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# CONVERGENCE CRITERIA OF A VISCOSITY COMMON FIXED-POINT ITERATIVE PROCESS FOR ASYMPTOTICALLY NONEXPANSIVE SEMIGROUP IN BANACH SPACES

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Abstract. Let $X$ be a real arbitrary Banach space and let $C$ be a nonempty closed convex subset of $X$. Let $\mathcal{F} = \{T(t) : t > 0\}$ be the semigroup of asymptotically nonexpansive self-mappings on $C$ such that $F \neq \emptyset$, where $F$ is the set of fixed points of $T(t)$ for all $t > 0$. Let $f : C \to C$ be a contractive mapping, $\{\alpha_n\}, \{\beta_n\}$ be real sequences in $[0,1]$ and $\{t_n\}$ be a sequence of positive real numbers. Define $\{x_n\}$ and $\{y_n\}$ to be the iterative sequences

$$\begin{aligned} y_n &= \alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)(T(t_n))^n x_n), \\ x_{n+1} &= \gamma_n f(y_n) + (1-\gamma_n)(\delta_n y_n + (1-\delta_n)(T(t_n))^n y_n), \ \ n \geq 1. \end{aligned}$$

Some strong convergence theorems of the sequence $\{x_n\}$ to an element of $F$ are established under appropriate conditions.

## 1. Introduction

The concept of asymptotically nonexpansive self-mappings which is a generalization of the class of nonexpansive self-mappings was first introduced in 1972 by Goebel and Kirk [3]. They proved that any asymptotically nonexpansive self-mapping of a nonempty closed convex bounded subset of a uniformly convex Banach space possesses a fixed point. Since then, the weak and strong convergence problems of iterative sequences (with errors) for asymptotically nonexpansive self-mappings have been studied by many authors (see, for examples, [1-2, 5-6 and 9]). Recently, in 2008, Lou, Zhang and He [8] studied the viscosity approximation fixed point for asymptotically nonexpansive self-mappings in Banach spaces. They proved the following theorems.

**Theorem 1.1.** *Let $K$ be a nonempty closed convex subset of a Banach space $X$ which has a uniformly Gâteaux differentiable norm and $T : K \to K$ an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and $f$ a contraction on $C$. Let $\{\alpha_n\}, \{\beta_n\}$ be a sequence in $(0,1)$ satisfying*

$$C1 : \lim_{n\to\infty} \alpha_n = 0; \ \ C2 : \lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0.$$

*Then the sequence $\{z_n\}$ defined by*

$$z_{n+1} = \alpha_n f(z_n) + (1-\alpha_n)T^n z_n,$$

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*converges strongly to the unique solution of the variational inequality:*

$$p \in F(T) \ \textit{such that} \ \langle (I-f)p, j(p-x^*)\rangle \le 0 \ \ \forall x^* \in F(T).$$

**Theorem 1.2.** *Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $X$ which has a uniformly Gâteaux differentiable norm and $T : K \to K$ an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and $f$ a contraction on $C$. Let $\{\alpha_n\}, \{\beta_n\}$ be a sequence in $(0,1)$ satisfying*

$$C1 : \lim_{n\to\infty} \alpha_n = 0; \ \ C2 : \sum_{n=1}^{\infty} \alpha_n = \infty \ \ C3 : \lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0.$$

*For arbitrary $x_0 \in K$, let the sequence $\{x_n\}$ be defined iteratively by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n x_n.$$

*Assume*
(i) $\alpha_n, \beta_n, \gamma_n \in [0,1], \alpha_n + \beta_n + \gamma_n = 1$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(iii) *$T$ satisfies the asymptotically regularity;* $\lim_{n\to\infty} \|T^{n+1}x_n - T^n x_n\| = 0$.
*Then the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality:*

$$p \in F(T) \ \textit{such that} \ \langle (I-f)p, j(p-x^*)\rangle \le 0 \ \ \forall x^* \in F(T).$$

Let $X$ be a Banach space and let $C$ be a nonempty closed convex subset of $X$. A one parameter nonexpansive semigroup is a family $\mathcal{F} = \{T(t) : t \ge 0\}$ of self-mappings of $C$ such that
(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(t+s)x = T(t)T(s)x$ for $t, s > 0$ and $x \in C$;
(iii) $\lim_{t\to 0^+} T(t)x = x$ for $x \in C$;
(iv) for each $t > 0$, $T(t)$ is nonexpansive, i.e.,

$$\|T(t)x - T(t)y\| \le \|x - y\|,$$

for all $x, y \in C$.

In [8], Song and Xu give some strong convergence theorems for the viscosity iteration process

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n,$$

in a real reflexive strictly convex Banach space with a uniformly *Gâteaux* differentiable norm. Here $f$ is a contractive mapping on $C$, i.e., a mapping for which there is some $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\|,$$

for all $x, y \in C$. They proved the following theorem.

**Theorem 1.3.** *Let $E$ be a real reflexive strictly convex Banach space with a uniformly Gâteaux differentiable norm, and $K$ a nonempty closed convex subset of $E$ and $\{T(t)\}$ a uniformly asymptotically regular nonexpansive semigroup from $K$ into itself such that $F = \bigcap_{t>0} F(T(t)) \neq \emptyset$, and $f : K \to K$ a fixed contractive mapping* *with contractive coefficient* $\beta \in (0,1)$. *Suppose* $\lim_{n\to\infty} t_n = \infty$, *and* $\alpha_n \in (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *If* $\{x_n\}$ *is given by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \ \ n \geq 1,$$

*then the sequence* $\{x_n\}$ *converges strongly to some common fixed point* $p$ *of solution of* $\mathcal{F}$ *such that* $p$ *is the unique solution in* $F$ *to the co-variational inequality:*

$$\langle f(p) - p, J(y-p)\rangle \leq 0 \ \ \forall y \in F.$$

In this paper, we study a one parameter asymptotically nonexpansive semigroup. Let $X$ be a Banach space and let $C$ be a nonempty closed convex subset of $X$. A one parameter asymptotically nonexpansive semigroup is a family $\mathcal{F} = \{T(t) : t \geq 0\}$ of self-mappings of $C$ such that

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(t+s)x = T(t)T(s)x$ for $t, s > 0$ and $x \in C$;

(iii) $\lim_{t\to 0^+} T(t)x = x$ for $x \in C$;

(iv) for each $t > 0$, $T(t)$ is asymptotically nonexpansive, i.e., there exists a sequence $\{r_n^{(t)}\} \subset [0,1)$ with $r_n^{(t)} \to 0$ as $n \to \infty$ such that

$$\|(T(t))^n x - (T(t))^n y\| \leq (1 + r_n^{(t)})\|x - y\|,$$

for all $x, y \in C$ and $n \geq 1$.

We give some strong convergence for the viscosity iterative process defined by

$$\begin{aligned} y_n &= \alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)(T(t_n))^n x_n), \\ x_{n+1} &= \gamma_n f(y_n) + (1-\gamma_n)(\delta_n y_n + (1-\delta_n)(T(t_n))^n y_n), \ \ n \geq 1, \end{aligned} \tag{1.1}$$

where $\{t_n\}$ is a sequence of positive real numbers.

We need the following lemmas for our main results.

**Lemma 1.4** (See [4, Lemma 2.1])**.** *Let* $\{a_n\}, \{b_n\}$ *and* $\{\delta_n\}$ *be sequences of non-negative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n \ \ \textit{for all } n.$$

*If* $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, *then*

(1) $\lim_{n\to\infty} a_n$ *exists.*

(2) $\lim_{n\to\infty} a_n = 0$ *if* $\{a_n\}$ *has a subsequence converging to zero.*

**Lemma 1.5.** Let $C$ be a nonempty closed subset of a Banach space $X$ and $T(t) : C \to C$ be an asymptotically nonexpansive self-mapping for each $t > 0$ with the fixed point set $F = \bigcap_{t>0} F(T(t)) \neq \emptyset$, where $F(T)$ is the set of all fixed points of the mapping $T$. Then $F$ is a closed subset in $C$.

*Proof.* Let $\{p_n\}$ be a sequence in $F$ such that $p_n \to p$ as $n \to \infty$. Since $C$ is closed and $\{p_n\}$ is a sequence in $C$, we must have $p \in C$. Since $T(t) : C \to C$ is asymptotically nonexpansive, we obtain

$$\|T(t)p - p_n\| = \|T(t)p - T(t)p_n\| \leq (1 + r_1)\|p - p_n\|,$$

for each $t > 0$. Taking limit as $n \to \infty$, we get

$$\|T(t)p - p\| \leq 0,$$

for each $t > 0$, which implies that $T(t)p = p$, for each $t > 0$. Hence $p \in F$. The proof is complete. □

## 2. Main results

In this section, we present our main results. The first theorem gives the necessary and sufficient condition for the convergence of the sequence $\{x_n\}$ defined in (1.1).

**Theorem 2.1.** *Let $X$ be a real Banach space and let $C$ be a nonempty closed convex subset of $X$. Let $\mathcal{F} = \{T(t) : t \geq 0\}$ be an asymptotically nonexpansive semigroup of self-mappings of $C$ such that $F = \bigcap_{t>0} F(T(t)) \neq \emptyset$ in $C$. Assume that, for each $t > 0$, $T(t)$ is an asymptotically nonexpansive mapping with respect to $r_n^{(t)}$ such that $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = \sup_{t>0} r_n^{(t)}$. Let $f : C \to C$ be a contractive mapping and let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ be real sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$.*
*Then, the iterative sequence $\{x_n\}$ defined in (1.1) converges to a common fixed point of $T(t_n)$ if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0.$$

*Proof.* The necessity is obvious, so it is omitted. We now proof the sufficiency. Let $p \in F$. Since $T(t) : C \to C$ is an asymptotically nonexpansive mapping for $t > 0$ and $C$ is a nonempty closed convex subset of $X$, we have

$$\begin{aligned}
\|y_n - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)(T(t_n))^n x_n) - p\| \\
&\leq \alpha_n \|f(x_n) - p\| + (1-\alpha_n)\beta_n\|x_n - p\| \\
&\quad + (1-\alpha_n)(1-\beta_n)\|(T(t_n))^n x_n - p\| \\
&\leq \alpha_n\alpha\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\beta_n\|x_n - p\| \\
&\quad + (1-\alpha_n)(1-\beta_n)(1 + r_n^{(t_n)})\|x_n - p\| \\
&\leq \alpha_n\alpha\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\beta_n\|x_n - p\| \\
&\quad + (1-\alpha_n)(1-\beta_n)\|x_n - p\| + r_n^{(t_n)}(1-\alpha_n)(1-\beta_n)\|x_n - p\| \\
&\leq (1 - (1-\alpha)\alpha_n + r_n)\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&\leq (1 + r_n)\|x_n - p\| + \alpha_n\|f(p) - p\|. \qquad (2.1)
\end{aligned}$$

Similarly we have that

$$\|x_{n+1} - p\| \leq (1 + r_n)\|y_n - p\| + \gamma_n\|f(p) - p\|.$$

From this and (2.1), we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq (1 + r_n)\{(1 + r_n)\|x_n - p\| + \alpha_n\|f(p) - p\|\} + \gamma_n\|f(p) - p\| \\
&\leq (1 + r_n)(1 + r_n)\|x_n - p\| + [(1 + r_n)\alpha_n + \gamma_n]\|f(p) - p\| \\
&= (1 + r_n(2 + r_n))\|x_n - p\| + [(1 + r_n)\alpha_n + \gamma_n]\|f(p) - p\| \\
&= (1 + c_n)\|x_n - p\| + b_n, \qquad (2.2)
\end{aligned}$$

where $c_n = r_n(2 + r_n)$ and $b_n = [(1 + r_n)\alpha_n + \gamma_n]\|f(p) - p\|$. Since $\sum_{n=1}^{\infty} r_n < \infty$, we have that $\{2 + r_n\}$ and $\{1 + r_n\}$ are bounded. Thus $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, imply that $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. Hence Lemma 1.4

implies that $\lim_{n\to\infty}\|x_n - p\|$ exists. Thus $\{x_n\}$ is bounded and so are $\{(T(t_n))^n x_n\}$ and $\{f(x_n)\}$ because $T(t_n)$ is asymptotically nonexpansive and $f$ is contractive. Now since $\{x_n\}$ is bounded and from (2.1), we conclude that $\{y_n\}$ is bounded and so are $\{(T(t_n))^n y_n\}$ and $\{f(y_n)\}$.

We next turn to another calculation for $\|y_n - p\|$ and $\|x_{n+1} - p\|$ as follows.

$$\begin{aligned}
\|y_n - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)(T(t_n))^n x_n) - p\| \\
&\le \alpha_n \|f(x_n) - (T(t_n))^n x_n\| + (1-\alpha_n)\beta_n\|x_n - p\| \\
&\quad + (1-\beta_n+\alpha_n\beta_n)\|(T(t_n))^n x_n - p\| \\
&\le \alpha_n \|f(x_n) - (T(t_n))^n x_n\| + (1-\alpha_n)\beta_n\|x_n - p\| \\
&\quad + (1-\beta_n+\alpha_n\beta_n)(1+r_n^{(t_n)})\|x_n - p\| \\
&= \alpha_n \|f(x_n) - (T(t_n))^n x_n\| + (1-\alpha_n)\beta_n\|x_n - p\| \\
&\quad + (1-\beta_n+\alpha_n\beta_n)\|x_n - p\| + r_n^{(t)}(1-\beta_n+\alpha_n\beta_n)\|x_n - p\| \\
&= (1 + r_n(1+\alpha_n\beta_n))\|x_n - p\| + \alpha_n\|f(x_n) - (T(t_n))^n x_n\| \\
&\le (1+2r_n)\|x_n - p\| + \alpha_n\|f(x_n) - (T(t_n))^n x_n\|. \qquad (2.3)
\end{aligned}$$

Similarly, we have that

$$\|x_{n+1} - p\| \le (1+2r_n)\|y_n - p\| + \gamma_n\|f(y_n) - (T(t))^n y_n\|. \qquad (2.4)$$

Putting (2.3) in (2.4), we obtain that

$$\begin{aligned}
\|x_{n+1} - p\| &\le (1+2r_n)^2\|x_n - p\| + (1+2r_n)\alpha_n\|f(x_n) - (T(t_n))^n x_n\| \\
&\quad + \gamma_n\|f(y_n) - (T(t_n))^n y_n\| \\
&= (1+d_n)\|x_n - p\| + e_n, \qquad (2.5)
\end{aligned}$$

where $d_n = 4r_n(1+r_n)$ and $e_n = (1+2r_n)\alpha_n\|f(x_n) - (T(t_n))^n x_n\| + \gamma_n\|f(y_n) - (T(t_n))^n y_n\|$. By the assumption that $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty}\alpha_n < \infty$, $\sum_{n=1}^{\infty}\gamma_n < \infty$, and $\{(T(t_n))^n x_n\}$, $\{(T(t_n))^n y_n\}, \{f(x_n)\}$ and $\{f(y_n)\}$ are bounded, we have that $\sum_{n=1}^{\infty} d_n < \infty$ and $\sum_{n=1}^{\infty} e_n < \infty$. Hence Lemma 1.4 tells us that $\lim_{n\to\infty}\|x_n - p\|$ exists. Thus $\{\|x_n - p\|\}$ is bounded. Let $L = \sup_n \|x_n - p\|$. We can rewrite (2.5) as

$$\|x_{n+1} - p\| \le \|x_n - p\| + Ld_n + e_n \quad \text{for} \quad n \ge 1. \qquad (2.6)$$

From this and by induction, we obtain, for $m, n \ge 1$ and $p \in F$, that

$$\|x_{n+m} - p\| \le \|x_n - p\| + L\sum_{i=n}^{n+m-1} d_i + \sum_{i=n}^{n+m-1} e_i. \qquad (2.7)$$

Also from (2.6), we obtain

$$d(x_{n+1}, F) \le d(x_n, F) + Ld_n + e_n.$$

By the assumption $\liminf_{n\to\infty} d(x_n, F) = 0$ and because $\sum_{n=1}^{\infty}(Ld_n + e_n) < \infty$, Lemma 1.4 tells us that

$$\lim_{n\to\infty} d(x_n, F) = 0. \qquad (2.8)$$

We now show that $\{x_n\}$ is a Cauchy sequence in $X$. Let $\epsilon > 0$. From (2.7) and since $\sum_{n=1}^{\infty} d_n < \infty$ and $\sum_{n=1}^{\infty} e_n < \infty$, there exists $n_0$ such that, for $n \geq n_0$, we have

$$d(x_n, F) < \epsilon/6, \sum_{i=n}^{\infty} d_i < \epsilon/(3L) \text{ and } \sum_{i=n}^{\infty} e_i < \epsilon/3. \tag{2.9}$$

By the first inequality in (2.9) and the definition of infimum, there exists $p_0 \in F$ such that

$$\|x_{n_0} - p_0\| < \epsilon/6. \tag{2.10}$$

Combining (2.7),(2.9) and (2.10), we obtain

$$\begin{aligned} \|x_{n_0+m} - x_{n_0}\| &\leq \|x_{n_0+m} - p_0\| + \|x_{n_0} - p_0\| \\ &\leq 2\|x_{n_0} - p_0\| + L \sum_{i=n_0}^{n_0+m-1} d_i + \sum_{i=n_0}^{n_0+m-1} e_i \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon, \end{aligned}$$

which implies that $\{x_n\}$ is a Cauchy sequence in $X$. But $X$ is a Banach space, so there must be some $q \in X$ such that $x_n \to q$. Since $C$ is closed and $\{x_n\}$ is a sequence in $C$, we have that $q \in C$. Now $d(x_n, F) \to 0$ and $x_n \to q$ as $n \to \infty$, the continuity of $d(\cdot, F)$ implies that $d(q, F) = 0$. Thus $q \in F$ because $F$ is closed, by Lemma 1.5. Therefore $\{x_n\}$ converges to a common fixed point of $T(t)$, as desired. □

If $\beta_n = 0 = \delta_n$ for all $n$, then the iterative sequences in (1.1) become

$$\begin{aligned} y_n &= \alpha_n f(x_n) + (1 - \alpha_n)(T(t_n))^n x_n, \\ x_{n+1} &= \gamma_n f(y_n) + (1 - \gamma_n)(T(t_n))^n y_n, \quad n \geq 1, \end{aligned} \tag{2.11}$$

and we have the following result for a fixed point of $T(t)$.

**Corollary 2.2.** *Let $X$ be a real Banach space and let $C$ be a nonempty closed convex subset of $X$. Let $\mathcal{F} = \{T(t) : t \geq 0\}$ be an asymptotically nonexpansive semigroup of self-mappings of $C$ such that $F = \bigcap_{t>0} F(T(t)) \neq \emptyset$ in $C$. Assume that, for each $t > 0$, $T(t)$ is an asymptotically nonexpansive mapping with respect to $r_n^{(t)}$ such that $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = \sup_{t>0} r_n^{(t)}$. Let $f : C \to C$ be a contractive mapping and let $\{\alpha_n\}$ and $\{\gamma_n\}$ be real sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then, the sequence $\{x_n\}$ defined in (2.11) converges to a fixed point of $T(t)$ if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0.$$

We also have the following results involving asymptotic regularity and an auxiliary strictly increasing nonnegative function as in Ayaragarnchanakul [1].

**Corollary 2.3.** *Let $X, C, T(t)(t > 0)$ and the iterative sequence $\{x_n\}$ be as in Theorem 2.1. Suppose that the conditions in Theorem 2.1 hold and*
(1) *the mapping $T(t)(t > 0)$ is asymptotically regular in $x_n$, i.e.,*

$$\liminf_{n\to\infty} \|x_n - T(t)x_n\| = 0, \quad t > 0;$$

(2) $\liminf_{n\to\infty} \|x_n - T(t)x_n\| = 0$ *implies that*

$$\liminf_{n\to\infty} d(x_n, F) = 0.$$

*Then the sequences* $\{x_n\}$ *converges to a common fixed point of* $T(t)$.

For a sequence $\{t_n\}$, where $t_n > 0$ for all $n \geq 1$.

$$\begin{aligned} y_n &= \alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)(T(t_n))^n x_n), \\ x_{n+1} &= \gamma_n f(y_n) + (1-\gamma_n)(\delta_n y_n + (1-\delta_n)(T(t_n))^n y_n), \quad n \geq 1. \end{aligned} \tag{2.12}$$

**Theorem 2.4.** *Let* $X, C$ *be as in Theorem 2.1 and the iterative sequence* $\{x_n\}$ *be as in (2.12). Suppose that the mapping* $T(t_i)$ *is asymptotically nonexpansive and asymptotically regular in* $x_n$*, the conditions in Theorem 2.1 hold, and there exists an increasing function* $g : R^+ \to R^+$ *with* $g(r) > 0$ *for all* $r > 0$ *such that*

$$\|x_n - T(t_i)x_n\| \geq g(d(x_n, F)), \quad \forall i \geq 1 \quad \forall n \geq 1.$$

*Then the sequences* $\{x_n\}$ *defined in (2.12) converges to a common fixed point of* $T(t_n)$.

*Proof.* To apply Theorem 2.1, we prove that $\liminf_{n\to\infty} d(x_n, F) = 0$. From the assumption that $\|x_n - T(t_i)x_n\| \geq g(d(x_n, F))$ for $i \geq 1$ and for $n \geq 1$, we have

$$\frac{1}{m}\sum_{i=1}^{m} \|x_n - T(t_i)x_n\| \geq g(d(x_n, F)),$$

for $n \geq 1$ and $m \geq 1$. Since $T(t_i)$ is asymptotically regular in $x_n$, this implies that

$$\liminf_{n\to\infty} g(d(x_n, F)) = 0. \tag{2.13}$$

Suppose that $\liminf_{n\to\infty} d(x_n, F)) = L > 0$. By definition of infimum, there exists an $N$ such that

$$|\inf_{n\geq m} d(x_n, F)) - L| < \frac{L}{2},$$

for all $m \geq N$. Equivalently,

$$d(x_n, F) > \frac{L}{2},$$

for all $n \geq m \geq N$. Since $g$ is increasing, we have that

$$g(d(x_n, F)) \geq g\Big(\frac{L}{2}\Big),$$

for all $n \geq m \geq N$. This implies that

$$\liminf_{n\to\infty} g(d(x_n, F)) \geq g\Big(\frac{L}{2}\Big) > 0,$$

which contradicts (2.13). Hence $\liminf_{n\to\infty} d(x_n, F) = 0$, as desired. □

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