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BEST PROXIMITY POINT THEOREMS FOR A NEW CLASS OF α-ψ-PROXIMAL CONTRACTIVE MAPPINGS

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Abstract. Recently, Jleli and Samet [3] introduced a new concept of α-ψ-contractive mappings and they studied the existence and uniqueness of best proximity points. In this paper, we follow their work by relaxing some assumptions and considering a new family of the mappings ψ.

1. Introduction and preliminaries

Fixed point theory focusses on the strategies for solving nonlinear equations of the kind $Tx = x$ in which T is a self mapping defined on a subset of a metric space, a normed linear space, a topological vector space or some pertinent framework. But, when T is not a self mapping, it is plausible that $Tx = x$ has no solution. Subsequently, one targets to determine an element x that is in some sense closest to Tx. In fact, best approximation theorems and best proximity point theorems are suitable to be explored in this direction. A well known best approximation theorem, due to Fan [2], ascertains that if K is a nonempty compact convex subset of a Hausdorff locally convex topological vector space E and $T : K \to E$ is a continuous non-self mapping, then there exists an element x in such a way that $d(x, Tx) = d(Tx, K)$. Several authors, including Prolla [5], Reich [7] and Sehgal and Singh [11, 12], have accomplished extensions of this theorem in various directions. Moreover, a result that unifies all such best approximation theorems has been obtained by Vetrivel et al. [13]. Despite the fact that the best approximation theorems are befitting for furnishing an approximate solution to the equation $Tx = x$, such results may not afford an approximate solution that is optimal. On the other hand, best proximity point theorems offer an approximate solution that is optimal. Indeed, a best proximity point theorem details sufficient conditions for the existence of an element x such that the error $d(x, Tx)$ is minimum. A best proximity point theorem is fundamentally concerned with the global minimization of the real valued function $x \to d(x, Tx)$ that is an indicator of the error involved for an approximate solution of the equation $Tx = x$ (see, for example, [1]).

In 2012, Samet et al. [10] introduced the concepts of α-ψ-contractive and admissible mappings and established various fixed point theorems for such mappings in complete metric spaces. Afterwards Karapinar and Samet [4] generalized these notions to obtain fixed point results. The aim of this paper is to modify further

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the notions of α-ψ-contractive and α-admissible mappings and establish fixed point theorems for such mappings in complete metric spaces.

Very recently, Jleli and Samet [3] introduced a new concept of α-ψ-contractive mappings and using the results given in [10] they studied the existence and uniqueness of best proximity points. In this paper we follow their work by relaxing some assumptions and considering a new family of the mappings ψ.

The rest of this section for the sake of convenience, we recall some notations and definitions that will be used in the sequel.

Let A and B be nonempty subsets of a metric space (X, d). In this paper, we use the following notations:

$$d(A, B) = \inf\{d(a, b) : a \in A \text{ and } b \in B\},$$

$$A_0 = \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\},$$

and

$$B_0 = \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\}.$$

Definition 1.1. Let A and B be nonempty subsets of a metric space (X, d) and $T : A \to B$. An element $x^* \in A$ is said to be a best proximity point of T if $d(x^*, Tx^*) = d(A, B)$.

Raj [6] introduced the following concept.

Definition 1.2. Let A and B be nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. The pair (A, B) is said to satisfy the P-property if for all $x_1, x_2 \in A$ and $y_1, y_2 \in B$,

$$\left.\begin{array}{l} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{array}\right\} \text{ imply } d(x_1, x_2) = d(y_1, y_2).$$

Recently, Jleli and Samet [3] introduced the α-proximal admissible mappings and α-ψ-proximal contractions where $\psi \in \Psi_1$ and

$$\Psi_1 = \{\psi : \psi : [0, +\infty) \to [0, +\infty) \text{ is nondecreasing with } \sum_{n=1}^{\infty} \psi^n(t) < \infty, \; \forall t \in (0, +\infty)\}.$$

Definition 1.3. Let A and B be nonempty subsets of a metric space (X, d). Assume that $T : A \to B$ and $\alpha : A \times A \to [0, +\infty)$. We say that T is α-proximal admissible if for all $x_1, x_2, u_1, u_2 \in A$,

$$\left.\begin{array}{c} \alpha(x_1, x_2) \geq 1 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{array}\right\} \text{ imply } \alpha(u_1, u_2) \geq 1.$$

Remark 1.4. If $A = B$, then every α-proximal admissible mapping is an α-admissible mapping.

Definition 1.5. Let A and B be nonempty subsets of a metric space (X, d). Assume that $T : A \to B$, $\alpha : A\times A \to [0, +\infty)$ and $\psi \in \Psi_1$. We say that T is an α-ψ-proximal contraction if for all $x, y \in A$,

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)).$$

Jleli and Samet [3] proved the existence of the best proximity point theorems for α-ψ-proximal admissible mappings as the following.

Theorem 1.6 ([3])**.** *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty and $\psi \in \Psi_1$. Suppose that $T : A \to B$ is a mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the P-property;*
(ii) *T is an α-proximal admissible mapping;*
(iii) *T is an α-ψ-proximal contraction;*
(iv) *there exist x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B) \text{ and } \alpha(x_0, x_1) \geq 1;$$

(v) *T is continuous or if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$.*

Then there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A, B)$.

In order to assure the uniqueness of the best proximity point, [3] introduced the following definition.

Definition 1.7. Let A and B be nonempty subsets of a metric space (X, d). Suppose that $T : A \to B$ and $\alpha : A \times A \to [0, +\infty)$. We say that T is (α, d) regular if for all $x, y \in A$ with $\alpha(x, y) < 1$, there exists $z \in A_0$ such that

$$\alpha(x, z) \geq 1 \text{ and } \alpha(y, z) \geq 1.$$

Theorem 1.8 ([3])**.** *Suppose all hypotheses of Theorem 1.6 hold and T is (α, d) regular. Then T has a unique best proximity point.*

We now introduce the concepts of α-proximal admissible mappings with respect to η and (α, η, d) regularity for non-self mappings.

Definition 1.9. Let A and B be nonempty subsets of a metric space (X, d). Assume that $T : A \to B$, $\alpha : A \times A \to [0, +\infty)$ and $\eta : A \times A \to [0, +\infty)$. We say that T is α-proximal admissible with respect to η if for all $x_1, x_2, u_1, u_2 \in A$,

$$\left.\begin{array}{l} \alpha(x_1, x_2) \geq \eta(x_1, x_2) \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{array}\right\} \text{ imply } \alpha(u_1, u_2) \geq \eta(u_1, u_2).$$

Definition 1.10. Let A and B be nonempty subsets of a metric space (X, d). Assume that $T : A \to B$, $\alpha : A \times A \to [0, +\infty)$ and $\eta : A \times A \to [0, +\infty)$. We say that T is (α, η, d) regular if for all $x, y \in A$ with $\alpha(x, y) < \eta(x, y)$, there exists $z \in A_0$ such that

$$\alpha(x, z) \geq \eta(x, z) \text{ and } \alpha(y, z) \geq \eta(y, z).$$

Remark 1.11. If we suppose that $\eta(x,y)=1$ for all $x,y\in A$, then the Definition 1.9 and Definition 1.10 are reduced to Definition 1.3 and Definition 1.7, respectively.

Lemma 1.12 ([10]). *Suppose that $\psi:[0,+\infty)\to[0,+\infty)$. If ψ is nondecreasing, then for each $t\in(0,+\infty),\lim_{n\to\infty}\psi^n(t)=0$ implies $\psi(t)<t$.*

Remark 1.13. It is easily seen that if $\psi:[0,+\infty)\to[0,+\infty)$ is nondecreasing and $\psi(t)<t$ for all $t\in(0,+\infty)$, then $\psi(0)=0$.

Remark 1.14. By Lemma 1.12 for each $\psi\in\Psi_1$, we have $\psi(t)<t$ for all $t\in(0,+\infty)$ and by Remark 1.13 we obtain that $\psi(0)=0$.

Remark 1.15. Since every nondecreasing mapping is differentiable almost everywhere (see [8]), we observe that nondecreasing condition is closed to continuity and it is restrictive.

We denote with Ψ_2 the family of mappings $\psi:[0,+\infty)\to[0,+\infty)$ such that

(i) ψ is an upper semicontinuous mapping from the right;
(ii) $\psi(t)<t$ for all $t\in(0,+\infty)$;
(iii) $\psi(0)=0$.

Example 1.16. Let $\psi:[0,+\infty)\to[0,+\infty)$ be a mapping defined by

$$\psi(t)=\begin{cases}\frac{1}{3}, & t\in\mathbb{N};\\ 0, & \text{otherwise.}\end{cases}$$

We obtain that ψ is upper semicontinuous from the right, $\psi(t)<t$ for all $t\in(0,+\infty)$ and $\psi(0)=0$. Moreover ψ is not nondecreasing.

In this paper, we introduce a new class of α-ψ-proximal contractive type mappings with respect to η where $\psi\in\Psi_2$. We prove the existence of the the uniqueness best proximity point theorems for such mappings. Furthermore, we also present the applications using the our obtained results.

2. Main results

We now assure the existence of a best proximity point for a new class of α-ψ-proximal contractive type mapping with respect to η where $\psi\in\Psi_2$.

Theorem 2.1. *Let A and B be nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty and $\psi\in\Psi_2$. Suppose that $T:A\to B$ is a mapping satisfying the following conditions:*

(i) *$T(A_0)\subseteq B_0$ and (A,B) satisfies the P-property;*
(ii) *T is α-proximal admissible with respect to η;*
(iii) *if $x,y\in A$ and $\alpha(x,y)\geq\eta(x,y)$, then $d(Tx,Ty)\leq\psi(d(x,y))$;*
(iv) *there exist x_0 and x_1 in A_0 such that*
$$d(x_1,Tx_0)=d(A,B)\text{ and }\alpha(x_0,x_1)\geq\eta(x_0,x_1);$$
(v) *T is continuous or if $\{x_n\}$ is a sequence in A such that $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ for all $n\in\mathbb{N}$ and $x_n\to x\in X$ as $n\to\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x)\geq\eta(x_{n_k},x)$ for all $k\in\mathbb{N}$.*

Then there exists an element $x^*\in A_0$ such that $d(x^*,Tx^*)=d(A,B)$.

Proof. Since $T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$. Therefore

$$d(x_1, Tx_0) = d(A, B),\ d(x_2, Tx_1) = d(A, B) \text{ and } \alpha(x_0, x_1) \geq \eta(x_0, x_1).$$

Since T is α-proximal admissible with respect to η, we obtain that

$$\alpha(x_1, x_2) \geq \eta(x_1, x_2).$$

By continuing the process as above, we can construct a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = d(A, B) \text{ and } \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}), \tag{2.1}$$

for all $n \in \mathbb{N} \cup \{0\}$. Using the P-property of (A, B), we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \text{ for all } n \in \mathbb{N}. \tag{2.2}$$

Using (iii) and (2.1), this yields

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \psi(d(x_{n-1}, x_n)), \tag{2.3}$$

for all $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N} \cup \{0\}$, then by (2.1) we have x_n is a best proximity point. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\psi(t) < t$ for all $t \in (0, +\infty)$ and using (2.3), we have

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n), \tag{2.4}$$

for all $n \in \mathbb{N}$. Therefore $\{d(x_n, x_{n+1})\}$ is a nonincreasing sequence. It follows that there exists $c \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = c.$$

We will prove that $c = 0$. Suppose that $c > 0$. Since ψ is upper semicontinuous from the right and by using (2.4), we have

$$c = \limsup_{n\to\infty} d(x_n, x_{n+1}) \leq \limsup_{n\to\infty} \psi(d(x_{n-1}, x_n)) \leq \psi(c) < c,$$

which leads to a contradiction. Therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

This implies that for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that

$$d(x_{n_k}, x_{n_k+1}) < \frac{1}{2^k}.$$

We obtain that

$$\sum_{k=1}^{\infty} d(x_{n_k}, x_{n_k+1}) < \infty.$$

Therefore $\{x_{n_k}\}$ is a Cauchy sequence in A_0. Since X is complete and A is closed, we have $\{x_{n_k}\}$ converges to some $x^* \in A$. By continuity of T, we have

$$\lim_{k\to\infty} Tx_{n_k} = Tx^*.$$

Using the continuity of a metric d, we obtain that

$$d(A, B) = \lim_{n\to\infty} d(A, B) = \lim_{k\to\infty} d(x_{n_k+1}, Tx_{n_k}) = d(x^*, Tx^*).$$

On the other hand, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that

$$\alpha(x_{n_{k_j}}, x) \geq \eta(x_{n_{k_j}}, x) \text{ for all } j \in \mathbb{N}. \tag{2.5}$$

Using (2.1) and (2.5), for each $j \in \mathbb{N}$, we obtain that

$$\begin{aligned} d(Tx^*, x^*) &\leq d(Tx, Tx_{n_{k_j}}) + d(Tx_{n_{k_j}}, x_{n_{k_j}+1}) + d(x_{n_{k_j}+1}, x^*) \\ &\leq \psi(d(x_{n_{k_j}}, x)) + d(A, B) + d(x_{n_{k_j}+1}, x^*). \end{aligned}$$

Since ψ is upper semicontinuous from the right, we obtain that

$$\limsup_{j\to\infty} \psi(d(x_{n_{k_j}}, x)) \leq \psi(0) = 0.$$

Therefore $d(Tx^*, x^*) \leq d(A, B)$. From the fact that $d(A, B) \leq d(Tx^*, x^*)$, we obtain the desired result. □

Theorem 2.2. *Suppose all hypotheses of Theorem 2.1 hold. Assume that T is (α, η, d) regular. Then T has a unique best proximity point.*

Proof. Assume that x^* and y^* are two best proximity points of T. This implies that

$$d(Tx^*, x^*) = d(A, B) = d(Ty^*, y^*). \tag{2.6}$$

Since (A, B) satisfies the P-property, we obtain that

$$d(Tx^*, Ty^*) = d(x^*, y^*). \tag{2.7}$$

We prove the result in two cases.
Case I. Suppose that $\alpha(x^*, y^*) \geq \eta(x^*, y^*)$. By the assumption and (2.7), we obtain that

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq \psi(d(x^*, y^*)). \tag{2.8}$$

By the fact that $\psi(t) < t$ for all $t \in (0, +\infty)$, we have (2.8) holds when $d(x^*, y^*) = 0$ and so $x^* = y^*$.
Case II. Suppose that $\alpha(x^*, y^*) < \eta(x^*, y^*)$. Since T is (α, η, d) regular, there exists $z_0 \in A_0$ such that

$$\alpha(x^*, z_0) \geq \eta(x^*, z_0) \text{ and } \alpha(y^*, z_0) \geq \eta(y^*, z_0). \tag{2.9}$$

Since $T(A_0) \subseteq B_0$, there exists $z_1 \in A_0$ such that

$$d(z_1, Tz_0) = d(A, B). \tag{2.10}$$

Using α-proximal admissibility with respect to η of T together with (2.6),(2.9) and (2.10), we have

$$\alpha(x^*, z_1) \geq \eta(x^*, z_1).$$

By continuing the process as before, we con construct a sequence $\{z_n\}$ in A_0 such that

$$d(z_{n+1}, Tz_n) = d(A, B) \text{ and } \alpha(x^*, z_n) \geq \eta(x^*, z_n), \tag{2.11}$$

for all $n \in \mathbb{N} \cup \{0\}$. Since (A, B) satisfies the P-property and by using (2.11), it follows that

$$d(z_{n+1}, x^*) = d(Tz_n, Tx^*) \tag{2.12}$$

Using (2.11), this yields

$$d(z_{n+1}, x^*) = d(Tz_n, Tx^*) \leq \psi(d(z_n, x^*)), \tag{2.13}$$

for all $n \in \mathbb{N} \cup \{0\}$. If $z_k = x^*$ for some $k \in \mathbb{N} \cup \{0\}$, then by (2.12) we obtain that $z_n = x^*$ for all $n \geq k$. Therefore $\lim_{n\to\infty} z_n = x^*$. Assume that $z_n \neq x^*$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\psi(t) < t$ for all $t \in (0, +\infty)$ and by using (2.13), we have

$$d(z_{n+1}, x^*) \leq \psi(d(z_n, x^*)) < d(z_n, x^*),$$

for all $n \in \mathbb{N} \cup \{0\}$. Therefore $\{d(z_n, x^*)\}$ is a nonincreasing sequence and then converges to some $c \in \mathbb{R}$. We will show that $c = 0$. Suppose that $c > 0$. Since ψ is upper semicontinuous from the right, we have

$$c = \limsup_{n\to\infty} d(z_{n+1}, x^*) \leq \limsup_{n\to\infty} \psi(d(z_n, x^*)) \leq \psi(c) < c,$$

which leads to contradiction. It follows that

$$\lim_{n\to\infty} d(z_n, x^*) = 0.$$

This yields $\lim_{n\to\infty} z_n = x^*$. Similarly, by the same argument we can prove that $\lim_{n\to\infty} z_n = y^*$. Since the limit of the sequence is unique, we can conclude that $x^* = y^*$. ☐

Applying Theorem 2.1 and Theorem 2.2, we immediately obtain the following result.

Corollary 2.3. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty and $\psi \in \Psi_2$. Suppose that $T : A \to B$ is a mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the P-property;*
(ii) *T is an α-proximal admissible mapping;*
(iii) *T is an α-ψ-proximal contraction;*
(iv) *there exist x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B) \text{ and } \alpha(x_0, x_1) \geq 1;$$

(v) *T is continuous or if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$;*
(vi) *T is (α, d) regular.*

Then there exists a unique element $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A, B)$.

Letting $A = B$ in Theorem 2.1, we have the following result.

Corollary 2.4. *Let A be a nonempty closed subset of a complete metric space (X, d) and $\psi \in \Psi_2$. Suppose that $T : A \to A$ is a mapping satisfying the following conditions:*

(i) *T is α-admissible with respect to η;*
(ii) *if $x, y \in A$ and $\alpha(x, y) \geq \eta(x, y)$, then $d(Tx, Ty) \leq \psi(d(x, y))$;*
(iii) *there exists x_0 in A such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*

(iv) *T is continuous or if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq \eta(x_{n_k}, x)$ for all $k \in \mathbb{N}$.*

Then T has a fixed point.

Corollary 2.5. *Suppose all hypotheses of Theorem 2.4 hold. Assume that for all $x, y \in A$ with $\alpha(x, y) < \eta(x, y)$, there exists $z \in A$ such that*

$$\alpha(x, z) \geq \eta(x, z) \text{ and } \alpha(y, z) \geq \eta(y, z).$$

Then T has a unique fixed point.

3. Applications

Using Theorem 2.2, we obtain the standard best proximity point theorem.

Theorem 3.1. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty and $\psi \in \Psi_2$. Suppose that $T : A \to B$ is a mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the P-property;*
(ii) *for each $x, y \in A$, there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$.*

Then there exists a unique element $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A, B)$.

Proof. Let $\alpha, \eta : X \times X \to [0, +\infty)$ be mappings defined by

$$\alpha(x, y) = 1 \text{ and } \eta(x, y) = 1 \quad \text{for all } x, y \in A.$$

It follows that T is α-proximal admissible with respect to η. Suppose that $\psi : [0, +\infty) \to [0, +\infty)$ defined by $\psi(t) = kt$ for all $t \in [0, +\infty)$. This implies that ψ is upper semicontinuous from the right, $\psi(t) < t$ for all $t \in (0, +\infty)$ and $\psi(0) = 0$. Let $x \in A_0$. Since $T(A_0) \subseteq B_0$, there exists $y \in A_0$ such that $d(Tx, y) = d(A, B)$. Furthermore, we can see that all assumptions in Theorem 2.2 are now satisfied. This completes the proof. □

We next prove the existence of the best proximity points on a metric space endowed with an arbitrary binary relation. Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X. Suppose that $\mathcal{S}$ is a symmetric relation attached to $\mathcal{R}$. Therefore $\mathcal{S} = \mathcal{R} \cup \mathcal{R}^{-1}$. It follows that for all $x, y \in X$,

$$x\mathcal{S}y \text{ if and only if } x\mathcal{R}y \text{ or } y\mathcal{R}x.$$

Jleli and Samet [3] introduced the concept of proximal comparative mappings and proved the best proximity point results for such mappings.

Definition 3.2. Let A and B be nonempty subsets of a metric space (X, d). We say that a mapping $T : A \to B$ is a proximal comparative mapping if for all $x_1, x_2, u_1, u_2 \in A$,

$$\left.\begin{array}{c} x_1\mathcal{S}x_2 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{array}\right\} \text{ imply } u_1\mathcal{S}u_2.$$

Theorem 3.3. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty and $\psi \in \Psi_2$. Suppose that $\mathcal{R}$ be a binary relation over X and $T : A \to B$ is a mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the P-property;*
(ii) *T is a proximal comparative mapping;*
(iii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B) \text{ and } x_0 \mathcal{S} x_1;$$

(iv) *for all $x, y \in A$, $x\mathcal{S}y$ implies $d(Tx, Ty) \leq \psi(d(x, y))$;*
(v) *T is a continuous mapping.*

Then there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A, B)$.

Proof. Suppose that $\alpha, \eta : A \times A \to [0, +\infty)$ are mappings defined by

$$\alpha(x,y) = \begin{cases} 1, & x\mathcal{S}y; \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \eta(x,y) = \begin{cases} \frac{1}{2}, & x\mathcal{S}y; \\ 2, & \text{otherwise.} \end{cases}$$

Let $x_1, x_2, u_1, u_2 \in A$ be such that

$$\alpha(x_1, x_2) \geq \eta(x_1, x_2),\ d(u_1, Tx_1) = d(A, B) \text{ and } d(u_2, Tx_2) = d(A, B).$$

This implies that

$$x_1 \mathcal{S} x_2,\ d(u_1, Tx_1) = d(A, B) \text{ and } d(u_2, Tx_2) = d(A, B).$$

Since T is a proximal comparative mapping, we obtain that $u_1 \mathcal{S} u_2$. Therefore $\alpha(u_1, u_2) \geq \eta(u_1, u_2)$ and then T is α-proximal admissible with respect to η. Using (iii), we have

$$d(x_1, Tx_0) = d(A, B) \text{ and } \alpha(x_0, x_1) \geq \eta(x_0, x_1).$$

Assume that $x, y \in A$ and $\alpha(x, y) \geq \eta(x, y)$. It follows that $x\mathcal{S}y$. By (iv), we get that

$$d(Tx, Ty) \leq \psi(d(x, y)).$$

Hence all assumptions in Theorem 2.1 are now satisfied. Thus we obtain the desired result. □

Theorem 3.4. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty and $\psi \in \Psi_2$. Suppose that $\mathcal{R}$ be a binary relation over X and $T : A \to B$ is a mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the P-property;*
(ii) *T is a proximal comparative mapping;*
(iii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B) \text{ and } x_0 \mathcal{S} x_1;$$

(iv) *for all $x, y \in A$, $x\mathcal{S}y$ implies $d(Tx, Ty) \leq \psi(d(x, y))$;*
(v) *if $\{x_n\}$ is a sequence in A such that $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \mathcal{S} x$ for all $k \in \mathbb{N}$.*

Then there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A, B)$.

Proof. Suppose that $\alpha, \eta : X \times X \to [0, +\infty)$ defined as in Theorem 3.3. Assume that $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$. This implies that $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N}$. Using (v), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \mathcal{S} x$ for all $k \in \mathbb{N}$. Therefore $\alpha(x_{n_k}, x) \geq \eta(x_{n_k}, x)$ for all $k \in \mathbb{N}$. Hence all assumptions in Theorem 2.1 are now satisfied. Thus we obtain the desired result. □

Theorem 3.5. *Suppose all hypotheses of Theorem 3.3 (resp. Theorem 3.4) hold. Assume that for all $x, y \in A$ with $(x, y) \notin \mathcal{S}$, there exists $z \in A_0$ such that $x \mathcal{S} z$ and $y \mathcal{S} z$. Then T has a unique best proximity point.*

Proof. Suppose that $\alpha, \eta : X \times X \to [0, +\infty)$ defined as in Theorem 3.3. Let $x, y \in A$ and $\alpha(x, y) < \eta(x, y)$. This implies that $(x, y) \notin \mathcal{S}$. By assumption, there exists $z \in A_0$ such that $x \mathcal{S} z$ and $y \mathcal{S} z$. Therefore

$$\alpha(x, z) \geq \eta(x, z) \text{ and } \alpha(y, z) \geq \eta(y, z).$$

Thus T is (α, η, d) regular. Hence all assumptions in Theorem 2.2 are now satisfied. So the proof is complete. □

The concept of comparative mappings is introduced by Samet and Turinici [9]. They also assured the unique fixed point theorem for such mappings in complete metric spaces.

Definition 3.6. Let A be a nonempty subset of a metric space (X, d). We say that $T : A \to A$ is a comparative mapping if for all $x, y \in A$, $x \mathcal{S} y$ implies $Tx \, \mathcal{S} \, Ty$.

Remark 3.7. If $T : A \to A$ is a comparative mapping, then it is a proximal comparative mapping.

Corollary 3.8. *Let A be a nonempty closed subset of a complete metric space (X, d) and $\psi \in \Psi_2$. Suppose that $\mathcal{R}$ be a binary relation over X and $T : A \to A$ is a mapping satisfying the following conditions:*

(i) *T is a comparative mapping;*
(ii) *there exists $x_0 \in X$ such that $x_0 \mathcal{S} T x_0$.*
(iii) *for all $x, y \in A$, $x \mathcal{S} y$ implies $d(Tx, Ty) \leq \psi(d(x, y))$;*
(iv) *T is continuous or if $\{x_n\}$ is a sequence in A such that $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \mathcal{S} x$ for all $k \in \mathbb{N}$.*

Then T has a fixed point.

Corollary 3.9. *Suppose all hypotheses of Theorem 3.8 hold. Assume that for all $x, y \in A$ with $(x, y) \notin \mathcal{S}$, there exists $z \in A$ such that $x \mathcal{S} z$ and $y \mathcal{S} z$. Then T has a unique fixed point.*

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