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A PROJECTED LANDWEBER METHOD WITH VARIABLE STEPS FOR THE SPLIT EQUALITY PROBLEM

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Dedicated to Prof. Sompong Dhompongsa on the occasion of his 65th birthday

Abstract. In this paper we consider the split equality problem (SEP): find $x \in C, y \in Q$ such that $Ax = By$, where C and Q are nonempty closed convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, A is a $p \times n$ matrix, and B is a $p \times m$ matrix, respectively. One drawback of the existing algorithm for the SEP is that one has to compute or estimate the matrix norms. In this paper, we construct a new algorithm in a way that the implementation of the algorithm does not need any prior information of the matrix norms. Under some mild conditions, we establish the convergence of the proposed algorithm to a solution of the SEP.

1. Introduction

We consider a problem that consists of finding a pair $(\hat{x}, \hat{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ such that

$$\hat{x} \in C, \ \hat{y} \in Q, \ \text{and } A\hat{x} = B\hat{y}, \tag{1.1}$$

where C and Q are nonempty closed convex subset of $\mathbb{R}^n$ and $\mathbb{R}^m$, A is a $p \times n$ matrix, and B is a $p \times m$ matrix, respectively. There are several methods solving this problem. In particular, Byrne and Moudafi [3] proposed the following algorithm: choose an arbitrary initial guess x_1, calculate:

$$\left[\begin{array}{l} x_{k+1} = P_C(x_k - \varrho A^\top (Ax_k - By_k)) \\ y_{k+1} = P_Q(x_k - \varrho B^\top (By_k - Ax_k)), \end{array} \right. \tag{1.2}$$

which is an application of the well-known projected Landweber method. Here ϱ is the constant-step whose choice is mainly relying on the norms of matrixes A and B.

Problem (1.1) is called the split equality problem (SEP) [8], which can be regraded as an extension of the the well-known split feasibility problem (SFP). Indeed when B is the identity matrix, the SEP is then reduced to the SFP: find $\hat{x}$ satisfying the property:

$$\hat{x} \in C \quad \text{and} \quad A\hat{x} \in Q. \tag{1.3}$$

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The SFP has been proved very useful in dealing with a variety of signal processing and image recovery [4, 7]. An efficient method that solves the SFP is due to Byrne's CQ algorithm [1]:

$$x_{k+1} = P_C(x_k - \varrho A^\top (I - P_Q)Ax_k), \tag{1.4}$$

where the stepsize ϱ is a fixed real number in $(0, 2\|A\|^{-2})$. Compared with the original algorithm in [5], the CQ algorithm (1.4) is more easily performed because it dose not involve matrix inverses. However, to implement the CQ algorithm, one has to compute or estimate the value of $\|A\|$, which is not always possible in practice. To overcome this drawback, many authors have conducted worthwhile research on the CQ algorithm so that the choice of the step does not depend on the matrix morms (see for instance [7, 11, 13, 10, 14, 15, 16]).

A similar question of Byrne and Moudafi's method also arises: Does there exist a way to select the step in algorithm (1.2) that dose not depend on the matrixes norms? It is the purpose of this paper to answer the above question affirmatively. By using the idea in the variable-step CQ algorithm, we construct a new method in a way that the implementation of algorithm (1.2) does not need any prior information of matrix norms. Under some mild conditions, we establish the convergence of the proposed algorithm.

2. The algorithms

In this section we will introduce our iterative scheme to solve the SEP. Choose an arbitrary initial guess x_1, calculate:

$$\left[\begin{array}{l} x_{k+1} = P_C(x_k - \varrho_k A^\top (Ax_k - By_k)) \\ y_{k+1} = P_Q(x_k - \varrho_k B^\top (By_k - Ax_k)), \end{array}\right. \tag{2.1}$$

where ϱ_k is a sequence of positive real numbers such that

$$\sum_{k=0}^{\infty} \varrho_k = \infty, \quad \sum_{k=0}^{\infty} \varrho_k^2 < \infty. \tag{2.2}$$

It is clear our choice of the step does not need any information on the matrix norms of A and B.

The following lemmas play an important role in our subsequent convergence analysis.

Lemma 2.1 ([9])**.** *Let (ϵ_k) and (s_k) be positive real sequences. Assume that $\sum_k \epsilon_k < \infty$. If either (i) $s_{k+1} \leq (1+\epsilon_k)s_k$, or (ii) $s_{k+1} \leq s_k + \epsilon_k$, then the limit of (s_k) exists.*

Let C be a nonempty closed convex subset of $\mathbb{R}^n$. Denote by P_C the projection from $\mathbb{R}^n$ onto C; that is,

$$P_C x = \arg\min_{y \in C} \|x - y\|, \quad x \in \mathbb{R}^n.$$

The projection operator has the following properties (see [6]).

Lemma 2.2. *Let P_C be the projection operator onto C. Then for any $x, y \in \mathbb{R}^n$,*

(i) *P_C is nonexpansive, i.e.,*

$$\|P_C x - P_C y\| \leq \|x - y\|,$$

(ii) P_C *is firmly nonexpansive, i.e.,*

$$\|P_Cx - P_Cy\|^2 \le \langle x - y, P_Cx - P_Cy\rangle,$$

(iii) $I - P_C$ *is firmly nonexpansive.*

Denote by S the solution set of the SEP, namely

$$S = \{(x, y) : x \in C, y \in Q, Ax = By\}.$$

For $x \in \mathbb{R}^n, y \in \mathbb{R}^m$, let $z = (x, y)$ be an element in the product space $\mathbb{R}^n \times \mathbb{R}^m$ with the norm

$$\|z\| = \sqrt{\|x\|^2 + \|y\|^2}.$$

3. CONVERGENCE ANALYSIS

Let us now establish the convergence analysis of the proposed algorithm.

Theorem 3.1. *If the SEP is consistent, namely* $S \neq \emptyset$*, then the sequence* $z_n = (x_n, y_n)$ *generated by* (2.1) *converges to an element in* S.

Proof. Let $\delta = \max(\|A\|^2, \|B\|^2)$. Taking $z^* = (x^*, y^*) \in S$, we have that

$$\begin{aligned}
\|x_{k+1} - z\|^2 &= \|P_C(x_k - \varrho_k A^\top(Ax_k - By_k)) - x^*\|^2 \\
&\le \|(x_k - x^*) - \varrho_k A^\top(Ax_k - By_k)\|^2 \\
&= \|x_k - x^*\|^2 - 2\varrho_k\langle A(x_k - x^*), Ax_k - By_k\rangle \\
&\quad + \varrho_k^2\|A^\top(Ax_k - By_k)\|^2 \\
&\le \|x_k - x^*\|^2 - 2\varrho_k\langle A(x_k - x^*), Ax_k - By_k\rangle \\
&\quad + \varrho_k^2\delta\|Ax_k - By_k\|^2,
\end{aligned}$$

and also that

$$\begin{aligned}
\|y_{k+1} - y^*\|^2 &= \|P_Q(y_k + \varrho_k B^\top(Ax_k - By_k)) - y^*\|^2 \\
&\le \|(y_k - y^*) + \varrho_k B^\top(Ax_k - By_k)\|^2 \\
&= \|y_k - y^*\|^2 + 2\varrho_k\langle B(y_k - y^*), Ax_k - By_k\rangle \\
&\quad + \varrho_k^2\|B^\top(Ax_k - By_k)\|^2 \\
&\le \|y_k - y^*\|^2 + 2\varrho_k\langle B(y_k - y^*), Ax_k - By_k\rangle \\
&\quad + \varrho_k^2\delta\|Ax_k - By_k\|^2.
\end{aligned}$$

Since $Ax^* = By^*$, adding up the last two inequalities yields

$$\begin{aligned}
\|z_{k+1} - z^*\|^2 \le \|z_k - z^*\|^2 &- 2\varrho_k\delta\|Ax_k - By_k\|^2 \\
&+ 2\varrho_k^2\delta\|Ax_k - By_k\|^2.
\end{aligned} \tag{3.1}$$

On the other hand, it follows that

$$\begin{aligned}
\|Ax_k - By_k\|^2 &= \|A(x_k - x^*) + B(y^* - y_k)\|^2 \\
&\le (\|A\|\|x_k - x^*\| + \|B\|\|y^* - y_k\|)^2 \\
&\le 2\delta(\|x_k - x^*\|^2 + \|y^* - y_k\|^2) \\
&= 2\delta\|z_k - z^*\|^2.
\end{aligned}$$

Substituting this into (3.1), we have

$$\|z_{k+1} - z^*\|^2 \le (1 + \sigma_k)\|z_k - z^*\|^2 - 2\delta\varrho_k\|Ax_k - By_k\|^2 \tag{3.2}$$

where $\sigma_k = 4\delta^2\varrho_k^2$. It is readily seen that $\sum \sigma_k < \infty$ due to (2.2). By Lemma 2.1, we conclude that the sequence $(\|z_k - z^*\|)$ is convergent; in particular, (z_k) is bounded.

We next prove that $\lim_k \|Ax_k - By_k\| = 0$. Take $M > 0$ so that

$$\|z_k - z\| \le M, \ \forall\, k \in \mathbb{N}.$$

From (3.2), it follows that

$$\begin{aligned} 2\delta\varrho_k\|Ax_k - By_k\|^2 &\le \|z_k - z\|^2 - \|z_{k+1} - z\|^2 + \sigma_k\|z_k - z^*\|^2 \\ &\le \|z_k - z\|^2 - \|z_{k+1} - z\|^2 + M^2\sigma_k, \end{aligned}$$

which immediately implies that

$$2\delta\sum_{j=1}^{k}\varrho_j\|Ax_j - By_j\|^2 \le \|z_1 - z^*\|^2 + M^2\sum_{j=1}^{k}\sigma_j^2.$$

Taking the limit by letting $k \to \infty$ in the last relation yields

$$\sum_{k=1}^{\infty}\varrho_k\|Ax_k - By_k\|^2 < \infty. \tag{3.3}$$

This together with the assumption $\sum_k \varrho_k = \infty$ particularly implies that

$$\liminf_{k\to\infty}\|Ax_k - By_k\| = 0.$$

To prove $\lim_k \|Ax_k - By_k\| = 0$, it suffices to verify the existence of the $\lim_k \|Ax_k - By_k\|$. Actually, we have that

$$\begin{aligned} \|A(x_{k+1} - x_k)\| &\le \|A\|\|P_C(x_k - \varrho_k A^\top(Ax_k - By_k)) - x_k\| \\ &\le \|A\|\|\varrho_k A^\top(Ax_k - By_k)\| \\ &\le \delta\varrho_k\|Ax_k - By_k\|, \end{aligned}$$

and also that

$$\begin{aligned} \|B(y_{k+1} - y_k)\| &\le \|B\|\|P_Q(y_k + \varrho_k B^\top(Ax_k - By_k)) - y_k\| \\ &\le \|B\|\|\varrho_k B^\top(Ax_k - By_k)\| \\ &\le \delta\varrho_k\|Ax_k - By_k\|. \end{aligned}$$

Let $a_k = Ax_k - By_k$. By the last two inequalities, we have

$$\begin{aligned} \|a_{k+1} - a_k\| &= \|(Ax_{k+1} - By_{k+1}) - (Ax_k - By_k)\| \\ &\le \|A(x_k - x_{k+1})\| + \|B(y_{k+1} - y_k)\| \\ &\le 2\delta\varrho_k\|Ax_k - By_k\| \\ &= 2\delta\varrho_k\|a_k\|. \end{aligned}$$

Hence, we have

$$\|a_{k+1}\|^2 = \|a_k\|^2 + 2\langle a_k, a_{k+1} - a_k\rangle + \|a_{k+1} - a_k\|^2$$

$$\leq \|a_k\|^2 + 2\|a_k\|\|a_{k+1} - a_k\| + \|a_{k+1} - a_k\|^2$$
$$\leq \|a_k\|^2 + 4\delta\varrho_k\|a_k\|^2 + 4\delta^2\varrho_k^2\|a_k\|^2$$
$$\leq \|a_k\|^2 + 4\delta\varrho_k\|a_k\|^2 + 8\delta^3 M^2\varrho_k^2,$$

where the last inequality follows from the following estimates:

$$\begin{aligned}\|a_k\| &= \|Ax_k - Ax^* + By^* - By_k\| \\ &\leq \sqrt{\delta}(\|x_k - x^*\| + \|y_k - y^*\|) \\ &\leq \sqrt{2\delta}(\|x_k - x^*\|^2 + \|y_k - y^*\|^2)^{1/2} \\ &= \sqrt{2\delta}\|z_k - z^*\| \leq \sqrt{2\delta}M.\end{aligned}$$

Setting $\eta_k = 4\delta\varrho_k\|a_k\|^2 + 8\delta^3M^2\varrho_k^2$, we have

$$\|a_{k+1}\|^2 \leq \|a_k\|^2 + \eta_k. \tag{3.4}$$

It is clear that $\sum_k \eta_k < \infty$ due to (3.3) and (2.2). We can therefore apply Lemma 2.1 to (3.4) to get the existence of the $\lim_k \|Ax_k - By_k\|$. Hence $\lim_k \|Ax_k - By_k\| = 0$.

Finally, we show that every cluster point of (z_k) is in the set S. Suppose that a subsequence $(z_{k_j}) = (x_{k_j}, y_{k_j})$ of (z_k) converges to a point $\hat{z} = (\hat{x}, \hat{y})$. It is readily seen that $\hat{x} \in C, \hat{y} \in Q$. Consequently

$$\begin{aligned}\|A\hat{x} - B\hat{y}\| &= \lim_{j\to\infty} \|Ax_{k_j} - By_{k_j}\| \\ &= \lim_{k\to\infty} \|Ax_k - By_k\| = 0,\end{aligned}$$

that is, $\hat{z} = (\hat{x}, \hat{y}) \in S$. Note that $\lim \|z_k - z\|$ exists for all $z \in S$. In particular, we have that $\lim \|z_k - \hat{z}\|$ exists. Since, however, the subsequence (z_{k_j}) converges to $\hat{z}$, we must have $\lim \|z_k - \hat{x}\| = 0$. Therefore $z_k \to \hat{z} \in S$, or equivalently $x_k \to \hat{x}$ and $y_k \to \hat{y}$. □

Remark 3.2. The above result can be easily extended to the infinite dimensional Hilbert spaces. The only difference is that one can establish the weak convergence of the proposed algorithm to a solution of the SEP.

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