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COMMON FIXED POINT FOR (ψ,φ)-WEAK CONTRACTIONS ON ORDERED GAUGE SPACES AND APPLICATIONS

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Abstract. In this paper, we establish a fixed point theorem for four mappings satisfying (ψ,φ)-weakly contractive conditions on ordered Gauge spaces. We also give applications to integral equations.

1. Introduction and preliminaries

Fixed point theory became one of the most interesting area of research in the last fifty years. Contractice type mappings on a complete metric space which are generalizations of Banach contraction principles was studied [1,3,11,13,15–17,22,23, 27]. Existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [21] and then by Nieto and Lopez [19]. In this direction many authors gived some generalisations of fixed point theorems [2,5,6,8,14,18,20, 22,24–26]. In 2011 Chifu and Petruşel [10] presented some fixed point theorems for $\varphi-$contractions on ordered and complete gauge space. After Cherichi and Samet [9] proved new coincidence and fixed point theorems for mappings satisfying generalized weakly contractive conditions on the setting of ordered gauge spaces and gived some applications for integral equations. These theorems generalized many existing results in the literature.

The purpose of this paper is to present a new fixed point theorem for four mappings that satisfy generalized weak contractions in ordered gauge space.

Definition 1.1. Let X be a nonempty set. A map $d : X \times X \to [0,+\infty)$ is called a pseudo-metric in X whenever

(i) $d(x,x) = 0$ for all $x \in X$,
(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$,
(iii) $d(x,y) \le d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Definition 1.2. Let X be a nonempty set endowed with a pseudo-metric d. The $d-ball$ of radius $\varepsilon > 0$ centered at $x \in X$ is the set

$$B(x,d,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}.$$

Definition 1.3. A family $\mathcal{F} = \{d_\lambda : \lambda \in \Lambda\}$ of pseudo-metrics is called separating if for each pair $x \neq y$, there is a $d_\lambda \in \mathcal{F}$ such that $d_\lambda(x,y) \neq 0$.

Definition 1.4. Let X be a nonempty set and $\mathcal{F} = \{d_\lambda : \lambda \in \Lambda\}$ be a separating family of pseudo-metrics on X. The topology $\tau(\mathcal{F})$ having for a subbasis the family $B(\mathcal{F}) = \{B(x,d_\lambda,\varepsilon) : x \in X,\ d_\lambda \in \mathcal{F},\ \varepsilon > 0\}$ of the balls is called the topology in

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X induced by the family $\mathcal{F}$. The pair $(X, \tau(\mathcal{F}))$ is called a gauge space. Note that $(X, \tau(\mathcal{F}))$ is Hausdorff because we require $\mathcal{F}$ to be separating.

Definition 1.5. Let $(X, \tau(\mathcal{F}))$ be a gauge space with respect to the family $\mathcal{F} = \{d_\lambda : \lambda \in \Lambda\}$ of pseudo-metrics on X. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(i) The sequence $\{x_n\}$ converges to x if and only if

$$\forall \lambda \in \Lambda, \quad \forall \varepsilon > 0, \ \exists N \in \mathbb{N} \quad \ni \quad d_\lambda(x_n, x) < \varepsilon, \ \forall n \geq N.$$

In this case, we denote $x_n \xrightarrow{\mathcal{F}} x$.

(ii) The sequence $\{x_n\}$ is Cauchy if and only if

$$\forall \lambda \in \Lambda, \quad \forall \varepsilon > 0, \ \exists N \in \mathbb{N} \quad \ni \quad d_\lambda(x_{n+p}, x_n) < \varepsilon, \ \forall n \geq N, \ p \in \mathbb{N}.$$

(iii) $(X, \tau(\mathcal{F}))$ is complete if and only if any Cauchy sequence in $(X, \tau(\mathcal{F}))$ is convergent to an element of X.

(iv) A subset of X is said to be closed if it contains the limit of any convergent sequence of its elements.

Definition 1.6. Let $\mathcal{F} = \{d_\lambda : \lambda \in \Lambda\}$ be a family of of pseudo-metrics on X. $(X, \mathcal{F}, \preceq)$ is called an ordered gauge space if $(X, \tau(\mathcal{F}))$ is a gauge space and $(X, \preceq)$ is a partially ordered set.

Definition 1.7 ([9])**.** Let $(X, \tau(\mathcal{F}))$ be a gauge space and $f, g : X \to X$ are giving mappings. We say that the pair $\{f, g\}$ is compatible if for all $\lambda \in \Lambda$, $d_\lambda(fgx_n, gfx_n) \to 0$ as $n \to \infty$ whenever $\{x_n\}$ is a sequence in X such that $fx_n \xrightarrow{\mathcal{F}} t$ and $gx_n \xrightarrow{\mathcal{F}} t$ for some $t \in X$.

For more information about gauge space, the reader can refer to [12].

Throughout this paper we need following definitions.

Definition 1.8 ([4])**.** Let $(X, \preceq)$ be a partially ordered set. A pair (f, g) of selfmaps of X said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$.

Definition 1.9 ([2])**.** Let $(X, \preceq)$ be a partially ordered set and f and g be two selfmaps on X . An ordered pair (f, g) said to be partially weakly increasing if $fx \preceq gfx$ for all $x \in X$.

Note that a pair (f, g) is weakly increasing if and only if ordered pair (f, g) and (g, f) are partially weakly increasing.

Definition 1.10 ([2])**.** Let $(X, \preceq)$ be a partially ordered set. A mapping f is called weak annihilator of g if $fgx \preceq x$ for all $x \in X$.

Definition 1.11 ([2])**.** Let $(X, \preceq)$ be a partially ordered set. A mapping f is called dominating if $x \preceq fx$ for each x in X.

Definition 1.12. X be a non-empty set and let $f, T : X \to X$. The mappings f, T are said to be weakly compatible if they commute at their coincidence points, that is, if $fx = Tx$ for some $x \in X$ implies that $fTx = Tfx$.

We also use two classes of function in this paper [11],

(C1) $\Phi = \{\varphi \mid \varphi : [0, \infty) \to [0, \infty)$ is lower semi continuous, $\varphi(t) > 0$ for all $t > 0$, $\varphi(0) = 0\}$.

(C2) $\Psi = \{\psi \mid \psi : [0, \infty) \to [0, \infty)$ is continuous and non-decreasing with $\psi(t) = 0$ if and only if $t = 0\}$.

2. Main theorem

Theorem 2.1. *Let X be nonempty set and $(X, \digamma, \preceq)$ be an ordered complete gauge space. Suppose that f, g, S and T are four self mappings on X, $\{T, f\}$ and $\{S, g\}$ be partially weakly increasing with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$ dominating maps f and g are weak annihilators of T and S, respectively. Suppose that there exist control functions ψ and φ such that for every comparable elements $x, y \in X$ and for all $\lambda \in \Lambda$*

$$\psi(d_\lambda(fx, gy)) \leq \psi(M(x, y)) - \varphi(M(x, y)), \tag{2.1}$$

where,

$$M(x, y) = \max\left\{d_\lambda(Sx, Ty), d_\lambda(fx, Sx), d_\lambda(gy, Ty), \frac{d_\lambda(Sx, gy) + d_\lambda(fx, Ty)}{2}\right\}.$$

If for a nondecreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n and $y_n \to u$ implies that $x_n \preceq u$ and eitherakly compatible

(a) *$\{f, S\}$ are compatible and one of f or S is continuous and $\{g, T\}$ are weakly compatible*

(b) *$\{g, T\}$ are compatible, g or T is continuous and $\{f, S\}$ are weakly compatible*

then f, g, S and T have common fixed point. Moreover, the set of common fixed points of f, g, S and T is well ordered if and only if f, g, S and T have one and only one common fixed point.

Proof. Let x_0 be an arbitrary point in X. Costruct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n-1} = fx_{2n-2} = Tx_{2n-1}$ and $y_{2n} = gx_{2n-1} = Sx_{2n}$. By given assumptions, $x_{2n-2} \preceq fx_{2n-2} = Tx_{2n-1} \preceq fTx_{2n-1} \preceq x_{2n-1}$ and $x_{2n-1} \preceq gx_{2n-1} = Sx_{2n} \preceq Sgx_{2n} \preceq x_{2n}$. Thus for all $n \geq 1$ we have $x_n \preceq x_{n+1}$.

Step 1. We will prove that $\lim_{n\to\infty} d_\lambda(y_{n+1}, y_n) = 0$, for all $\lambda \in \Lambda$.

Let $\lambda \in \Lambda$. We distinguish two cases.

Firstly, we suppose that there exists $n \in \mathbb{N}$ such that $d_\lambda(y_{2n}, y_{2n+1}) = 0$. From (2.1)

$$\begin{aligned} \psi(d_\lambda(y_{2n+1}, y_{2n+2})) &= \psi(d_\lambda(fx_{2n}, gx_{2n+1})) \\ &\leq \psi(M(x_{2n}, x_{2n+1})) - \varphi(M(x_{2n}, x_{2n+1})), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\left\{\begin{array}{c} d_\lambda(Sx_{2n}, Tx_{2n+1}), d_\lambda(fx_{2n}, Sx_{2n}), \\ d_\lambda(gx_{2n+1}, Tx_{2n+1}), \\ \frac{d_\lambda(Sx_{2n}, gx_{2n+1}) + d_\lambda(fx_{2n}, Tx_{2n+1})}{2} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} d_\lambda(y_{2n}, y_{2n+1}), d_\lambda(y_{2n+1}, y_{2n}), \\ d_\lambda(y_{2n+2}, y_{2n+1}), \frac{d_\lambda(y_{2n}, y_{2n+2}) + d_\lambda(y_{2n+1}, y_{2n+1})}{2} \end{array}\right\} \\ &= \max\left\{d_\lambda(y_{2n+2}, y_{2n+1}), \frac{d_\lambda(y_{2n+2}, y_{2n+1})}{2}\right\} \end{aligned}$$

$$= \quad d_\lambda\left(y_{2n+2}, y_{2n+1}\right).$$

Hence, by (2.2)

$$\psi\left(d_\lambda\left(y_{2n+1}, y_{2n+2}\right)\right) \leq \psi\left(d_\lambda\left(y_{2n+1}, y_{2n+2}\right)\right) - \varphi\left(d_\lambda\left(y_{2n+1}, y_{2n+2}\right)\right).$$

Thus $\varphi\left(d_\lambda\left(y_{2n+1}, y_{2n+2}\right)\right) = 0$. As $\varphi\left(t\right) = 0$ if and only if $t = 0$. Continuing this process $d_\lambda\left(y_{2n+2}, y_{2n+3}\right) = 0$. Then our claim holds.

In second case, we take $d_\lambda\left(y_{2n}, y_{2n+1}\right) > 0$ for each $n \in \mathbb{N}$. Since x_{2n} and x_{2n+1} are comparable, by (2.1)

$$\begin{aligned} \psi\left(d_\lambda\left(y_{2n+1}, y_{2n+2}\right)\right) &= \psi\left(d_\lambda\left(fx_{2n}, gx_{2n+1}\right)\right) \\ &\leq \psi\left(M\left(x_{2n}, x_{2n+1}\right)\right) - \varphi\left(M\left(x_{2n}, x_{2n+1}\right)\right) \\ &< \psi\left(M\left(x_{2n}, x_{2n+1}\right)\right), \end{aligned}$$

which implies that

$$d_\lambda\left(y_{2n+1}, y_{2n+2}\right) < M\left(x_{2n}, x_{2n+1}\right), \tag{2.3}$$

where

$$\begin{aligned} M\left(x_{2n}, x_{2n+1}\right) &= \max\left\{ \begin{array}{c} d_\lambda\left(Sx_{2n}, Tx_{2n+1}\right), d_\lambda\left(fx_{2n}, Sx_{2n}\right), \\ d_\lambda\left(gx_{2n+1}, Tx_{2n+1}\right), \\ \frac{d_\lambda(Sx_{2n}, gx_{2n+1}) + d_\lambda(fx_{2n}, Tx_{2n+1})}{2} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d_\lambda\left(y_{2n}, y_{2n+1}\right), d_\lambda\left(y_{2n+1}, y_{2n}\right), \\ d_\lambda\left(y_{2n+2}, y_{2n+1}\right), \frac{d_\lambda(y_{2n}, y_{2n+2}) + d_\lambda(y_{2n+1}, y_{2n+1})}{2} \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} d_\lambda\left(y_{2n}, y_{2n+1}\right), d_\lambda\left(y_{2n+2}, y_{2n+1}\right), \\ \frac{d_\lambda(y_{2n}, y_{2n+1}) + d_\lambda(y_{2n+1}, y_{2n+2})}{2} \end{array} \right\} \\ &= \max\left\{d_\lambda\left(y_{2n}, y_{2n+1}\right), d_\lambda\left(y_{2n+2}, y_{2n+1}\right)\right\}. \end{aligned}$$

If $M\left(x_{2n}, x_{2n+1}\right) = d_\lambda\left(y_{2n+2}, y_{2n+1}\right)$, this is contradiction. Therefore

$$M\left(x_{2n}, x_{2n+1}\right) = d_\lambda\left(y_{2n}, y_{2n+1}\right).$$

Hence from (2.3), we obtain $\psi(d_\lambda\left(y_{2n+1}, y_{2n+2}\right)) < \psi\left(d_\lambda\left(y_{2n}, y_{2n+1}\right)\right)$.

Since ψ is nondecreasing $d_\lambda\left(y_{2n+1}, y_{2n+2}\right) < d_\lambda\left(y_{2n}, y_{2n+1}\right)$.

Similarly we can show that $d_\lambda\left(y_{2n}, y_{2n+1}\right) < d_\lambda\left(y_{2n-1}, y_{2n}\right)$.

Therefore $\{d_\lambda\left(y_{2n+1}, y_{2n}\right)\}$ is monotone decreasing sequence which is bounded below by 0. Since $\{d_\lambda\left(y_{2n+1}, y_{2n}\right)\}$ is monotone and bounded there exists $r \geq 0$ such that $d_\lambda\left(y_{n+1}, y_n\right) \to r$ as $n \to \infty$.

From (2.1) and lower semicontinuity of φ,

$$\lim_{n\to\infty} \sup \psi(d_\lambda\left(y_{n+1}, y_n\right)) \leq \lim_{n\to\infty} \sup \psi(M\left(x_{n+1}, x_n\right)) - \lim_{n\to\infty} \inf \varphi(M\left(x_{n+1}, x_n\right)),$$

which implies that $\psi\left(r\right) \leq \psi\left(r\right) - \varphi\left(r\right)$ .So $r = 0$, and $\lim_{n\to\infty} d_\lambda\left(y_{n+1}, y_n\right) = 0$.

Step 2. Now, we will show $\{y_n\}$ is Cauchy sequence in the gauge space $(X, T\left(\digamma\right))$. For this it is sufficient to show that $\{y_{2n}\}$ is Cauchy in $\left(X, T\left(\digamma\right)\right)$. Suppose to the contrary. Then there exists a $(\lambda, \varepsilon) \in \Lambda \times (0, \infty)$ for which we can find two subsequences $\{x_{2m_k}\}$ and $\{x_{2n_k}\}$ of $\{x_n\}$ with $2m_k > 2n_k > k$ such that

$$d_\lambda\left(y_{2m_k}, y_{2n_k}\right) \geq \varepsilon, \tag{2.4}$$

and

$$d_\lambda\left(y_{2m_k-2}, y_{2n_k}\right) < \varepsilon. \tag{2.5}$$

Using (2.4), (2.5) and triangular inequality

$$\begin{aligned}\varepsilon \leq & d_\lambda\left(y_{2m_k}, y_{2n_k}\right) \\ \leq & d_\lambda(y_{2m_k}, y_{2m_k-1}) + d_\lambda(y_{2m_k-1}, y_{2m_k-2}) + d_\lambda\left(y_{2m_k-2}, y_{2n_k}\right).\end{aligned}$$

Letting $k \to \infty$, $\lim_{k\to\infty} d_\lambda\left(y_{2n_k}, y_{2m_k}\right) = \varepsilon$. Again, the triangular inequality

$$\begin{aligned}d_\lambda\left(y_{2m_k}, y_{2n_k}\right) &\leq d_\lambda\left(y_{2m_k}, y_{2m_k-1}\right) + d_\lambda(y_{2m_k-1}, y_{2n_k}), \\ d_\lambda(y_{2m_k-1}, y_{2n_k}) &\leq d_\lambda\left(y_{2m_k-1}, y_{2m_k}\right) + d_\lambda\left(y_{2m_k}, y_{2n_k}\right).\end{aligned}$$

Letting $k \to \infty$,we have $\lim_{k\to\infty} d_\lambda\left(y_{2m_{k-1}}, y_{2n_k}\right) = \varepsilon$. As

$$\begin{aligned}M(x_{2n_k}, x_{2m_k-1}) &= \max\left\{\begin{array}{c} d_\lambda\left(Sx_{2n_k}, Tx_{2m_k-1}\right), d_\lambda\left(fx_{2n_k}, Sx_{2n_k}\right), \\ d_\lambda\left(gx_{2m_k-1}, Tx_{2m_k-1}\right), \\ \frac{d_\lambda\left(Sx_{2n_k}, gx_{2m_k-1}\right) + d_\lambda\left(fx_{2n_k}, Tx_{2n_k}\right)}{2} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} d_\lambda\left(y_{2n_k}, y_{2m_k-1}\right), d_\lambda\left(y_{2n_k+1}, y_{2n_k}\right), \\ d_\lambda\left(y_{2m_k}, y_{2m_k-1}\right), \\ \frac{d_\lambda\left(y_{2n_k}, y_{2m_k}\right) + d_\lambda\left(y_{2n_k+1}, y_{2n_k}\right)}{2} \end{array}\right\},\end{aligned}$$

thus $\lim_{k\to\infty} M(x_{2n_k}, x_{2m_k-1}) = \max\left\{\varepsilon, 0, 0, \frac{\varepsilon}{2}\right\} = \varepsilon$. From (2.1),

$$\begin{aligned}\psi\left(d_\lambda\left(y_{2n_k+1}, y_{2m_k}\right)\right) &= \psi\left(d_\lambda\left(fx_{2n_k}, gx_{2m_k-1}\right)\right) \\ &\leq \psi\left(M(x_{2n_k}, x_{2m_k-1})\right) - \varphi\left(M(x_{2n_k}, x_{2m_k-1})\right).\end{aligned}$$

Taking limit as $k \to \infty$ implies that $\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon)$ which yields that $\varphi(\varepsilon) = 0$. So $\varepsilon = 0$ which is a contradiction. Hence $\{y_{2n}\}$ is a Cauchy sequence. Since $(X, T(\mathcal{F}))$ is complete gauge space, there exists $z \in X$ such that $y_{2n} \overset{\mathcal{F}}{\to} z$ as $n \to \infty$. Therefore as $n \to \infty$

$$y_{2n+1} \overset{\mathcal{F}}{\to} z,\ Tx_{2n+1} \overset{\mathcal{F}}{\to} z,\ fx_{2n} \overset{\mathcal{F}}{\to} z \text{ and } y_{2n+2} \overset{\mathcal{F}}{\to} z,\ Sx_{2n+2} \overset{\mathcal{F}}{\to} z,\ gx_{2n+1} \overset{\mathcal{F}}{\to} z.$$

Step 3. We will show that z is a common fixed point of f, g, S and T. Assume that S is continuous. Since $\{f, S\}$ are compatible, we have

$$fSx_{2n+2} \overset{\mathcal{F}}{\to} Sz = Sfx_{2n+2} \overset{\mathcal{F}}{\to} Sz \qquad \text{as } n \to \infty.$$

Also, $x_{2n+1} \preceq gx_{2n+1} = Sx_{2n+2}$. Now

$$\psi(d_\lambda(fSx_{2n+2}, gx_{2n+1})) \leq \psi\left(M\left(Sx_{2n+2}, x_{2n+1}\right)\right) - \varphi\left(M\left(Sx_{2n+2}, x_{2n+1}\right)\right),$$

where,

$$M\left(Sx_{2n+2}, x_{2n+1}\right) = \max\left\{\begin{array}{c} d_\lambda\left(SSx_{2n+2}, Tx_{2n+1}\right), \\ d_\lambda\left(fSx_{2n+2}, SSx_{2n+2}\right), d_\lambda\left(gx_{2n+1}, Tx_{2n+1}\right), \\ \frac{d_\lambda(SSx_{2n+2}, gx_{2n+1}) + d_\lambda(fSx_{2n+2}, Tx_{2n+1})}{2} \end{array}\right\}.$$

On taking limit as $n \to \infty$, we obtain

$$\psi\left(d_\lambda\left(Sz, z\right)\right) \leq \psi(d_\lambda\left(Sz, z\right)) - \varphi\left(d_\lambda\left(Sz, z\right)\right).$$

Thus $Sz = z$.

Now, $x_{2n+1} \preceq gx_{2n+1}$ and $gx_{2n+1} \xrightarrow{F} z$ as $n \to \infty$, $x_{2n+1} \preceq z$ and (2.1)

$$\psi(d_\lambda(fz, gx_{2n+1})) \leq \psi\left(M\left(z, x_{2n+1}\right)\right) - \varphi\left(M\left(z, x_{2n+1}\right)\right),$$

where,

$$M\left(z, x_{2n+1}\right) = \max\left\{\begin{array}{c} d_\lambda\left(Sz, Tx_{2n+1}\right), d_\lambda\left(fz, Sz\right), \\ d_\lambda\left(gx_{2n+1}, Tx_{2n+1}\right), \frac{d_\lambda(Sz, gx_{2n+1}) + d_\lambda(fz, Tx_{2n+1})}{2} \end{array}\right\}.$$

On taking limit as $n \to \infty$, we have $\psi(d_\lambda(fz, z)) \leq \psi\left(d_\lambda\left(fz, z\right)\right) - \varphi\left(d_\lambda\left(fz, z\right)\right)$ and $d_\lambda(fz, z) = 0$ for all $\lambda \in \Lambda$. In the virtue of the separating structure of F, this implies that $fz = z$.

Since $f\left(X\right) \subseteq T\left(X\right)$, there exists a $u \in X$ such that $fz = Tu$. Suppose that $gu \neq Tu$.Since $z \preceq fz = Tu \preceq fTu \preceq u$ implies $z \preceq u$. From (2.1), we obtain

$$\psi\left(d_\lambda\left(Tu, gu\right)\right) = \psi\left(d_\lambda\left(fz, gu\right)\right) \leq \psi\left(M\left(z, u\right)\right) - \varphi\left(M\left(z, u\right)\right),$$

where

$$\begin{aligned} M\left(z, u\right) &= \max\left\{\begin{array}{c} d_\lambda\left(Sz, Tu\right), d_\lambda\left(fz, Sz\right), d_\lambda\left(gu, Tu\right), \\ \frac{d_\lambda(Sz, gu) + d_\lambda(fz, Tu)}{2} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} d_\lambda\left(z, z\right), d_\lambda\left(z, z\right), d_\lambda\left(gu, Tu\right), \\ \frac{d_\lambda(gu, Tu) + d_\lambda(Tu, Tu)}{2} \end{array}\right\} \\ &= d_\lambda\left(gu, Tu\right). \end{aligned}$$

Thus, $\psi\left(d_\lambda\left(Tu, gu\right)\right) \leq \psi\left(d_\lambda\left(Tu, gu\right)\right) - \varphi\left(d_\lambda\left(Tu, gu\right)\right)$, this a contradiction with definition of φ. Hence, $d_\lambda(Tu, gu) = 0$. From separating structure of F, $Tu = gu$. Since g and T are weakly compatible, $gz = gfz = gTu = Tgu = Tfz = Tz$. Thus z is a coincidence point of g and T.

Now, since $x_{2n} \preceq fx_{2n}$ and $fx_{2n} \xrightarrow{F} z$ as $n \to \infty$, implies that $x_{2n} \preceq z$, so

$$\psi\left(d_\lambda\left(fx_{2n}, gz\right)\right) \leq \psi\left(M\left(x_{2n}, z\right)\right) - \varphi\left(M\left(x_{2n}, z\right)\right),$$

where

$$M\left(x_{2n}, z\right) = \max\left\{\begin{array}{c} d_\lambda\left(Sx_{2n}, Tz\right), d_\lambda\left(fx_{2n}, Sx_{2n}\right), d_\lambda\left(gz, Tz\right), \\ \frac{d_\lambda(Sx_{2n}, gz) + d_\lambda(fx_{2n}, Tz)}{2} \end{array}\right\},$$

on taking limit as $n \to \infty$, we have $\psi\left(d_\lambda\left(z, gz\right)\right) \leq \psi\left(d_\lambda\left(z, gz\right)\right) - \varphi\left(d_\lambda\left(z, gz\right)\right)$ and $d_\lambda\left(z, gz\right) = 0$. By separating structure of F, $z = gz$. Therefore $fz = gz = Sz = Tz = z$.

The proof is similar when f is continuous.

Similarly, the results follow when (b) holds.

Now suppose that the set of f, g, S and T is well ordered. We claim that common fixed point of f, g, S and T is unique.Assume on the contrary that, $fw = gw = Sw = Tw = w$ and $fv = gv = Sv = Tv = v$ but $w \neq v$. From (2.1)

$$\psi\left(d_\lambda\left(w, v\right)\right) = \psi\left(d_\lambda\left(fw, gv\right)\right) \leq \psi\left(M\left(w, v\right)\right) - \varphi\left(M\left(w, v\right)\right),$$

where

$$M\left(w, v\right) = \max\left\{\begin{array}{c} d_\lambda\left(Sw, Tv\right), d_\lambda\left(fw, Sw\right), d_\lambda\left(gv, Tv\right), \\ \frac{d_\lambda(Sw, gv) + d_\lambda(fw, Tv)}{2} \end{array}\right\}$$

$$= \max\left\{d_\lambda(w,v), 0, 0, \frac{d_\lambda(w,v)+d_\lambda(w,v)}{2}\right\}$$
$$= d_\lambda(w,v),$$

and $\psi(d_\lambda(w,v)) \leq \psi(d_\lambda(w,v)) - \varphi(d_\lambda(w,v))$. This is a contradiction. So $d_\lambda(w,v) = 0$. Thus $w = v$. Conversely, if f, g, S and T have only one common fixed point then the set of common fixed point of f, g, S and T being singleton is well ordered. □

Corollary 2.2. *Let X be a nonempty set and $(X, \digamma, \preceq)$ be an ordered complete gauge space.Suppose that f, S, T are three selfmappings on X, (T, f) and (S, f) be partially weakly increasing with $f(X) \subseteq T(X)$, $f(X) \subseteq S(X)$ and dominating map f is weak annihilator of T and S. Suppose that there exist control functions ψ and φ such that for every two comparable elements $x, y \in X$ and for all $\lambda \in \Lambda$*

$$\psi(d_\lambda(fx, fy)) \leq \psi(M(x,y)) - \varphi(M(x,y)),$$

where

$$M(x,y) = \max\left\{d_\lambda(Sx,Ty), d_\lambda(fx,Sx), d_\lambda(fy,Ty), \frac{d_\lambda(Sx,fy)+d_\lambda(fx,Ty)}{2}\right\},$$

is satisfied. If for a nondecreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n and $y_n \to u$ implies that $x_n \preceq u$ and either

(a) *$\{f, S\}$ are compatible, f or S is continuous and $\{f, T\}$ are weakly compatible or*
(b) *$\{f, T\}$ are compatible, f or T is continuous and $\{f, S\}$ are weakly compatible*

then f, S and T have a common fixed point. Moreover, the set of common fixed points of f, g, S and T is well ordered if and only if f, S and T have one and only one common fixed point.

Corollary 2.3. *Let X be a nonempty set and $(X, \digamma, \preceq)$ be an ordered complete gauge space. Suppose that f, g, T are three selfmappings on X. (T, f) and (T, g) be partially weakly increasing with $f(X) \subseteq T(X)$, $g(X) \subseteq T(X)$ and dominating map f and g are weak annihilators of T. Suppose that there exist control functions ψ and φ such that for every two comparable elements $x, y \in X$ and for all $\lambda \in \Lambda$,*

$$\psi(d_\lambda(fx, gy)) \leq \psi(M(x,y)) - \varphi(M(x,y)),$$

where

$$M(x,y) = \max\left\{\begin{array}{c} d_\lambda(Tx,Ty), d_\lambda(fx,Tx), d_\lambda(gy,Ty), \\ \frac{d_\lambda(Tx,gy)+d_\lambda(fx,Ty)}{2} \end{array}\right\},$$

is satisfied. If for a nondecreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n and $y_n \to u$ implies that $x_n \preceq u$ and either

(a) *$\{f, T\}$ are compatible, f or T is continuous and $\{g, T\}$ are weakly compatible or*
(b) *$\{g, T\}$ are compatible, g or T is continuous and $\{f, T\}$ are weakly compatible,*

then f, g and T have a common fixed point. Moreover, the set of common fixed points of f, g and T is well ordered if and only if f, g and T have one and only one common fixed point.

Example 2.4. Let $X = [0,1] \cup \{2,3,4,\dots\}$ with usual ordering and $A = \{1/2^n : n \in \mathbb{N}\}$ and $\digamma = \{d_\lambda\}_{\lambda\in\Lambda}$ where $d_\lambda : X \times X \to [0,\infty)$

$$d_\lambda(x,y) = \begin{cases} 0, & \text{if } x = y \text{ or } \{x,y\} \cap A = \{x,y\}, \\ 1, & \text{if } x \neq y \text{ and } \{x,y\} \cap A \neq \{x,y\}, \end{cases} \quad x,y \in X.$$

Then d_λ is pseudometric on X and $(X, \digamma, \preceq)$ be an ordered complete gauge space. Let $\psi, \varphi : [0,\infty) \to [0,\infty)$ be defined by $\psi(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}, \\ 1 & \text{if } x \in \left(\frac{1}{2}, 1\right], \\ x, & \text{otherwise} \end{cases}$ and $\varphi(x) = \begin{cases} \frac{1}{4} - x^2 & \text{if } 0 \le x \le \frac{1}{2}, \\ 0, & \text{otherwise} \end{cases}$ and selfmappings f, g, S, T on X be given by

$$f(x) = \begin{cases} 0, & \text{if } x \le \frac{1}{2}, \\ \frac{1}{2}, & \text{if } x \in \left(\frac{1}{2}, 1\right], \\ 1, & \text{if } x \in \left(\frac{1}{2}, 1\right], \\ x, & \text{if } x \in \{2,3,\dots\} \end{cases}, \qquad g(x) = \begin{cases} 0, & \text{if } x = 0, \\ \frac{1}{2}, & \text{if } x \in \left(0, \frac{1}{2}\right], \\ x, & \text{if } x \in \left(\frac{1}{2}, 1\right] \cup \{2,3,\dots\} \end{cases},$$

$$T(x) = \begin{cases} 0, & \text{if } x \le \frac{1}{2}, \\ \frac{1}{2}, & \text{if } x \in \left(\frac{1}{2}, 1\right], \\ x-1, & \text{if } x \in \{2,3,\dots\} \end{cases}, \quad S(x) = \begin{cases} 0, & \text{if } x \le \frac{1}{2}, \\ 2x-1, & \text{if } x \in \left(\frac{1}{2}, 1\right], \\ x, & \text{if } x \in \{2,3,\dots\} \end{cases}.$$

f, g, S and T satisfy all conditions of Theorem 2.1 and 0 is a unique common fixed point.

3. Applications

Let Γ be set of functions $a : [0,\infty) \to [0,\infty)$ satisfying

(i) a is locally integrable on $[0,\infty)$,

(ii) for all $\varepsilon > 0$, we have $\int_0^\varepsilon a(t)dt > 0$.

Theorem 3.1. *Let X be a nonempty set and $(X, \digamma, \preceq)$ be an ordered complete gauge space. Suppose that f, g, S and T are four self mappings on X, $\{T, f\}$ and $\{S, g\}$ be partially weakly increasing with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$ dominating maps f and g are weak annihilators of T and S, respectively. Suppose that*

$$\int_0^{d_\lambda(fx,gy)} a_\lambda(t)dt \le \int_0^{M(x,y)} a_\lambda(t)dt - \int_0^{M(x,y)} b_\lambda(t)dt,$$

where,

$$M(x,y) = \max\left\{ \begin{array}{c} d_\lambda(Sx,Ty), d_\lambda(fx,Sx), d_\lambda(gy,Ty), \\ \frac{d_\lambda(Sx,gy)+d_\lambda(fx,Ty)}{2} \end{array} \right\},$$

for every comparable elements $x, y \in X$ and for all $\lambda \in \Lambda$, where $a_\lambda, b_\lambda \in \Gamma$. If for a nondecreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n and $y_n \to u$ implies that $x_n \preceq u$ and either

(a) *$\{f, S\}$ are compatible and one of f or S is continuous and $\{g, T\}$ are weakly compatible*

(b) $\{g,T\}$ *are compatible,* g *or* T *is continuous and* $\{f,S\}$ *are weakly compatible*

then f,g,S *and* T *have a common fixed point. Moreover, the set of common fixed points of* f,g,S *and* T *is well ordered if and only if* f,g,S *and* T *have one and only one common fixed point.*

Proof. It follows from Theorem 2.1, by taking for all $\lambda \in \Lambda$,

$$\psi(t) = \int_0^t a_\lambda(s)ds \quad \text{and} \quad \varphi(t) = \int_0^t b_\lambda(s)ds, \quad t \geq 0.$$

It is clear that ψ and φ satisfy condition (C1) and (C2). □

Corollary 3.2. *Let* $(X, \mathcal{F}, \preceq)$ *be an ordered complete gauge space and* $f,\ S,T : X \to X$ *be mappings,* (T,f) *and* (S,f) *be partially weakly increasing with* $f(X) \subseteq T(X)$, $f(X) \subseteq S(X)$ *and dominating map* f *is weak annihilator of* T *and* S*. Suppose that*

$$\int_0^{d_\lambda(fx,fy)} a_\lambda(t)dt \leq \int_0^{M(x,y)} a_\lambda(t)dt - \int_0^{M(x,y)} b_\lambda(t)dt,$$

where

$$M(x,y) = \max \left\{ \begin{array}{c} d_\lambda(Sx,Ty), d_\lambda(fx,Sx), d_\lambda(fy,Ty), \\ \frac{d_\lambda(Sx,fy)+d_\lambda(fx,Ty)}{2} \end{array} \right\},$$

for every comparable elements $x,y \in X$ *and for all* $\lambda \in \Lambda$*, where* $a_\lambda, b_\lambda \in \Gamma$*. If for a nondecreasing sequence* $\{x_n\}$ *with* $x_n \preceq y_n$ *for all* n *and* $y_n \to u$ *implies that* $x_n \preceq u$ *and either*

(a) $\{f,S\}$ *are compatible,* f *or* S *is continuous and* $\{f,T\}$ *are weakly compatible or*

(b) $\{f,T\}$ *are compatible,* f *or* T *is continuous and* $\{f,S\}$ *are weakly compatible*

then f,S *and* T *have a common fixed point. Moreover, the set of common fixed points of* f,g,S *and* T *is well ordered if and only if* f,S *and* T *have one and only one common fixed point.*

Corollary 3.3. *Let* $(X, \mathcal{F}, \preceq)$ *be an ordered complete gauge space and* $f,\ g,T : X \to X$ *be mappings,* (T,f) *and* (T,g) *be partially weakly increasing with* $f(X) \subseteq T(X)$, $g(X) \subseteq T(X)$ *and dominating map* f *and* g *are weak annihilators of* T*. Suppose that*

$$\int_0^{d_\lambda(fx,gy)} a_\lambda(t)dt \leq \int_0^{M(x,y)} a_\lambda(t)dt - \int_0^{M(x,y)} b_\lambda(t)dt,$$

where

$$M(x,y) = \max \left\{ \begin{array}{c} d_\lambda(Tx,Ty), d_\lambda(fx,Tx), d_\lambda(gy,Ty), \\ \frac{d_\lambda(Tx,gy)+d_\lambda(fx,Ty)}{2} \end{array} \right\},$$

for every comparable elements $x,y \in X$ *and for all* $\lambda \in \Lambda$*, where* $a_\lambda, b_\lambda \in \Gamma$*. If for a nondecreasing sequence* $\{x_n\}$ *with* $x_n \preceq y_n$ *for all* n *and* $y_n \to u$ *implies that* $x_n \preceq u$ *and either*

(a) $\{f, T\}$ *are compatible,* f *or* T *is continuous and* $\{g, T\}$ *are weakly compatible or*

(b) $\{g, T\}$ *are compatible,* g *or* T *is continuous and* $\{f, T\}$ *are weakly compatible,*

then f, g *and* T *have a common fixed point. Moreover, the set of common fixed points of* f, g *and* T *is well ordered if and only if* f, g *and* T *have one and only one common fixed point.*

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