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THE DENJOY-WOLFF ITERATION PROPERTY IN THE HILBERT BALL

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Dedicated to Professor Sompong Dhompongsa on his 65th birthday

Abstract. We first present new conditions which are equivalent to the Denjoy-Wolff iteration property in the Hilbert ball and then use them to find new classes of mappings with this property.

1. Introduction

Let B_H denote the Hilbert ball, that is, the open unit ball of a complex Hilbert space H, and let k_{B_H} denote the Kobayashi distance in B_H. This paper concerns the asymptotic behavior of the iterates of holomorphic self-mappings and, more generally, of k_{B_H}-nonexpansive self-mappings of the Hilbert ball B_H in the spirit of the celebrated Denjoy-Wolff theorem, which we now recall.

Theorem 1.1 ([15, 58, 59]; see also [11, 55, 60] and [57]). *Let Δ be the open unit disc in the complex plane $\mathbb{C}$. If a holomorphic function $F : \Delta \to \Delta$ does not have a fixed point, then there is a unique point ξ on the boundary $\partial\Delta$ of Δ such that the sequence of iterates $\{F^n\}$ of F converges pointwise to ξ, uniformly on compact subsets of Δ.*

The most general result of this type in the case of $\mathbb{C}^k$ is due to the first author [6].

Before stating it, we recall that a bounded and convex domain D in a complex Banach space $(X, \|\cdot\|)$ is said to be *strictly convex* if for each $x, y \in \overline{D}$, the open segment

$$(x, y) = \{z \in X : z = sx + (1 - s)\, y \quad \text{for some } \ 0 < s < 1\}$$

lies in D ([16]).

Theorem 1.2 ([6]; see also [7, 10] and [3]). *If D is a bounded and strictly convex domain in $\mathbb{C}^k$ and $F : D \to D$ is holomorphic and fixed point free, then there exists a point $\xi \in \partial D$ such that the sequence $\{F^n\}$ of iterates of F converges in the compact-open topology to the constant mapping taking the value ξ.*

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But in the case of an *infinite dimensional* complex Hilbert space, the situation changes radically, that is, the Denjoy-Wolff theorem no longer holds. More precisely, in a complex infinite dimensional Hilbert space H, this convergence result fails even for biholomorphic self-mappings of the Hilbert ball B_H ([56]; see also [19]). Stachura's example shows that in order to obtain a generalization of the Denjoy-Wolff theorem, we not only need additional properties of the boundary of the domain D, but we also have to impose some restrictions on the holomorphic self-mapping $F : D \to D$ itself. Therefore the following notions were introduced in [39] and [51, page 224]. Let B_H be the open unit ball in an infinite dimensional complex Hilbert space H. We say that a self-mapping F of B_H has the *Denjoy-Wolff iteration property* (DWIP) if F has no fixed point in B_H and for each $x \in B_H$, the sequence of iterates $\{F^n(x)\}$ strongly converges to a unique point ξ on the boundary ∂B_H of B_H. We will also say that a class $\mathcal{G}$ of self-mappings of B_H has the Denjoy-Wolff iteration property if whenever $F \in \mathcal{G}$ has no fixed point in B_H, then for each $x \in B_H$, the sequence of iterates $\{F^n(x)\}$ strongly converges to a unique point ξ on the boundary ∂B_H of B_H. Next, a self-mapping F of B_H has the compact Denjoy-Wolff iteration property (cDWIP) if F has no fixed point in B_H and the sequence of iterates $\{F^n\}$ converges in the compact-open topology to a unique point ξ on the boundary ∂B_H of B_H. A class $\mathcal{G}$ of self-mappings of B_H has the compact-open Denjoy-Wolff iteration property if each $F \in \mathcal{G}$ has the cDWIP. Finally, if a self-mapping F of B_H has no fixed point in B_H and the sequence of iterates $\{F^n\}$ converges uniformly on each closed ball $B(0, r) \subset B_H$, $0 < r < 1$, to a unique point ξ on the boundary ∂B_H of B_H, then we say that F has the strong Denjoy-Wolff iteration property (sDWIP). A class $\mathcal{G}$ of self-mappings of B_H has the strong Denjoy-Wolff iteration property if each $F \in \mathcal{G}$ has the sDWIP.

The following classes of self-mappings of B_H are known to have the Denjoy-Wolff iteration property:

1) ([30]; see also [7–9], [14, 29, 34] and [38]) the class $\mathcal{G}_1$ consisting of mappings which are condensing with respect to the Kuratowski measure of noncompactness ([43]; see also [4, 44] and [5]);

2) ([21, 22, 48] and [49]) the class $\mathcal{G}_2$ of firmly k_{B_H}-nonexpansive mappings of the first kind;

3) ([21, 22, 48] and [49]) the class $\mathcal{G}_3$ of firmly k_{B_H}-nonexpansive mappings of the second kind;

4) ([47]) the class $\mathcal{G}_4$ consisting of the averaged mappings of the first kind, that is, $F = (1 - c)I \oplus cT$, where T is k_{B_H}-nonexpansive and $c \in (0, 1)$ (see Theorem 2.2 below for the definition of the operation $\oplus$);

5) ([47]) the class $\mathcal{G}_5$ consisting of the averaged mappings of the second kind, that is, $F = (1 - c)I + cT$, where T is k_{B_H}-nonexpansive and $c \in (0, 1)$;

6) ([25, 37] and [41]) the class $\mathcal{G}_6$ consisting of mappings F of $\overline{B_H}$ onto $\overline{B_H}$ which are k_{B_H}-isometries in B_H, have exactly two fixed points in $\overline{B_H}$ and these fixed points lie on the boundary ∂B_H.

In this paper we first present new conditions which are equivalent to the Denjoy-Wolff iteration property in the Hilbert ball B_H and then use them to find new classes of mappings with this property (see Section 3 below).

2. Preliminaries

We use the following concepts and notations. Let D_1 and D_2 be domains in the complex Banach spaces X_1 and X_2, respectively. The set of all holomorphic mappings from D_1 into D_2 is denoted by $H(D_1, D_2)$. A mapping $f \in H(D_1, D_2)$ is said to be biholomorphic if $f(D_1) = D_2$, f is one-to one, and $f^{-1} \in H(D_2, D_1)$. If such a mapping exists, then we say that D_1 is biholomorphically equivalent to D_2.

The Kobayashi distance ([31, 32] and [33]; see also [2, 21, 26] and [27]) plays an important role in the fixed point theory of holomorphic mappings. So first we recall its definition and a few of its important properties, which will be used in the proof of our theorems.

Let Δ be the open unit disc in the complex plane $\mathbb{C}$. The Poincaré distance $k_\Delta = \omega$ on Δ is given by

$$k_\Delta(z, w) = \omega(z, w) := \arg\tanh\left|\frac{z - w}{1 - z\overline{w}}\right|$$
$$= \arg\tanh\left(1 - \sigma(z, w)\right)^{\frac{1}{2}},$$

where

$$\sigma(z, w) := \frac{\left(1 - |z|^2\right)\left(1 - |w|^2\right)}{|1 - z\overline{w}|^2}, \qquad z, w \in \Delta$$

([22, 23] and [51]).

Next, we recall the Lempert function.

Definition 2.1 ([45])**.** The Lempert $\delta : D \to [0, \infty)$ is defined by the following formula:

$$\delta_D(x, y) := \inf\{\omega(0, \lambda) : \lambda \in [0, 1) \text{ and there exists } f \in H(\Delta, D)$$
$$\text{so that } f(0) = x \text{ and } f(\lambda) = y\},$$

where $x, y \in D$.

Definition 2.2 ([31, 32] and [33])**.** Let D be a domain in a Banach space X. The Kobayashi pseudodistance k_D in D is defined by

$$k_D(x, y) := \inf\left\{\sum\nolimits_{j=1}^{n} \delta_D(x_j, x_{j+1}) : n \in \mathbb{N}, \{x = x_1, ..., x_{n+1} = y\} \subset D\right\}.$$

Now we recall connections between holomorphic mappings and the Kobayashi pseudodistances. Namely, if D_1 and D_2 are domains in the complex Banach spaces X_1 and X_2, respectively, then each holomorphic mapping $f : D_1 \to D_2$ is nonexpansive, that is,

$$k_{D_2}(f(x), f(y)) \le k_{D_1}(x, y)$$

for all $x, y \in D_1$ ([31, 32] and [33]; see also [22, 24, 38] and [51]).

Observe that if D_1 is biholomorphically equivalent to D_2 by the biholomorphic function $f : D_1 \to D_2$, then

$$k_{D_2}(f(x), f(y)) = k_{D_1}(x, y)$$

for all $x, y \in D_1$.

Next, if D is a domain in a complex Banach space X and a mapping $f : D \to D$ satisfies

$$k_D(f(x), f(y)) \leq k_D(x, y)$$

for all $x, y \in D$, then f is called k_D-nonexpansive ([20]). Directly from this definition we get that if D_1 is biholomorphically equivalent to D_2, by the biholomorphic mapping $f : D_1 \to D_2$, then a mapping $g : D_1 \to D_1$ is k_{D_1}-nonexpansive if and only if the mapping $h := f \circ g \circ f^{-1} : D_2 \to D_2$ is k_{D_2}-nonexpansive.

If the Kobayashi pseudodistance k_D is a metric in the topological sense, then it is called the *Kobayashi distance*. For example, if D is a bounded and convex domain, then k_D is the Kobayashi distance which coincides with the Lempert function δ_D ([17] and [45]).

In general, there are no explicit formulae for the Kobayashi pseudodistance k_D of domains D. However, in the case of the Hilbert ball B_H, we do have the following explicit formula for the Kobayashi distance k_{B_H}. Namely,

$$k_{B_H}(x, y) = \arg\tanh\left(1 - \sigma(x, y)\right)^{\frac{1}{2}},$$

where $x, y \in B_H$ and

$$\sigma(x, y) = \frac{\left(1 - \|x\|^2\right)\left(1 - \|y\|^2\right)}{|1 - (x, y)|^2},$$

where $(\cdot, \cdot)$ is the inner product in the Hilbert space H ([22] and [23]).

Observe that if Δ is the open unit disc in the complex plane $\mathbb{C}$, then the Kobayashi distance k_Δ is simply the Poincaré distance on Δ.

Now we recall the following very important property of the Kobayashi distance k_{B_H}.

Lemma 2.3 ([22] and [24])**.** *The Kobayashi distance k_{B_H} is locally equivalent to the norm $\|\cdot\|$ in the Hilbert space H, that is, the following inequalities are valid:*

(i)

$$\arg\tanh\left(\frac{\|x - y\|}{2}\right) \leq k_{B_H}(x, y)$$

for all $x, y \in B_H$;

(ii)

$$k_{B_H}(x, y) \leq \arg\tanh\left(\frac{\|x - y\|}{\operatorname{dist}_{\|\cdot\|}(x, \partial B_H)}\right)$$

whenever $\|x - y\| < \operatorname{dist}_{\|\cdot\|}(x, \partial B_H)$, where

$$\operatorname{dist}_{\|\cdot\|}(x, \partial B_H) := \inf\{\|x - y\| : y \in \partial B_H\}.$$

Next, we present further properties of the distance k_{B_H}.

Theorem 2.4. *The metric space (B_H, k_{B_H}) has the following properties:*

(i) ([23] and [22]) *For each pair of distinct points $x, y \in B_H$, there exist a unique geodesic line passing through them and a unique geodesic segment $[x, y]$ joining them. For each $0 \leq t \leq 1$, there is a unique point $z = (1 - t)\, x \oplus ty$ satisfying $k_{B_H}(x, z) = tk_{B_H}(x, y)$ and $k_{B_H}(z, y) = (1 - t)\, k_{B_H}(x, y)$.*

(ii) ([22]) *If* x, y, w *and* z *are in* B_H *and* $0 \le t \le 1$, *then*

$$k_{B_H}((1-t)\,x \oplus tw, (1-t)\,x \oplus tz) \le tk_{B_H}(w,z)$$

and

$$k_{B_H}((1-t)\,x \oplus tw, (1-t)\,y \oplus tz) \le (1-t)\,k_{B_H}(x,y) + tk_{B_H}(w,z).$$

Strict inequalities occur if $0 < t < 1$ *and the relevant points do not lie on the same geodesic.*

(iii) ([23] and [22]) *The metric space* (B_H, k_{B_H}) *has the Opial property with respect to the weak topology in* H *([46]), that is, if a* k_{B_H}*-bounded sequence* $\{x_n\}$ *in* B_H *tends weakly to* $x \in B_H$, *then*

$$\limsup_n k_{B_H}(x_n, x) < \limsup_n k_{B_H}(x_n, y)$$

for each $y \in B_H \setminus \{x\}$.

(iv) ([28] and [36]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences in* B_H *and let* $\{x_n\}$ *converge strongly to* $\xi \in \partial B_H$. *If*

$$\sup\{k_D\,(x_n, y_n) : n \in \mathbb{N}\} = c < \infty,$$

then $\{y_n\}$ *is also strongly convergent to* ξ.

Before reviewing a few results from the fixed point theory of holomorphic self-mappings of B_H (or, more generally, k_{B_H}-nonexpansive self-mappings of B_H), we recall the following notion. A mapping $T : B_H \to B_H$ is said to map B_H *strictly inside* B_H if $\sup_{x \in B_H} \|T(x)\| < 1$. Now we are ready to present the Earle-Hamilton theorem for B_H.

Theorem 2.5 ([18] (see also [52] and [53]))**.** *If a holomorphic* $T : B_H \to B_H$ *maps* B_H *strictly inside itself, then there exists a number* $0 \le s < 1$ *such that*

$$k_{B_H}(T(x), T(y)) \le sk_{B_H}(x,y)$$

for all $x, y \in B_H$, *and therefore* T *has a unique fixed point. Moreover, for each* $x \in B_H$, *the sequence of its iterates* $\{T^n(x)\}$ *converges to this point.*

Using Theorems 2.4 and 2.5 and the explicit formula for the Kobayashi distance k_{B_H}, we get the following fixed point theorem for the Hilbert ball.

Theorem 2.6 ([22,23,40] and [35])**.** *Let* $T : B_H \to B_H$ *be a holomorphic mapping or, more generally, a* k_{B_H}*-nonexpansive mapping. Then the following statements are equivalent:*

(i) T *has a fixed point.*

(ii) *For each point* $x \in B_H$, *the sequence of its iterates* $\{T^n\,(x)\}$ *lies strictly inside* B_H *(this means that* $\{T^n\,(x)\}$ *is* k_{B_H}*-bounded).*

(iii) *There exists a ball* $B\,(x, r)$ *in* (B_H, k_{B_H}) *which is* T*-invariant.*

(iv) *There exists a nonempty,* k_{B_H}*-bounded,* k_{B_H}*-closed and* k_{B_H}*-convex subset of* B_H *which is* T*-invariant.*

(v) *There exists a nonempty,* k_{B_H}*-bounded,* k_{B_H}*-closed and convex subset of* B_H *which is* T*-invariant.*

(vi) *There exists a* k_{B_H}*-bounded sequence* $\{x_n\}$ *in* B_H *with* $x_n - T(x_n) \to 0$.

When we study the behavior of iterates of k_{B_H}-nonexpansive mappings, then the following theorem is very useful.

Theorem 2.7 ([23] and [22]; see also [1,2,50,51] and [60]). *If a k_{B_H}-nonexpansive mapping $T : B_H \to B_H$ is fixed point free, then there exists a unique point ξ_T of norm one such that all the "ellipsoids"*

$$E(\xi_T, R) := \left\{ x \in B_H : \frac{|1-(x,\xi_T)|^2}{1-\|x\|^2} < R \right\}, \qquad R > 0,$$

are invariant under T, $\overline{E(\xi_T, R)} \cap \partial B_H = \{\xi_T\}$ and for every $x \in B_H$, there exists $R > 0$ such that $x \in E(\xi_T, R)$. Moreover, all the approximating curves defined by

$$w_a(t) := (1-t)\, a \oplus tT(w_a(t))$$

and

$$z_a(t) := (1-t)\, a + tT(z_a(t)),$$

where $t \in [0,1)$ and $a \in B_H$, converge strongly to ξ_T as $t \to 1^-$.

Directly from Theorems 2.4–2.7 and the explicit formula for k_{B_H} we get the following corollary.

Corollary 2.8 ([47]). *Let B_H be the Hilbert ball and let $T : B_H \to B_H$ be k_{B_H}-nonexpansive. If T is fixed point free and $\lim_n \|T^n(\tilde{x})\| = 1$ for some $\tilde{x} \in B_H$, then T has the DWIP.*

At this point we need the notions of total boundedness and finite total boundedness of a metric space. Recall that a metric space is said to be totally bounded if for each $\varepsilon > 0$, it can be decomposed into a finite number of sets of diameter $< \varepsilon$ ([43]; see also [44]). We also say that a metric space is finitely totally bounded if each nonempty and bounded subset of X is totally bounded. Now we are able to recall Całka's theorem regarding the behavior of the sequence of iterates of a nonexpansive mapping on a finitely totally bounded metric space X.

Theorem 2.9 ([12]). *Let f be a nonexpansive mapping of a finitely totally bounded metric space X into itself. If for some $x_0 \in X$, the sequence $\{f^n(x_0)\}$ contains a bounded subsequence, then for each $x \in X$, the sequence $\{f^n(x)\}$ is bounded.*

.

3. Main result

In this section we present the main result of our paper and some of its applications. First we observe the following fact.

Theorem 3.1. *Let B_H be the Hilbert ball and let $T : B_H \to B_H$ be k_{B_H}-nonexpansive. Then the following statements are equivalent:*

1) *T has the DWIP.*
2) *T has the cDWIP.*
3) *T has the sDWIP.*

Proof. The implications 3) $\Rightarrow$ 2) and 2) $\Rightarrow$ 1) are obvious. To get the implication 1) $\Rightarrow$ 3), it is sufficient to apply Theorem 2.4 *(iv)* and the inequality

$$k_{B_H}(T^n(x), T^n(y)) \leq k_{B_H}(x, y),$$

which is valid for all $n \in \mathbb{N}$ and all $x, y \in B_H$. □

Now we are ready to state and prove the main theorem of our paper.

Theorem 3.2. *Let B_H be the Hilbert ball and let $T : B_H \to B_H$ be k_{B_H}-nonexpansive. The following statements are equivalent:*

1) *T has the DWIP.*
2) *There exists a point $\tilde{x} \in B_H$ such that $\lim_n \|T^n(\tilde{x})\| = 1$.*
3) *There exists a point $\tilde{x} \in B_H$ such that the sequence of its iterates $\{T^n(\tilde{x})\}$ is relatively compact in the Hilbert space H and $\limsup_n \|T^n(\tilde{x})\| = 1$.*
4) *T has no fixed point and there exists a point $\tilde{x} \in B_H$ such that the sequence of its iterates $\{T^n(\tilde{x})\}$ is relatively compact in the Hilbert space H.*

Proof. The implications 1) $\Rightarrow$ 2) and 1) $\Rightarrow$ 3) are obvious. Next, it follows from Theorem 2.6 *(ii)* and Corollary 2.8 that 2) $\Rightarrow$ 1) and by Theorem 2.9 we get 3) $\Rightarrow$ 2). Finally, using Theorem 2.6 *(ii)*, we obtain 3) $\Leftrightarrow$ 4). □

Before presenting a few applications of this theorem, we recall that the Hilbert ball B_H is biholomorfically equivalent to the Siegel upper half-space Ω ([13, 22, 54] and [56]). Indeed, let B_H be the open unit ball in the complex infinite-dimensional Hilbert space H and let e be a vector in H of norm 1. Then the Hilbert space H can be written as the orthogonal direct sum decomposition $\mathbb{C} \times H^{\perp}$, where $H^{\perp}$ is the subspace of H orthogonal to e with codimension 1, that is,

$$H \ni w = \lambda e + w' = \left(\lambda, w'\right) \in \mathbb{C} \times H^{\perp}.$$

The Hilbert ball B_H is biholomorphically equivalent to the domain

$$\Omega := \left\{ w = \left(\lambda, w'\right) \in \mathbb{C} \times H^{\perp} : \operatorname{Im} \lambda > \left\|w'\right\|^2 \right\}$$

by the Cayley transform $\mathcal{C} : B_H \to \Omega$ given by

$$B_H \ni z = (\xi, z') \mapsto \mathcal{C}(z) = \mathcal{C}\left(\xi, z'\right) = \left(i\frac{1 + (z, e)}{1 - (z, e)}, \frac{i\left(z - (z, e)\, e\right)}{1 - (z, e)} \right)$$
$$= \left(i\frac{1 + \xi}{1 - \xi}, \frac{iz'}{1 - \xi} \right) = \left(\lambda, w'\right) = w \in \Omega.$$

As mentioned above, the mapping $\mathcal{C}$ is invertible and

$$z = (\xi, z') = \mathcal{C}^{-1}(w) = \mathcal{C}^{-1}(\lambda, w') = \frac{2w}{\lambda + i} - e = \Big(\frac{\lambda - i}{\lambda + i}, \frac{2w'}{\lambda + i} \Big).$$

Observe that the boundary of Ω in H is given by

$$\partial\Omega = \left\{ w = \left(\lambda, w'\right) \in \mathbb{C} \times H^{\perp} : \operatorname{Im} \lambda = \left\|w'\right\|^2 \right\}$$

and therefore $\overline{\Omega} = \Omega \cup \partial_H \Omega$ is the closure of Ω in H. If we add ∞ to H and introduce a basis for the open neighborhoods of ∞ in $H \cup \{\infty\}$ by $\{x \in H : \|x\| > \varepsilon\}$, where

$\varepsilon > 0$, then the closure $\overline{\Omega}^{\infty}$ of Ω in $H \cup \{\infty\}$ is equal to $\overline{\Omega} \cup \{\infty\}$. Next, we extend the Cayley transform $\mathcal{C}$ to the closure of B_H by setting

$$\overline{B_H} \setminus \{e\} \ni z = (\xi, z') \to \mathcal{C}(z) = \mathcal{C}(\xi, z') = \left(i\frac{1+(z,e)}{1-(z,e)}, \frac{i(z-(z,e)e)}{1-(z,e)} \right)$$
$$= \left(i\frac{1+\xi}{1-\xi}, \frac{iz'}{1-\xi} \right) = (\lambda, w') = w \in \overline{\Omega}$$

and

$$\mathcal{C}(e) = \infty.$$

Then this extended Cayley transform $\mathcal{C} : \overline{B_H} \to \overline{\Omega}^{\infty}$ is also invertible and $\mathcal{C}^{-1} : \overline{\Omega}^{\infty} \to \overline{B_H}$ is given by the following formula:

$$\overline{\Omega} \ni z = (\xi, z') = \mathcal{C}^{-1}(w) = \mathcal{C}^{-1}(\lambda, w') = \frac{2w}{\lambda + i} - e = \left(\frac{\lambda - i}{\lambda + i}, \frac{2w'}{\lambda + i} \right) \in \overline{B_H} \setminus \{e\}$$

and

$$\mathcal{C}^{-1}(\infty) = e.$$

Both mappings $\mathcal{C}$ and $\mathcal{C}^{-1}$ are continuous in $H \cup \{\infty\}$. Observe that if Y is a finite-dimensional subspace of H with $e \in Y$, then $B_Y := B_H \cap Y$ is biholomorphically equivalent to $\Omega_Y := \Omega \cap Y$ by the Cayley transform $\mathcal{C}$ restricted to B_Y, and $\overline{B_Y}^H$ is isomorphic to $\overline{\Omega_Y}^{\infty} = \overline{\Omega_Y} \cup \{\infty\}$ by the Cayley transform $\mathcal{C}$.

Next, if a set $A \subset \Omega$ is relatively compact in H, then its image $\mathcal{C}^{-1}(A) \subset B_H$ is also relatively compact in H.

Now we are ready to formulate the following Denjoy-Wolff theorem for holomorphic self-mappings of Ω or, more generally, k_Ω-nonexpansive self-mappings of Ω. This theorem is a direct consequence of Theorem 3.2 and the above remarks regarding the Cayley transform $\mathcal{C}$ and its extension.

Theorem 3.3. *Let $f : \Omega \to \Omega$ be k_Ω-nonexpansive. The following statements are equivalent:*

I) *The sequence of iterates $\{f^n\}$ converges pointwise to a point $\tilde{\xi}$ on the boundary $\partial_\infty \Omega = \partial\Omega \cup \{\infty\}$ of Ω in $\overline{\Omega}^{\infty}$.*

II) *For some $\tilde{w} \in \Omega$, the sequence of its iterates $\{f^n(\tilde{w})\}$ converges to a point $\tilde{\xi}$ on the boundary $\partial_\infty \Omega$ of Ω.*

III) *There exists $\tilde{w} \in \Omega$ such that the sequence of its iterates $\{f^n(\tilde{w})\}$ is either relatively compact in the Hilbert space H and*

$$\liminf_n \operatorname{dist}(\{f^n(\tilde{w})\}, \partial_H \Omega) = \inf\{\|f^n(\tilde{w}) - w\| : n \in \mathbb{N}, w \in \partial_H \Omega\} = 0,$$

or $\lim_n \|f^n(\tilde{w})\| = +\infty$.

IV) *Either f has no fixed point and there exists $\tilde{w} \in \Omega$ such that the sequence of its iterates $\{f^n(\tilde{w})\}$ is relatively compact in the Hilbert space H or there exists $\tilde{w} \in \Omega$ such that $\lim_n \|f^n(\tilde{w})\| = +\infty$.*

As a direct consequence of the above theorem we get the following corollary.

Corollary 3.4. *If $f : \Omega \to \Omega$ is fixed point free and there exists a finite-dimensional subspace Y of H such that the set $\Omega_Y = \Omega \cap Y$ is f-invariant, then for each $w \in \Omega$, the sequence of its iterates $\{f^n(w)\}$ converges to a unique point $\tilde{\xi}$ on the boundary $\partial_\infty \Omega$ of Ω.*

Hence if $f:\Omega\to\Omega$ satisfies one of the condition given in Theorem 3.3, then we see that the mapping $T=\mathcal{C}^{-1}\circ f\circ\mathcal{C}:B_H\to B_H$ has the DWIP.

Next, observe that sometimes it is easier to construct a function $f:\Omega\to\Omega$ such that the mapping $T=\mathcal{C}^{-1}\circ f\circ\mathcal{C}:B_H\to B_H$ has the claimed properties, than it is to seek such a function directly in B_H.

Now we are ready to present a few examples of mappings $f:\Omega\to\Omega$ which satisfy the conditions given either in Theorem 3.3 or in Corollary 3.4. Among them there are functions f such that the mappings $T=\mathcal{C}^{-1}\circ f\circ\mathcal{C}$ are not elements of any of the classes $\mathcal{G}_j$, $j=1,...,6$, mentioned in the Introduction.

Example 3.5. The mapping $f:\Omega\to\Omega$ is a translation defined by

$$f(w)=f(\lambda,w')=(\lambda+a,w')$$

for $w\in\Omega$, where a is a nonzero real number. The mapping f has no fixed points and $f(\Omega_{\mathrm{lin}\{e\}})$ is a subset of $\Omega_{\mathrm{lin}\{e\}}$.

As a matter of fact, we can present more general mappings of the above type.

Example 3.6. Let $\varphi:H^{\perp}\to H^{\perp}$ be holomorphic with $\|\varphi(w')\|\le\|w'\|$ for each $w'\in H^{\perp}$. The mapping $f:\Omega\to\Omega$ is of the form

$$f(w)=f(\lambda,w')=(\lambda+a,\varphi(w')),\quad w\in\Omega,$$

where a is a nonzero real number. Assume that there exists a finite-dimensional subspace Y of $H^{\perp}$ such that the set Ω_Y is φ-invariant. It is obvious that f is fixed point free.

Example 3.7. Let $\varphi:H^{\perp}\to H^{\perp}$ be holomorphic with $\|\varphi(w')\|\le\|w'\|$ for each $w'\in H^{\perp}$. The mapping $f:\Omega\to\Omega$ is of the form

$$f(w)=f(\lambda,w')=(\lambda+ia,\varphi(w')),\quad w\in\Omega,$$

where a is a positive real number and there exists a finite-dimensional subspace Y of $H^{\perp}$ such that the set Ω_Y is φ-invariant. It is obvious that f is fixed point free.

Example 3.8. The mapping $f:\Omega\to\Omega$ is defined by

$$f(w)=f(\lambda,w')=(\lambda+i(\|a\|^2+2(w',a)),w'+a),\quad w\in\Omega,$$

where a is a nonzero element of $H^{\perp}$. It is not difficult to note that the sequence of iterates $\{\varphi\}^n(i,0)$ lies in a two dimensional subspace of H and f has no fixed points.

Example 3.9. Let $\varphi:H^{\perp}\to H^{\perp}$ be holomorphic with $\|\varphi(w')\|\le\|w'\|$ for each $w'\in H^{\perp}$. Assume that there exists $0\ne a\in H^{\perp}\setminus\varphi(H^{\perp})$. Then the mapping $f:\Omega\to\Omega$ given by

$$f(w)=f(\lambda,w')=(\lambda+i\|a\|^2,\varphi(w')+a),\quad w\in\Omega,$$

has no fixed points. Let $\Phi(w')=\varphi(w')+a$. Now it is sufficient to assume that for some $w'\in H^{\perp}$, the sequence of its iterates $\{\Phi\}^n(w')$ lies in a finite-dimensional subspace of $H^{\perp}$ to get a mapping $T:=\mathcal{C}^{-1}\circ f\circ\mathcal{C}:B_H\to B_H$ with the DWIP.

Example 3.10. Consider the "nonisotropic" dilation $D_t : \Omega \to \Omega$ introduced in the following way:

$$D_t(w) = D_t(\lambda, w') = (t^2\lambda, tw'), \quad w \in \Omega,$$

where $0 < t < \infty$ is a fixed constant ([54]). This holomorphic automorphism D_t is the analog of the Möbius transformation $M_{se} : B_H \to B_H$ with $-1 < s = \frac{t^2-1}{t^2+1} < 1$, where

$$M_{se}(z) = M_{se}(\xi, z') = \frac{1}{1+s\xi}(\xi + s, \sqrt{1-s^2}z')$$

for $z = (\xi, z') \in B_H$. The mapping D_t can be continuously extended to $\overline{\Omega}^{\infty}$ and for $t \neq 1$ this extended mapping D_t fixes only 0 and ∞. We also have $D_t(\Omega_{\mathrm{lin}\{e\}}) = \Omega_{\mathrm{lin}\{e\}}$.

The last example yields a mapping $T := \mathcal{C}^{-1} \circ f \circ \mathcal{C} : \overline{B_H} \to \overline{B_H}$ ($t > 0$ and $t \neq 1$), which is a particular case of the following one.

Example 3.11. Let T be a mapping of $\overline{B_H}$ onto $\overline{B_H}$ which is a k_{B_H}-isometry in B_H, has exactly two fixed points in $\overline{B_H}$, and these fixed points lie on the boundary ∂B_H of B_H. Then there exists a 2-dimensional T-invariant affine set in B_H and therefore the mapping T has the DWIP by Theorem 3.2 (see also example 6 in the Introduction).

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