $\gamma_n \in (0,\infty)$ such that $\lim\limits_{n\to\infty} \gamma_n = 0$. Choose $x_0 \in K$, then

$$\begin{array}{rcl} y_n & = & \beta_n z_n + (1-\beta_n)x_n, \\ x_{n+1} & = & \alpha_n z_n' + (1-\alpha_n)x_n, \end{array}$$

where $\|z_n - z_n'\| \leq H(Tx_n, Ty_n) + \gamma_n$ and $\|z_{n+1} - z_n'\| \leq H(Tx_{n+1}, Ty_n) + \gamma_n$ for $z_n \in Tx_n$ and $z_n' \in Ty_n$. Simultaneously, Shahzad and Zegeye [31] extended the results of Sastry and Babu and Song and Wang to quasi nonexpansive multivalued mappings and also relaxed the end point condition along with compactness of the domain by using the following modified iteration scheme and gave the affirmative answer to the Panyanak's question in a more general setting.

$$y_n = \beta_n z_n + (1-\beta_n)x_n,\ \beta_n \in [0,1],\ n \geq 0,$$

$$x_{n+1} = \alpha_n z_n' + (1-\alpha_n)x_n,\ \alpha_n \in [0,1],\ n \geq 0,$$

where $z_n \in P_T x_n$ and $z_n' \in P_T y_n$.

Recently, Sokhuma and Kaewkhao [32] introduced the following modified Ishikawa iteration scheme for a pair of hybrid mappings. Let K be a nonempty bounded, closed and convex subset of Banach space X. Furthur, let $t : K \to K$ be a single valued nonexpansive mapping and $T : K \to CB(K)$ be a multivalued nonexpansive mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by

$$\begin{cases} y_n = \beta_n z_n + (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n t y_n + (1-\alpha_n)x_n, \end{cases} \tag{2.1}$$

where $x_0 \in K$, $z_n \in Tx_n$ and $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$.

To make our presentation self contained, we also collect relevant definitions and results. In a metric space (X, d), a geodesic path joining $x \in X$ and $y \in X$ is a map c from a closed interval $[0, r] \subset R$ to X such that $c(0) = x,\ c(r) = y$ and $d(c(t),\ c(s)) = |s - t|$ for all $s,\ t \in [0,\ r]$. In particular, the mapping c is an isometry and $d(x,\ y) = r$. The image of c is called a geodesic segment joining x and y which is denoted by $[x,\ y]$ whenever such a segment exists uniquely. For any

$x,\ y \in X$, we denote the point $z \in [x,\ y]$ by $z = (1-\alpha)x \oplus \alpha y$, where $0 \le \alpha \le 1$ if $d(x,\ z) = \alpha d(x,\ y)$ and $d(z,\ y) = (1-\alpha)d(x,\ y)$.

The space $(X,\ d)$ is called a geodesic space if any two points of X are joined by a geodesic and X is said to be uniquely geodesic if there exists exactly one geodesic joining x and y for each $x,\ y \in X$. A subset K of X is called convex if K contains every geodesic segment joining any two points in K. A geodesic triangle $\triangle(x_1,\ x_2,\ x_3)$ in a geodesic metric space $(X,\ d)$ is consisted of three points of X (as the vertices of $\triangle$) and a geodesic segment between each pair of points (as the edges of $\triangle$). A comparison triangle for $\triangle(x_1,\ x_2,\ x_3)$ in $(X,\ d)$ is a triangle $\overline{\triangle}(x_1,\ x_2,\ x_3) := \triangle(\overline{x}_1,\ \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x}_i,\ \overline{x}_j) = d(x_i,\ x_j)$ for $i,\ j \in \{1,2,3\}$. A point $\overline{x} \in [\overline{x}_1,\ \overline{x}_2]$ is said to be comparison point for $x \in [x_1, x_2]$ if $d(x_1, x) = d(\overline{x}_1, \overline{x})$. Similarly, comparison points on $[\overline{x}_2, \overline{x}_3]$ and $[\overline{x}_3, \overline{x}_1]$ can also be defined.

A geodesic metric space X is called a CAT(0) space if all geodesic triangles satisfy the following comparison axiom namely: CAT(0) inequality. Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be its comparison triangle in $\mathbb{R}^2$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x,\ y \in \triangle$ and all comparison points $\overline{x},\ \overline{y}\ \in\ \overline{\triangle}$,

$$d(x,\ y) \le d_{\mathbb{R}^2}(\overline{x},\ \overline{y}).$$

Towards certain classes of examples, one may recall that every convex subset of Euclidean space $\mathbb{R}^n$ endowed with the induced metric is a CAT(0) space. Moreover, if any real normed space X is CAT(0) space, then it is a pre-Hilbert space. Furthermore, if X_1 and X_2 are CAT(0) spaces, then so is $X_1 \times X_2$. For further details on CAT(0) spaces, one can consult [4–7].

Now, we collect some basic geometric properties which are instrumental throughout the discussions. Let X be a complete CAT(0) space and $\{x_n\}$ be a bounded sequence in X. For $x \in X$, write:

$$r(x, (x_n)) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r((x_n))$ is given by

$$r((x_n)) = \inf\{r(x, x_n) : x\ \in X\},$$

and the asymptotic center $A((x_n))$ of (x_n) is defined as:

$$A((x_n)) = \{x \in X : r(x, x_n) = r((x_n))\}.$$

It is well known that in a CAT(0) space, $A((x_n))$ consists of exactly one point (see Proposition 5 of [14]). In 2008, Kirk and Panyanak [23] gave a concept of convergence in CAT(0) spaces which is an analogue of weak convergence in Banach spaces and restriction of Lim's concept of convergence [26] to CAT(0) spaces.

Definition 2.1 ([23])**.** A sequence (x_n) in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of u_n for every subsequence (u_n) of (x_n). In this case, we write $\Delta - \lim_n x_n = x$ and read as x is the Δ-limit of (x_n).

Notice that for a given $(x_n) \subset X$ which Δ-converges to x and for any $y \in X$ with $y \ne x$ (owing to uniqueness of asymptotic center), we have

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

Thus every CAT(0) space satisfies the Opial property. Now, we collect some basic facts about CAT(0) spaces which will be frequently used throughout the text.

Lemma 2.2 ([23]). *Every bounded sequence in a complete CAT(0) space admits a Δ-convergent subsequence.*

Lemma 2.3 ([13]). *If K is a closed convex subset of a complete CAT(0) space X and (x_n) is a bounded sequence in K, then the asymptotic center of (x_n) is in K.*

Lemma 2.4 ([15]). *Let $(X,\ d)$ be a CAT(0) space. For $x,\ y \in X$ and $t \in [0,1]$, there exists a unique $z \in [x,\ y]$ such that*

$$d(x,z) = td(x,y) \text{ and } d(y,z) = (1-t)d(x,y).$$

Notice that we use the notation $(1-t)x \oplus ty$ for the unique point z in the case of preceding lemma.

Lemma 2.5 ([15]). *For $x, y, z \in X$ and $t \in [0,1]$ we have*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

The following lemma is a consequence of Lemma 2.9 (contained in [24]) which will be utilized to prove our main results.

Lemma 2.6. *Let X be a complete $CAT(0)$ space with $x \in X$. Suppose that $\{t_n\}$ is a sequence in $[b,c]$ for some $b, c \in (0,1)$ and$\{x_n\}$, $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$, and $\lim_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r$ for some $r \geq 0$. Then*

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Lemma 2.7. *Let X be a CAT(0) space, and let K be a nonempty closed convex subset of X. Then,*

$$dist(y,Ty) \leq d(y,x) + dist(x,Tx) + H(Tx,Ty),$$

where $x, y \in K$ and T is a multivalued mapping from K to $CB(K)$.

Recently Garcia-Falset et al. [16] introduced two generalizations of the Condition (C) in Banach spaces. Now, we state Condition (E_μ) and Condition (C_λ) in the framework of CAT(0) spaces.

Definition 2.8. Let T be a mapping on a subset C of CAT(0) space X and $\mu \geq 1$.*Then* T is said to satisfy Condition (E_μ) if

$$d(x,Ty) \leq \mu d(x,Tx) + d(x,y),\ x, y \in C.$$

We say that T satisfies Condition (E) whenever T satisfies the Condition E_μ for some $\mu \geq 1$.

Definition 2.9. Let T be a mapping on a subset C of a CAT(0) space X and $\lambda \in (0,1)$.*Then* T is said to satisfy Condition (C_λ) if

$$\lambda d(x,Tx) \leq d(x,y) \ \Rightarrow\ d(Tx,Ty) \leq d(x,y),\ x, y \in C.$$

Notice that if $0 < \lambda_1 < \lambda_2 < 1$ then the Condition (C_{λ_1}) implies Condition (C_{λ_2}). The following example shows that the class of mappings satisfying Condition (E) and (C_λ) for some $\lambda \in (0,1)$ is larger than the class of mappings satisfying the Condition (C).

Example 2.10 ([16])**.** For a given $\lambda \in (0,1)$, define a mapping T on $[0,1]$ by

$$T(x) = \begin{cases} \frac{x}{2} \text{ if } x \neq 1 \\ \frac{1+\lambda}{2+\lambda} \text{ if } x = 1. \end{cases}$$

Then the mapping T satisfies Condition (C_λ) but it fails to satisfy Condition (C_{λ_1}) whenever $0 < \lambda < \lambda_1$. Moreover T satisfies Condition (E_μ) for $\mu = \frac{2+\lambda}{2}$.

In 2011, Abkar and Eslamian [1] proved the existence of fixed of two generalized nonexpansive mapping under weakly commuting condition:

Definition 2.11. Let K be a nonempty subset of a CAT(0) space X. Then the mappings $t : K \to K$ and $T : K \to CB(K)$ are said to be commuting weakly if $t(\partial_K T(x)) \subset T(t(x))$ for all $x \in K$, where $\partial_X Y$ denotes the relative boundary of Y in X.

Theorem 2.12. *Let K be a nonempty bounded, closed and convex subset of a complete CAT(0) space X. Let $t : K \to K$ be a quasinonexpansive single valued mapping, and let $T : K \to KC(K)$ be a multivalued mapping satisfying the conditions (E) and (C_λ) for some $\lambda \in (0,1)$. If t and T commute weakly, then they have a common fixed point, i.e. there exists a point $z \in K$ such that $z = t(z) \in T(z)$.*

Now, we present the iteration scheme of Sokhuma and Kaewkhao [32] in the setting of CAT(0) spaces which can be described as follows: Let K be a nonempty bounded, closed and convex subset of a CAT(0) space X, let $t : K \to K$ be a single valued mapping and let $T : K \to CB(K)$ be a multivalued mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by

$$\begin{cases} y_n = \alpha_n z_n \oplus (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n t y_n \oplus (1-\beta_n)x_n, \end{cases} \tag{2.2}$$

where $x_0 \in K$, $z_n \in Tx_n$ and $0 < a \le \alpha_n,\ \beta_n \le b < 1$.

The purpose of this paper is to study the convergence of iteration scheme (2.1) for generalized nonexpansive mapping in CAT(0) spaces which enables us to enlarge the class of spaces as well as class of mappings. Our results generalize and extend the corresponding relevant results due to Sokhuma and Kaewkhao [32], Akkasriworn et al. [2] and Izhar Uddin et al. [36].

3. Main results

We begin with the following lemma.

Lemma 3.1. *Let K be a nonempty closed convex subset of a CAT(0) space X. Let $t : K \to K$ and $T : K \to CB(K)$ be single-valued and multivalued mappings such that both satisfy the condition (C_λ) wherein $Fix(t) \cap Fix(T) \neq \emptyset$ with $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. If $\{x_n\}$ is the sequence involved in the modified Ishikawa iteration defined by (2.2), then $\lim_{n\to\infty} d(x_n, w)$ exists for all $w \in Fix(t) \cap Fix(T)$.*

Proof. Let $x_0 \in K$ and $w \in Fix(t) \cap Fix(T)$. As, $\lambda \in (0,\ 1)$, implies

$$\lambda d(w, tw) = 0 \leq d(ty_n, w),$$

owing to Condition (C_λ), we get

$$d(ty_n, tw) \leq d(y_n, w).$$

Similarly, in view of λ dist$(w, Tw) = 0 \leq d(x_n, w)$, we can have $H(Tx_n, Tw) \leq d(x_n, w)$. Now, consider

$$\begin{aligned} d(x_{n+1}, w) &= d((1-\beta_n)x_n \oplus \beta_n ty_n, w) \\ &\leq (1-\beta_n)d(x_n, w) + \beta_n d(ty_n, w) \\ &\leq (1-\beta_n)d(x_n, w) + \beta_n d(y_n, w) \\ &= (1-\beta_n)d(x_n, w) + \beta_n d((1-\alpha_n)x_n \oplus \alpha_n z_n, w) \\ &\leq (1-\beta_n)d(x_n, w) + \beta_n(1-\alpha_n)d(x_n, w) + \alpha_n\beta_n d(z_n, w) \\ &= (1-\beta_n)d(x_n, w) + \beta_n(1-\alpha_n)d(x_n, w) + \alpha_n\beta_n dist(z_n, Tw) \\ &\leq (1-\beta_n)d(x_n, w) + \beta_n(1-\alpha_n)d(x_n, w) + \alpha_n\beta_n H(Tx_n, Tw) \\ &\leq (1-\beta_n)d(x_n, w) + \beta_n(1-\alpha_n)d(x_n, w) + \alpha_n\beta_n d(x_n, w) \\ &= d(x_n, w) \end{aligned}$$

which implies that $\{d(x_n, w)\}$ is decreasing as well as bounded below ensuring the convergence of underlying sequence. □

Lemma 3.2. *Let K be a nonempty closed convex subset of a CAT(0) space X with $t : K \to X$ and $T : K \to CB(K)$ such that both the maps satisfy the Condition (C_λ) wherein $Fix(t) \cap Fix(T) \neq \emptyset$ and $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. If $\{x_n\}$ is the sequence involved in the modified Ishikawa iteration defined by (2.2), then $\lim_{n\to\infty} d(ty_n, w) = 0$.*

Proof. Let $w \in Fix(t) \cap Fix(T)$. In view of Lemma 3.1, we assume that $\lim_{n\to\infty} d(x_n, w) = c$. Consider,

$$\begin{aligned} d(ty_n, w) = d(ty_n, tw) &\leq d(y_n, w) \\ &= d((1-\alpha_n)x_n \oplus \alpha_n z_n, w) \\ &\leq (1-\alpha_n)d(x_n, w) + \alpha_n d(z_n, w) \\ &= (1-\alpha_n)d(x_n, w) + \alpha_n dist(z_n, Tw) \\ &\leq (1-\alpha_n)d(x_n, w) + \alpha_n H(Tx_n, Tw) \\ &\leq (1-\alpha_n)d(x_n, w) + \alpha_n d(x_n, w) = d(x_n, w) \end{aligned}$$

Also,

$$\lim_{n\to\infty} \sup d(ty_n, w) \leq \lim_{n\to\infty} \sup d(y_n, w) \leq \lim_{n\to\infty} \sup d(x_n, w) = c. \tag{3.1}$$

Now, we have

$$c = \lim_{n\to\infty} d(x_{n+1}, w) = \lim_{n\to\infty} d((1-\alpha_n)x_n + \alpha_n ty_n, w). \tag{3.2}$$

Owing to Lemma 2.6, from (3.1) and (3.2), we conclude that $d(ty_n, x) = 0$. □

Lemma 3.3. *Let K be a nonempty closed convex subset of a CAT(0) space X with $t : K \to X$ and $T : K \to CB(K)$ such that both the maps satisfy the Condition (C_λ) wherein $Fix(t) \cap Fix(T) \neq \emptyset$ and $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. If $\{x_n\}$ is the sequence involved in the modified Ishikawa iteration defined by (2.2), then* $\lim_{n\to\infty} d(x_n, z_n) = 0$.

Proof. Let $w \in Fix(t) \cap Fix(T)$. As earlier, we put $\lim_{n\to\infty} d(x_n, w) = c$. For $n \geq 0$, we have

$$\begin{aligned} d(x_{n+1}, w) &= d((1-\beta_n)x_n \oplus \beta_n t y_n, w) \\ &\leq (1-\beta_n)d(x_n, w) \oplus \beta_n d(ty_n, w) \\ &\leq (1-\beta_n)d(x_n, w) \oplus \beta_n d(y_n, w) \end{aligned}$$

and therefore,

$$d(x_{n+1}, w) - d(x_n, w) \leq \beta_n(d(y_n, w) - d(x_n, w)),$$

so that

$$\frac{d(x_{n+1}, w) - d(x_n, w)}{\beta_n} + d(x_n, w) \leq d(y_n, w).$$

On taking limit $n \to \infty$, we obtain

$$\lim_{n\to\infty} \inf \left\{ \frac{d(x_{n+1}, w) - d(x_n, w)}{\beta_n} + d(x_n, w) \right\} \leq \lim_{n\to\infty} \inf d(y_n, w)$$

implying thereby

$$c \leq \lim_{n\to\infty} \inf d(y_n, w).$$

Owing to Equation (3.1), we have $\lim_{n\to\infty} \sup d(y_n, w) \leq c$, which further implies that

$$c = \lim_{n\to\infty} d(y_n, w) = \lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n z_n, w) \tag{3.3}$$

In view of,

$$\begin{aligned} d(z_n, w) &= dist(z_n, Tw) \\ &\leq H(Tx_n, Tw) \\ &\leq d(x_n, w), \end{aligned}$$

we have

$$\lim_{n\to\infty} \sup d(z_n, w) \leq \lim_{n\to\infty} \sup d(x_n, w) = c. \tag{3.4}$$

On using Lemma 2.6, equations (3.3) and (3.4), we get $\lim_{n\to\infty} d(x_n, z_n) = 0$ as desired. □

Lemma 3.4. *Let K be a nonempty closed convex subset of a CAT(0) space X with $t : K \to K$ and $T : K \to CB(K)$ such that both the maps satisfy the Condition (C_λ) while t further satisfies Condition (E) wherein $Fix(t) \cap Fix(T) \neq \emptyset$ and $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. If $\{x_n\}$ is the sequence involved in the modified Ishikawa iteration defined by (2.2), then* $\lim_{n\to\infty} d(x_n, tx_n) = 0$.

Proof. As t satisfies the condition (E), there exists $\mu \geq 1$, such that

$$\begin{aligned} d(tx_n, x_n) &\leq d(tx_n, y_n) + d(y_n, x_n) \\ &\leq \mu d(ty_n, y_n) + d(y_n, x_n) + d(y_n, x_n) \\ &\leq \mu\{d(ty_n, x_n) + d(x_n, y_n)\} + 2d(y_n, x_n) \\ &= \mu d(ty_n, x_n) + (\mu + 2)d(y_n, x_n) \\ &= \mu d(ty_n, x_n) + (\mu + 2)d((1 - \beta_n)x_n \oplus \beta_n z_n, x_n) \\ &\leq \mu d(ty_n, x_n) + (\mu + 2)\beta_n d(z_n, x_n) \end{aligned}$$

On taking the limit of both the sides, we get

$$\lim_{n\to\infty} d(tx_n, x_n) \leq \lim_{n\to\infty} \mu d(ty_n, x_n) + \lim_{n\to\infty} (\mu + 2)\beta_n d(z_n, x_n)$$

so that in view of Lemma 3.2 and 3.3, $\lim_{n\to\infty} d(tx_n, x_n) = 0$. □

Theorem 3.5. *Let K be a nonempty closed convex subset of a CAT(0) space X with $t : K \to K$ and $T : K \to CB(K)$ such that both the maps satisfy the Condition (C_λ) while t further satisfies Condition (E) wherein $Fix(t) \cap Fix(T) \neq \emptyset$ and $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. If $\{x_n\}$ is the sequence involved in the modified Ishikawa iteration defined by (2.2). If $x_{n_i} \to y$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\} \in Tx_{n_i}$ for all n_i, then $y \in Fix(t) \cap Fix(T)$.*

Proof. Let us suppose that $\lim_{i\to\infty} d(x_{n_i}, y) = 0$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\} \in Tx_{n_i}$ for all n_i. Using Lemma 3.4, we have

$$\lim_{i\to\infty} d(tx_{n_i}, x_{n_i}) = 0.$$

Owing to condition (E), we have

$$d(x_{n_i}, ty) \leq \mu d(x_{n_i}, tx_{n_i}) + d(x_{n_i}, y).$$

On taking the limit of both the side, we get

$$\lim_{i\to\infty} d(x_{n_i}, ty) = 0.$$

Hence by uniqueness of the limit of a sequence, we obtain $y = ty$. Since

$$\lambda dist(x_{n_i}, Tx_{n_i}) = 0 \leq d(x_{n_i}, y),$$

we have,

$$H(Tx_{n_i}, Ty) \leq d(x_{n_i}, y).$$

Owing to Condition (C_λ), Lemma 2.7 and Lemma 3.4, we can have

$$\begin{aligned} dist(y, Ty) &\leq d(y, x_{n_i}) + dist(x_{n_i}, Tx_{n_i}) + H(Tx_{n_i}, Ty) \\ &\leq d(y, x_{n_i}) + d(x_{n_i}, z_{n_i}) + d(x_{n_i}, y) \to 0, as\ \ i \to \infty, \end{aligned}$$

so that $y \in Fix(T)$ and hence $y \in Fix(t) \cap Fix(T)$ as desired. □

Theorem 3.6. *Let K be a nonempty compact and convex subset of a CAT(0) space X with $t : K \to K$ and $T : K \to CB(K)$ such that both the maps satisfy the Condition (C_λ) while t further satisfies Condition (E) wherein $Fix(t) \cap Fix(T) \neq \emptyset$ and $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. If $\{x_n\}$ is the sequence involved in the modified Ishikawa iteration defined by (2.2), then $\{x_n\}$ converges strongly to a common fixed point of t and T.*

Proof. Due to the fact that K is compact and the sequence $\{x_n\}$ is contained in K, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $y \in K$, that is, $\lim_{i\to\infty} d(x_{n_i}, y) = 0$.

Owing to Theorem 3.5, we have $y \in Fix(t) \cap Fix(T)$ so that in view of Lemma 3.1, the existance of $\lim_{n\to\infty} d(x_n, y)$ is ensured and in all $\lim_{n\to\infty} d(x_n, y) = \lim_{i\to\infty} d(x_{n_i}, y)$. Therefore, $\{x_n\}$ converges strongly to a common fixed point y of t and T. □

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