Journal of Nonlinear and Convex Analysis
Volume 16, Number 3, 2015, 459–466

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL


# THE NEW VARIATIONAL INCLUSION PROBLEMS FOR FIXED POINT THEOREM

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Abstract. For the purpose of this article, we modify the variational inclusion problems and prove the strong convergence theorem for approximating a common element of two sets of solutions of variational inclusion problems, variational inequality problem and the set of fixed points of a nonexpansive mapping without the conditions $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$ in a framework of Hilbert space.

## 1. Introduction

Throughout this article, we assume that $H$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $C$ be a nonempty closed convex subset of $H$. Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is called a *fixed point* of $T$ if $Tx = x$. The set of fixed points of $T$ is denoted by $F(T) := \{x \in C : Tx = x\}$. A mapping $T$ of $C$ into itself is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C.$$

A mapping $A : C \to H$ is called $\alpha$-*inverse strongly monotone* if there exists a positive real number $\alpha$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2,$$

for all $x, y \in C$. A bounded linear operator $A : C \to H$ is called *strongly positive* with coefficient $\bar{\gamma}$ if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2,$$

for all $x \in C$. Let $B : H \to H$ be a mapping and $M : H \to 2^H$ be a multi-valued mapping. The *variational inclusion problem* is to find $u \in H$ such that

$$\theta \in Bu + Mu, \tag{1.1}$$

where $\theta$ is a zero vector in $H$. The set of the solution of (1.1) is denoted by $VI(H, B, M)$. Variational inclusion problem is widely studied in various problems such as mathematical programming, complementarity problems, variational inequalities, optimal control, mathematical economics and game theory, etc. Many authors have increasingly investigated the problem (1.1); see for instance, [2], [4] and the references therein.

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2010 *Mathematics Subject Classification.* 47H09, 47H10, 47J20, 49J40, 47J25.

*Key words and phrases.* Fixed point theorem, variational inclusion problems, variational inequality problems, nonexpansive mapping, resolvent operator.

This research was supported by the Research Administration Division of King Mongkut's Institute of Technology Ladkrabang.

Let $A : C \to H$. The *variational inequality problem* is to find a point $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \tag{1.2}$$

for all $v \in C$. The set of solutions of the variational inequality problem is denoted by $VI(C, A)$. The applications of the variational inequality problem have been expanded to problems for economics, finance, optimization and game theory. Some methods have been proposed to solve the variational inequality problem; see, for example, [3], [10] and the references therein.

Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping, then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \forall u \in H,$$

is called the *resolvent operator* associated with $M$, where $\lambda$ is a positive number and $I$ is a identity mapping, see [11].

In 2008, Zhang *et al.* [11] introduced the iterative scheme for finding a common element of the set of solutions of the variational inclusion problem with a multivalued maximal monotone mapping and inverse-strongly monotone mappings and the set of fixed points of nonexpansive mappings in Hilbert space as follows:

$$\begin{aligned} y_n &= J_{M,\lambda}(x_n - \lambda A x_n), \\ x_{n+1} &= \alpha_n x + (1 - \alpha_n) S y_n, \forall n \geq 0, \end{aligned}$$

and proved strong convergence theorem of the sequence $\{x_n\}$ under suitable conditions of parameter $\{\alpha_n\}$ and $\lambda$.

It is well-known that the Banach's contraction mapping principle is the basis theorem of fixed point theory. This theorem guarantees the existence and uniqueness of fixed points. One classical way to investigate nonexpansive mappings is to use contractions to approximate a nonexpansive mapping. For $t \in (0, 1)$, defined a contraction $T_t : C \to C$ by

$$T_t x = tu + (1 - t)Tx, x \in C,$$

where $u \in C$. Banach's contraction mapping principle guarantees that $T_t$ has a unique fixed point $x_t$ in $C$, see [1] and [8].

Recently, Kangtunyakarn [5] proved a strong convergence theorem of the sequence $\{x_n\}$ for finding a common element of the set of fixed point of a nonexpansive mapping and the set of solution of variational inequality problem without assumption on the set of fixed points of a nonexpansive mapping and the set of variational inequality. He defined the sequence $\{x_n\}$ as follows:

$$x_{n+1} = \alpha T x_n + (1 - \alpha) P_C(I - \rho A)x_n, \forall n = 0, 1, 2, \ldots,$$

where $T : C \to C$ is a nonexpansive mapping, $A : C \to H$ is a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ and positive real numbers $\alpha$, $\rho$.

Let $A, B : H \to H$ be mappings and $M : H \to 2^H$ be a multi-valued mapping. Motivated by (1.1), we introduce the new problem for finding a point $u \in H$ such that

$$\theta \in aAu + (1 - a)Bu + Mu, \forall a \in [0, 1], \tag{1.3}$$

where $\theta$ is a zero vector. The set of solutions of (1.3) is denoted by $VI(H, aA + (1-a)B, M)$. If $A = B$, then (1.3) reduces to (1.1).

From motivated by Zhang *et al.*[11], Kangtunyakarn [5], the concept of Banach's contraction mapping principle and (1.3), we prove a strong convergence theorem for approximating a common element of two sets of solutions of (1.1) and the set of solutions of variational inequality problem and the set of fixed points of a nonexpansive mapping without the conditions $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$.

## 2. Preliminaries

Let $H$ be a real Hilbert space and $C$ be a nonempty closed convex subset of $H$. We denote weak and strong convergence by notations " $\rightharpoonup$" and " $\rightarrow$", respectively. In a real Hilbert space $H$, it is well known that

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha \|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2,$$

for all $x, y \in H$ and $\alpha \in [0,1]$.

Let $P_C$ be the metric projection of $H$ onto $C$ i.e., for $x \in H, P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

**Lemma 2.1** ([7]). *Let $A$ be a strongly positive linear bounded operator on a Hilbert space $H$ with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

**Lemma 2.2** ([9]). *Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$ and let $A$ be a mapping of $C$ into $H$. Let $u \in C$. Then, for $\lambda > 0$,*

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

*where $P_C$ is the metric projection of $H$ onto $C$.*

**Lemma 2.3** ([11]). *Let $u \in H$ is a solution of variational inclusion (1.1) if and only if $u = J_{M,\lambda}(u - \lambda Bu), \forall \lambda > 0$, i.e.,*

$$VI(H, B, M) = F(J_{M,\lambda}(I - \lambda B)), \forall \lambda > 0.$$

*Further, if $\lambda \in (0, 2\alpha]$, then $VI(H, B, M)$ is closed convex subset in $H$.*

**Lemma 2.4** ([11]). *The resolvent operator $J_{M,\lambda}$ associated with $M$ is single-valued, nonexpansive for all $\lambda > 0$ and 1-inverse strongly monotone.*

**Lemma 2.5.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping and $A, B : C \to H$ be $\alpha$ and $\beta$-inverse strongly monotone mapping, respectively, with $F(J_{M,\lambda}(I - \lambda A)) \cap F(J_{M,\lambda}(I - \lambda B)) \neq \emptyset$. Then*

$$F(J_{M,\lambda}(I - \lambda(aA + (1-a)B))) = F(J_{M,\lambda}(I - \lambda A)) \cap F(J_{M,\lambda}(I - \lambda B)),$$

*for all $a \in (0,1)$ and $0 < \lambda < 2\eta$ where $\eta = min\{\alpha, \beta\}$. Moreover, $J_{M,\lambda}(I - \lambda(aA + (1-a)B))$ is a nonexpansive mapping.*

*Proof.* It is easy to see that $F(J_{M,\lambda}(I - \lambda A)) \cap F(J_{M,\lambda}(I - \lambda B)) \subseteq F(J_{M,\lambda}(I - \lambda(aA + (1-a)B)))$.

Let $x_0 \in F(J_{M,\lambda}(I-\lambda(aA+(1-a)B)))$ and $x^* \in F(J_{M,\lambda}(I-\lambda A)) \cap F(J_{M,\lambda}(I-\lambda B))$. From Lemma 2.3, we have

$$x_0 \in VI(H, aA + (1-a)B, M). \tag{2.1}$$

Since $F(J_{M,\lambda}(I-\lambda A)) \cap F(J_{M,\lambda}(I-\lambda B)) \subseteq F(J_{M,\lambda}(I-\lambda(aA+(1-a)B)))$, we have

$$x^* \in F(J_{M,\lambda}(I-\lambda(aA+(1-a)B))).$$

From Lemma 2.3, we have

$$x^* \in VI(H, aA + (1-a)B, M). \tag{2.2}$$

From the nonexpansiveness of $J_{M,\lambda}$, we have

$$\begin{aligned} \|x^* - x_0\|^2 &= \|J_{M,\lambda}(I-\lambda(aA+(1-a)B))x^* - J_{M,\lambda}(I-\lambda(aA+(1-a)B))x_0\|^2 \\ &\le \|(I-\lambda(aA+(1-a)B))x^* - (I-\lambda(aA+(1-a)B))x_0\|^2 \\ &= \|x^* - x_0\|^2 - 2\lambda a\langle x^* - x_0, Ax^* - Ax_0\rangle \\ &\quad - 2\lambda(1-a)\langle x^* - x_0, Bx^* - Bx_0\rangle \\ &\quad + \lambda^2 \|a(Ax^* - Ax_0) + (1-a)(Bx^* - Bx_0)\|^2 \\ &\le \|x^* - x_0\|^2 - \lambda a(2\alpha - \lambda)\|Ax^* - Ax_0\|^2 \\ &\quad - \lambda(1-a)(2\beta - \lambda)\|Bx^* - Bx_0\|^2. \end{aligned} \tag{2.3}$$

This implies that

$$\lambda a(2\alpha - \lambda)\|Ax^* - Ax_0\|^2 \le 0.$$

Then

$$Ax^* = Ax_0. \tag{2.4}$$

Using the same method as (2.4), we have

$$Bx^* = Bx_0. \tag{2.5}$$

From (2.1), we have

$$\theta \in Mx_0 + (aA + (1-a)B)x_0. \tag{2.6}$$

From (2.2), we have

$$\theta \in Mx^* + (aA + (1-a)B)x^*. \tag{2.7}$$

From (2.6) and (2.7), we have

$$\theta \in Mx_0 + aAx_0 + (1-a)Bx_0 - Mx^* - aAx^* - (1-a)Bx^*.$$

From (2.4) and (2.5), we have

$$\theta \in Mx_0 - Mx^*. \tag{2.8}$$

Since $x^* \in F(J_{M,\lambda}(I-\lambda A))$ and Lemma 2.3, we have

$$x^* \in VI(H, A, M) \tag{2.9}$$

From (2.4), (2.8) and (2.9), we have

$$\begin{aligned} \theta &\in Mx_0 - Mx^* + Mx^* + Ax^* \\ &= Mx_0 + Ax_0. \end{aligned} \tag{2.10}$$

It implies that $x_0 \in VI(H,A,M)$. Using the same method as (2.10), we have $x_0 \in VI(H,B,M)$. Then

$$x_0 \in VI(H,A,M) \cap VI(H,B,M).$$

From Lemma 2.3, we have

$$x_0 \in F(J_{M,\lambda}(I-\lambda A)) \cap F(J_{M,\lambda}(I-\lambda B)).$$

Then

$$F(J_{M,\lambda}(I-\lambda(aA+(1-a)B))) \subseteq F(J_{M,\lambda}(I-\lambda A)) \cap F(J_{M,\lambda}(I-\lambda B)).$$

Applying (2.3), we have $J_{M,\lambda}(I-\lambda(aA+(1-a)B))$ is a nonexpansive mapping. □

**Remark 2.6.** From Lemma 2.3 and 2.5, we can conclude that

$$VI(H, aA+(1-a)B, M) = VI(H,A,M) \cap VI(H,B,M),$$

for all $a \in (0,1)$. From Lemma 2.3, we have

$$F(J_{M,\lambda}(I-\lambda(aA+(1-a)B))) = VI(H,A,M) \cap VI(H,B,M).$$

## 3. Main result

**Theorem 3.1.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping and $A,B : C \to H$ be $\alpha$ and $\beta$-inverse strongly monotone mapping, respectively, and let $D : C \to H$ be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$. Let $T : C \to C$ be a nonexpansive mapping with $\mathcal{F} := F(T) \cap VI(H,A,M) \cap VI(H,B,M) \cap VI(C,D) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is generated by $x_1 \in C$ and*

$$\begin{cases} y_n = J_{M,\lambda}(I-\lambda(aA+(1-a)B))x_n, \\ x_{n+1} = \alpha_n P_C(I-\rho D)y_n + (1-\alpha_n)Tx_n, \forall n \geq 1, \end{cases} \tag{3.1}$$

*where $a \in (0,1)$, $\{\alpha_n\} \subseteq [c,d] \subset [0,1]$, for all $n \in \mathbb{N}$, $0 < \rho \leq \|D\|^{-1}$ and $0 < \lambda < 2\eta$ with $\eta = min\{\alpha,\beta\}$. Then the sequence $\{x_n\}$ converges strongly to $x^* \in \mathcal{F}$.*

*Proof.* Let $x,y \in C$. From Lemma 2.1, we have

$$\begin{aligned} \|(I-\rho D)x - (I-\rho D)y\| =& \|(I-\rho D)(x-y)\| \\ \leq& (1-\rho\bar{\gamma})\|x-y\|. \end{aligned} \tag{3.2}$$

Let $x^* \in \mathcal{F}$. From the definition of $y_n$, Lemma 2.5 and Remark 2.6, we have

$$\begin{aligned} \|y_n - x^*\| =& \|J_{M,\lambda}(I-\lambda(aA+(1-a)B))x_n - x^*\| \\ \leq& \|x_n - x^*\|. \end{aligned} \tag{3.3}$$

From the definition of $x_n$, (3.2) and (3.3), we have

$$\begin{aligned} \|x_{n+1} - x^*\| =& \|\alpha_n(P_C(I-\rho D)y_n - x^*) + (1-\alpha_n)(Tx_n - x^*)\| \\ \leq& \alpha_n\|P_C(I-\rho D)y_n - x^*\| + (1-\alpha_n)\|Tx_n - x^*\| \\ \leq& \alpha_n\|(I-\rho D)y_n - (I-\rho D)x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ \leq& \alpha_n(1-\rho\bar{\gamma})\|y_n - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ \leq& \alpha_n(1-\rho\bar{\gamma})\|x_n - x^*\| + (1-\alpha_n)\|x_n - x^*\| \end{aligned}$$

$$
\begin{aligned}
&\leq (1-\rho\bar{\gamma}\alpha_n)\|x_n - x^*\| \\
&\leq (1-\rho\bar{\gamma}c)\|x_n - x^*\| \\
&= p\|x_n - x^*\| \\
&\leq p(p\|x_{n-1} - x^*\|) \\
&= p^2\|x_{n-1} - x^*\| \\
&\quad \vdots \\
&\leq p^n\|x_1 - x^*\|, \qquad (3.4)
\end{aligned}
$$

where $p = (1-\rho\bar{\gamma}c) \in (0,1)$.
Since $p^n \to 0$ as $n \to \infty$ and (3.4), we can conclude that the sequence $\{x_n\}$ converges strongly to $x^* \in \mathcal{F}$. □

## 4. Application

To prove a strong convergence theorem in this section, we need the definition and lemma as follows:

**Definition 4.1** ([6]). Let $C$ be a nonempty convex subset of a real Banach space $X$. Let $\{T_i\}_{n=1}^N$ be a finite family of nonexpansive mappings of $C$ into itself and let $\lambda_1, \lambda_2, \dots, \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, 2, \dots, N$. Define a mapping $K : C \to C$ as follows:

$$
\begin{aligned}
U_1 &= \lambda_1 T_1 + (1-\lambda_1)I, \\
U_2 &= \lambda_2 T_2 U_1 + (1-\lambda_2)U_1, \\
U_3 &= \lambda_3 T_3 U_2 + (1-\lambda_3)U_2, \\
&\vdots \\
U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1-\lambda_{N-1})U_{N-2}, \\
K = U_N &= \lambda_N T_N U_{N-1} + (1-\lambda_N)U_{N-1}.
\end{aligned}
$$

Such a mapping is called the K-mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_1, \lambda_2, \dots \lambda_N$.

**Lemma 4.2** (See [6]). *Let $C$ be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of $C$ into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\lambda_1, \dots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, \dots, N-1$ and $0 < \lambda_N \leq 1$. Let $K$ be the $K$-mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_1, \lambda_2, \dots \lambda_N$. Then $F(K) = \bigcap_{i=1}^N F(T_i)$.*

**Remark 4.3.** From the definition of $K$, it is obvious that $K$ is a nonexpansive mapping.

**Theorem 4.4.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping and $A, B : C \to H$ be $\alpha$ and $\beta$-inverse strongly monotone mapping, respectively, and let $D : C \to H$ be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of $C$ into itself and let $\lambda_1, \dots, \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, \dots, N-1$ and $0 < \lambda_N \leq 1$.*

*Let $K$ be the $K$-mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_1, \lambda_2, \dots \lambda_N$. Assume $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \cap VI(H, A, M) \cap VI(H, B, M) \cap VI(C, D) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is generated by $x_1 \in C$ and*

$$\begin{cases} y_n = J_{M,\lambda}(I - \lambda(aA + (1-a)B))x_n, \\ x_{n+1} = \alpha_n P_C(I - \rho D)y_n + (1 - \alpha_n)Kx_n, \forall n \geq 1, \end{cases} \tag{4.1}$$

*where $a \in (0,1)$, $\{\alpha_n\} \subseteq [c, d] \subset [0,1]$ for all $n \in \mathbb{N}$, $0 < \rho \leq \|D\|^{-1}$ and $0 < \lambda < 2\eta$ with $\eta = min\{\alpha, \beta\}$. Then the sequence $\{x_n\}$ converges strongly to $x^* \in \mathcal{F}$.*

*Proof.* From Theorem 3.1 and Lemma 4.2, we obtain the desired conclusion □

## Acknowledgements

The authors are greatly thankful to the referees for their useful comments and suggestions improving the content of this paper.

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*Manuscript received October 25, 2013*
*revised June 12, 2014*

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