Journal of Nonlinear and Convex Analysis
Volume 16, Number 3, 2015, 529–538

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

ON EXISTENCE AND UNIQUENESS OF BEST PROXIMITY POINTS UNDER A POPESCU'S TYPE CONTRACTIVITY CONDITION

ÁNGEL ALMEIDA, ANTONIO FRANCISCO ROLDÁN LÓPEZ DE HIERRO, AND KISHIN SADARANGANI

Abstract. The main purpose of this paper is to present a best proximity point theorem. The novelty of the result is that assumption relative to the contractivity condition is only satisfied by elements verifying a certain condition.

1. Introduction

In 1976, Bogin [4] proved the following fixed point theorem.

Theorem 1.1 (Bogin [4]). *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping verifying*

$$d(Tx, Ty) \leq ad(x, y) + b\left(d(x, Tx) + d(y, Ty)\right) + c\left(d(x, Ty) + d(y, Tx)\right),$$

for all $x, y \in X$, where $a \geq 0$, $b > 0$, $c > 0$ and $a + 2b + 2c = 1$. Then T has a fixed point.

Very recently, Popescu [15] extended the previous result using an antecedent condition to the contractive property.

Theorem 1.2 (Popescu [15]). *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping satisfying*

$$\frac{1}{2}\, d(x, Tx) \leq d(x, y)$$
$$\Longrightarrow\ d(Tx, Ty) \leq ad(x, y) + b\left(d(x, Tx) + d(y, Ty)\right) + c\left(d(x, Ty) + d(y, Tx)\right),$$

for all $x, y \in X$, where $a \geq 0$, $b > 0$, $c > 0$ and $a + 2b + 2c = 1$. Then T has a fixed point.

Currently, the fixed point theory has became into one of the most useful branches of Nonlinear Analysis due to its applications. Their results have inspired the modern field in which *best proximity points* are the essential key. This theory is based in the following idea: when a non-self-mapping $T : A \to B$ (between two subsets of a metric space) has no fixed points, then it is interesting to study if there exists a point $x \in A$ such that $d(x, Tx) = d(A, B)$. This kind of points are known as *best proximity points of T*. In this paper, inspired by the previous results, we show some theorems that guarantee existence and uniqueness of best proximity points using an antecedent condition.

2010 *Mathematics Subject Classification.* 46T99, 47H10, 47H09, 54H25.
Key words and phrases. Best proximity point, fixed point, P-property, weak P-property.
The second author has been partially supported by Junta de Andalucía by project FQM-268 of the Andalusian CICYE.

2. Preliminaries

Let $\mathbb{N}$ denote the set of all non-negative integers. Henceforth, n, m and k will denote non-negative integers and "*for all n*" will mean "*for all* $n \geq 0$". Throughout the manuscript, let (X, d) be a metric space, let A and B two nonempty subsets of X and let $T : A \to B$ be a mapping. Define:

$$d(A,B) = \inf\left(\{d(a,b) : a \in A, b \in B\}\right),$$
$$A_0 = \{\ a \in A :\ d(a,b) = d(A,B) \text{ for some } b \in B\ \},$$
$$B_0 = \{\ b \in B :\ d(a,b) = d(A,B) \text{ for some } a \in A\ \}.$$

Notice that if $a \in A$ and $b \in B$ verify $d(a,b) = d(A,B)$, then $a \in A_0$ and $b \in B_0$. Therefore, A_0 is nonempty if, and only if, B_0 is nonempty. Therefore, is A_0 is nonempty, then A, B and B_0 are nonempty subsets of X. It is clear that if $A \cap B \neq \emptyset$, then A_0 is nonempty. In [7], the authors discussed sufficient conditions in order to guarantee the nonemptiness of A_0. In general, if A and B are closed subsets of a normed linear space such that $d(A,B) > 0$, then A_0 is contained in the boundary of A (see [20]).

The main aim of this paper is to study sufficient conditions to ensure the existence and the unicity, of the following kind of points.

Definition 2.1. We will say a point $x \in A$ is a *best proximity point of* T if $d(x, Tx) = d(A,B)$. If $A = B$, a best proximity point of T is a *fixed point of* T (that is, $Tx = x$).

Definition 2.2. Let A and B be two subsets of a metric space (X, d) such that A_0 is nonempty. We say that the pair (A, B) has the *P-property* if

$$\left.\begin{array}{c} a_1, a_2 \in A_0, \quad b_1, b_2 \in B_0 \\ d(a_1, b_1) = d(A,B) \\ d(a_2, b_2) = d(A,B) \end{array}\right\} \Rightarrow\ d(a_1, a_2) = d(b_1, b_2).$$

We will consider a weaker condition than the P-property as follows.

Definition 2.3. Let A and B be two subsets of a metric space (X, d) such that A_0 is nonempty. We say that the pair (A, B) has the *weak P-property* if

$$\left.\begin{array}{c} a_1, a_2 \in A_0, \quad b_1, b_2 \in B_0 \\ d(a_1, b_1) = d(A,B) \\ d(a_2, b_2) = d(A,B) \end{array}\right\} \Rightarrow\ d(a_1, a_2) \leq d(b_1, b_2).$$

In the following Example, we show that the weak P-property does not imply the P-property.

Example 2.4. Let $X = \mathbb{R}^2$ provided with the Euclidean metric and, given $n \in \mathbb{N}$ with $n \geq 1$, consider

$$A = \{(x, x+2) : -1 \leq x \leq 0\} \cup \{(x, -x+2) : 0 \leq x \leq 1\},$$
$$B = \{\ (-n, 0),\ (n, 0)\ \}.$$

It is easy to prove that $d(A,B) = \sqrt{(n-1)^2 + 1}$, $A_0 = \{(-1,1), (1,1)\}$ and $B_0 = B$. Taking into account that

$$d((-1,1),(1,1)) = 2, \quad d((-n,0),(n,0)) = 2n,$$

$$d((1,1,),(n,0)) = d(A,B) = d((-1,1),(-n,0)),$$

we have that

$$d((-1,1),(1,1)) = 2 \leq 2n = d((-n,0),(n,0)).$$

Therefore, the pair (A,B) has the weak P-property but it only verifies the P-property if $n=1$.

We will consider the following kind of functions.

Definition 2.5. A *comparison function* is a non-decreasing function $\varphi:[0,\infty)\to[0,\infty)$ such that $\lim_{n\to\infty}\varphi^n(t)=0$ for all $t>0$, where $\varphi^n=\varphi\circ\varphi\circ\overset{(n)}{\cdots}\circ\varphi$ denotes the n-iterate of φ. Let $\mathcal{F}$ denote the family of all comparison functions.

It is clear that every comparison function φ verifies the following properties.

$$(i)\ \varphi(t)<t \text{ for all } t>0; \quad (ii)\ \varphi(0)=0;$$

$$(iii)\ \varphi(t)\leq t \text{ for all } t\geq 0; \quad (iv)\ \varphi(t)=t \ \Rightarrow\ t=0. \tag{2.1}$$

3. Main results

The following result is the main aim of the present paper.

Theorem 3.1. *Let A and B two closed subsets of a complete metric space (X,d) such that $A_0\neq 0$ and let $T:A\to B$ be a mapping such that $TA_0\subseteq B_0$. Suppose that there exists $\varphi\in\mathcal{F}$ verifying that, for all $x,y\in A_0$,*

$$d(x,Tx)\leq d(x,y)+d(A,B) \implies d(Tx,Ty)\leq\varphi\left(M(x,y)-d(A,B)\right), \tag{3.1}$$

where $M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty)\}$. Also assume that

(a)**:** *T is continuous and (A,B) has the weak* P*-property.*

Then T has a unique best proximity point.

Notice that we only suppose that the contractivity condition holds for all $x,y\in A_0$, but not necessarily in A.

Proof. Existence. Fix any $x_0\in A_0$. Since $Tx_0\in TA_0\subseteq B_0$, there exists $x_1\in A$ such that $d(x_1,Tx_0)=d(A,B)$. In particular, $x_1\in A_0$. Now, since $Tx_1\in TA_0\subseteq B_0$, there exists $x_2\in A$ such that $d(x_2,Tx_1)=d(A,B)$, and, in particular, $x_2\in A_0$. Repeating this process, we can consider a sequence $\{x_n\}\subseteq A_0$ verifying that

$$d(x_{n+1},Tx_n)=d(A,B) \qquad \text{for all } n\geq 0. \tag{3.2}$$

Suppose that there is $n_0\in\mathbb{N}$ such that $x_{n_0}=x_{n_0+1}$. In this case, $d(x_{n_0},Tx_{n_0})=d(x_{n_0+1},Tx_{n_0})=d(A,B)$, and the existence of a best proximity point of T is proved. On the contrary, suppose that $x_n\neq x_{n+1}$ for all $n\geq 0$, that is,

$$d(x_n,x_{n+1})>0 \qquad \text{for all } n\geq 0. \tag{3.3}$$

Using the weak P-property, for all $n,m\geq 0$,

$$\left.\begin{array}{c} x_{n+1},x_{m+1}\in A_0, \quad Tx_n,Tx_m\in B_0 \\ d(x_{n+1},Tx_n)=d(A,B) \\ d(x_{m+1},Tx_m)=d(A,B)\end{array}\right\} \Rightarrow\ d(x_{n+1},x_{m+1})\leq d(Tx_n,Tx_m). \tag{3.4}$$

Furthermore, for all n,

$$d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, x_{n+1}) + d(A, B). \tag{3.5}$$

Next we claim that

$$d(Tx_n, Tx_{n+1}) \leq \varphi(d(x_n, x_{n+1})) \qquad \text{for all } n \geq 0. \tag{3.6}$$

Taking into account (3.5) and applying the contractivity condition (3.1) to $x = x_n$ and $y = x_{n+1}$, we notice that, for all n,

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq \varphi\left(M(x_n, x_{n+1}) - d(A, B)\right) \\ &= \varphi\left(\max\left\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\right\} - d(A, B)\right) \end{aligned} \tag{3.7}$$

Consider the subsets

$$N_1 = \{\ n \in \mathbb{N} : \ \text{the maximum in (3.7) is } d(x_n, x_{n+1})\ \},$$

$$N_2 = \{\ n \in \mathbb{N} : \ \text{the maximum in (3.7) is } d(x_n, Tx_n)\ \},$$

$$N_3 = \{\ n \in \mathbb{N} : \ \text{the maximum in (3.7) is } d(x_{n+1}, Tx_{n+1})\ \}.$$

Clearly, $N_1 \cup N_2 \cup N_3 = \mathbb{N}$. We distinguish three cases.

- If $n \in N_1$, then

  $$d(Tx_n, Tx_{n+1}) \leq \varphi\left(d(x_n, x_{n+1}) - d(A, B)\right) \leq \varphi\left(d(x_n, x_{n+1})\right)$$

  since φ is non-decreasing, so (3.6) holds in this case.
- If $n \in N_2$, it follows from (3.5) that $M(x_n, x_{n+1}) - d(A, B) = d(x_n, Tx_n) - d(A, B) \leq d(x_n, x_{n+1})$, and taking into account that φ is non-decreasing, we deduce that

  $$d(Tx_n, Tx_{n+1}) \leq \varphi\left(d(x_n, Tx_n) - d(A, B)\right) \leq \varphi\left(d(x_n, x_{n+1})\right),$$

  which means that (3.6) also holds in this case.
- If $n \in N_3$, it also follows from (3.5) that $M(x_n, x_{n+1}) - d(A, B) = d(x_{n+1}, Tx_{n+1}) - d(A, B) \leq d(x_{n+1}, x_{n+2})$ but, in this case, applying (3.3) and (3.4),

  $$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq d(Tx_n, Tx_{n+1}) \leq \varphi\left(d(x_{n+1}, Tx_{n+1}) - d(A, B)\right) \\ &\leq \varphi\left(d(x_{n+1}, x_{n+2})\right) < d(x_{n+1}, x_{n+2}), \end{aligned}$$

  which is false. Therefore, the case $n \in N_3$ is impossible.

The previous cases show that (3.6) holds. Combining (3.4) and (3.6), we have that

$$d(x_{n+1}, x_{n+2}) \leq d(Tx_n, Tx_{n+1}) \leq \varphi(d(x_n, x_{n+1})) \qquad \text{for all } n \geq 0,$$

and repeating this process, $d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1))$ for all $n \geq 1$. As φ is a comparison function,

$$\{d(x_n, x_{n+1})\} \to 0. \tag{3.8}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence reasoning by contradiction. Assume that $\{x_n\}$ is not a Cauchy sequence and we are going to get a contradiction.

In such a case, there exists $\varepsilon_0 > 0$ and two partial subsequences $\{x_{m(k)}\}_{k\in\mathbb{N}}$ and $\{x_{n(k)}\}_{k\in\mathbb{N}}$ verifying that, for all $k \in \mathbb{N}$,

$$k \leq m(k) < n(k), \quad d(x_{m(k)}, x_{n(k)}) \geq \varepsilon_0, \quad d(x_{m(k)}, x_{n(k)-1}) < \varepsilon_0 \tag{3.9}$$

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon_0. \tag{3.10}$$

By (3.8), there exists $n_1 \in \mathbb{N}$ such that $d(x_n, x_{n+1}) \leq \varepsilon_0/2$ for all $n \geq n_1$. Moreover, by (3.10), there exists $n_2 \in \mathbb{N}$ such that $d(x_{m(k)}, x_{n(k)-1}) \geq \varepsilon_0/2$ for all $k \geq n_2$. Letting $n_0 = \max(n_1, n_2)$, we have that for all $k \geq n_0$, as $n(k)-1 \geq m(k) \geq k \geq n_0$,

$$d(x_k, x_{k+1}) \leq \frac{\varepsilon_0}{2} \leq d(x_{m(k)}, x_{n(k)-1}). \tag{3.11}$$

Therefore, for all $k \geq n_0$,

$$\begin{aligned} d(x_{m(k)}, Tx_{m(k)}) &\leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, Tx_{m(k)}) \\ &= d(x_{m(k)}, x_{m(k)+1}) + d(A, B) \\ &\leq \frac{\varepsilon_0}{2} + d(A, B) \leq d(x_{m(k)}, x_{n(k)-1}) + d(A, B). \end{aligned} \tag{3.12}$$

We are going to show that

$$d(x_{m(k)+1}, x_{n(k)}) \leq \varphi(\varepsilon_0) \quad \text{for all } k \geq n_0. \tag{3.13}$$

Applying (3.4), (3.12) and the contractivity condition (3.1) to $x = x_{m(k)}$ and $y = x_{n(k)-1}$, for all $k \geq n_0$,

$$\begin{aligned} d(x_{m(k)+1}, x_{n(k)}) &\leq d(Tx_{m(k)}, Tx_{n(k)-1}) \leq \varphi\left(M(x_{m(k)}, x_{n(k)-1}) - d(A, B)\right) \\ &= \varphi\Big(\max\big\{d(x_{m(k)}, x_{n(k)-1}), d(x_{m(k)}, Tx_{m(k)}), \\ &\qquad d(x_{n(k)-1}, Tx_{n(k)-1})\big\} - d(A, B)\Big). \end{aligned} \tag{3.14}$$

Consider the subsets

$$N_1' = \left\{ \ k \geq n_0 : \ \text{the maximum in (3.14) is } d(x_{m(k)}, x_{n(k)-1}) \ \right\},$$

$$N_2' = \left\{ \ k \geq n_0 : \ \text{the maximum in (3.14) is } d(x_{m(k)}, Tx_{m(k)}) \ \right\},$$

$$N_3' = \left\{ \ k \geq n_0 : \ \text{the maximum in (3.14) is } d(x_{n(k)-1}, Tx_{n(k)-1}) \ \right\}.$$

Clearly $N_1' \cup N_2' \cup N_3' = \mathbb{N}$.

- If $k \in N_1'$, it follows from (3.9) that

  $$\begin{aligned} d(x_{m(k)+1}, x_{n(k)}) &\leq \varphi\left(d(x_{m(k)}, x_{n(k)-1}) - d(A, B)\right) \\ &\leq \varphi\left(d(x_{m(k)}, x_{n(k)-1})\right) \leq \varphi(\varepsilon), \end{aligned}$$

  so (3.13) holds.
- If $k \in N_2'$, then by (3.2) and (3.11),

  $$\begin{aligned} M(x_{m(k)}, x_{n(k)-1}) - d(A, B) &= d(x_{m(k)}, Tx_{m(k)}) - d(A, B) \\ &\leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, Tx_{m(k)}) - d(A, B) \\ &= d(x_{m(k)}, x_{m(k)+1}) \leq \varepsilon_0/2, \end{aligned}$$

  which means that

  $$d(x_{m(k)+1}, x_{n(k)}) \leq \varphi\left(M(x_{m(k)}, x_{n(k)-1}) - d(A, B)\right)$$

$$\leq \varphi(\varepsilon_0/2) \leq \varphi(\varepsilon_0),$$

so (3.13) also holds.

- If $k \in N_3'$, then

$$\begin{aligned} M(x_{m(k)}, x_{n(k)-1}) - d(A,B) &= d(x_{n(k)-1}, Tx_{n(k)-1}) - d(A,B) \\ &\leq d(x_{n(k)-1}, x_{n(k)}) + d(x_{n(k)}, Tx_{n(k)-1}) - d(A,B) \\ &= d(x_{n(k)-1}, x_{n(k)}) \leq \varepsilon_0/2, \end{aligned}$$

and the same reasoning yields to (3.13).

In any case, (3.13) holds. But, in this case, taking limit as $k \to \infty$ in (3.13) and using (3.10), we deduce that $\varepsilon_0 \leq \varphi(\varepsilon_0) < \varepsilon_0$, which is impossible. This contradiction shows us that $\{x_n\}$ is a Cauchy sequence. Since (X,d) is complete, there exists $x \in X$ such that $\{x_n\} \to x$. Furthermore, $x \in A$ because A is closed and $x_n \in A_0 \subseteq A$ for all n. Applying the continuity of T and taking limit in (3.2), we conclude that $d(x,Tx) = d(A,B)$, that is, x is a best proximity point of T.

Uniqueness. Assume that $x, y \in A$ are two best proximity points of T, that is, $d(x,Tx) = d(y,Ty) = d(A,B)$, and we are going to show that $x = y$. In such a case, using that weak P-property,

$$\left.\begin{array}{c} x, y \in A_0, \quad Tx, Ty \in B_0 \\ d(x,Tx) = d(A,B) \\ d(y,Ty) = d(A,B) \end{array}\right\} \Rightarrow d(x,y) \leq d(Tx,Ty). \tag{3.15}$$

Notice that $d(x,Tx) = d(A,B) \leq d(x,y) + d(A,B)$. Applying (2.1) and the contractivity condition (3.1) to x and y, we have that

$$\begin{aligned} d(x,y) \leq d(Tx,Ty) &\leq \varphi\left(M(x,y) - d(A,B)\right) \\ &= \varphi\left(\max\{d(x,y), d(x,Tx), d(y,Ty)\} - d(A,B)\right) \\ &= \varphi\left(\max\{d(x,y), d(A,B)\} - d(A,B)\right) \\ &= \varphi\left(\max\{d(x,y) - d(A,B), 0\}\right). \end{aligned}$$

If $\max\{d(x,y) - d(A,B), 0\} = 0$, then $d(x,y) \leq \varphi(0) = 0$, so $x = y$. And if $\max\{d(x,y) - d(A,B), 0\} = d(x,y) - d(A,B)$, then

$$d(x,y) \leq \varphi\left(d(x,y) - d(A,B)\right) \leq \varphi\left(d(x,y)\right) \leq d(x,y).$$

The equality $\varphi\left(d(x,y)\right) = d(x,y)$ also yields to $d(x,y) = 0$, that is, $x = y$. This finishes the proof. □

Remark 3.2. Notice that in the previous proof we have showed that, starting from any $x_0 \in A_0$, it is possible to consider a sequence $\{x_n\} \subseteq A_0$ verifying that $d(x_{n+1}, Tx_n) = d(A,B)$ for all $n \geq 0$, and any sequence verifying this property converges to the unique best proximity point of T.

In the following result, we replace the continuity of T by other hypotheses.

Theorem 3.3. *Let A and B two closed subsets of a complete metric space (X,d) such that $A_0 \neq 0$ and let $T : A \to B$ be a mapping such that $TA_0 \subseteq B_0$. Suppose that there exists $\varphi \in \mathcal{F}$ verifying that, for all $x \in A_0$ and all $y \in A$,*

$$rd(x,Tx) \leq d(x,y) + d(A,B) \implies d(Tx,Ty) \leq \varphi\left(M(x,y) - d(A,B)\right), \tag{3.16}$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$ *and* r *verifies* $r < 1$ *if* $d(A,B) > 0$ *and* $r \leq 1/2$ *if* $d(A,B) = 0$. *Also assume that*

(a')**:** (A,B) *has the* P*-property.*

Then T *has a unique best proximity point.*

Proof. If $r_1, r_2 \in \,]-\infty, 1/2[$ verify $r_1 \leq r_2$ and the result holds for r_2, then it is also valid for r_1. Then, it is only necessary to prove it when $r > 0$. Taking into account that the P-property implies the weak P-property and following the lines of the proof of Theorem 3.1, we can deduce that $\{x_n\} \subseteq A_0$ is a convergent sequence, and there is $x \in A$ such that $\{x_n\} \to x$. Using the P-property as in (3.4), we deduce that, for all $n, m \geq 0$,

$$\left.\begin{array}{c} x_{n+1}, x_{m+1} \in A_0, \quad Tx_n, Tx_m \in B_0 \\ d(x_{n+1}, Tx_n) = d(A,B) \\ d(x_{m+1}, Tx_m) = d(A,B) \end{array}\right\} \;\Rightarrow\; d(x_{n+1}, x_{m+1}) = d(Tx_n, Tx_m).$$

This means that $\{Tx_n\} \subseteq B_0 \subseteq B$ is also a Cauchy sequence. Thus, there is $z \in B$ such that $\{Tx_n\} \to z$. Taking limit in $d(x_{n+1}, Tx_n) = d(A,B)$ as $n \to \infty$, we deduce that

$$d(x,z) = d(A,B). \tag{3.17}$$

Next, we distinguish whether $z = x$ or not.

Case 1: $z = x$. In this case, by (3.17), $d(A,B) = 0$, so $r \leq 1/2$ and $x_{n+1} = Tx_n$ for all n. We are going to show that the set $N = \{n \in \mathbb{N} : d(x_n, x_{n+1}) \leq 2d(x_n, x)\}$ is not finite reasoning by contradiction. If N is finite, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+1}) > 2d(x_n, x)$ for all $n \geq n_0$. Therefore, for $n \geq n_0$,

$$2d(x_n, x) < d(x_n, x_{n+1}) \leq d(x_n, x) + d(x, x_{n+1}) \;\Rightarrow\; d(x_n, x) < d(x_{n+1}, x).$$

As $0 \leq d(x_{n_0}, x) < d(x_{n_0+1}, x) < d(x_n, x)$ for all $n \geq n_0 + 2$, we have that $\varepsilon_0 = d(x_{n_0+1}, x) > 0$ verifies $\varepsilon_0 < d(x_n, x)$ for all $n \geq n_0 + 2$, which contradicts the fact that $\{x_n\} \to x$. This proves that N is not finite. Therefore, there exists a partial subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$r d(x_{n(k)}, Tx_{n(k)}) \leq \frac{1}{2} d(x_{n(k)}, x_{n(k)+1}) \leq d(x_{n(k)}, x) \quad \text{for all } k.$$

Applying the contractive condition (3.16) to $x_{n(k)} \in A_0$ and $x \in A$, we have that, for all k,

$$\begin{aligned} d(x_{n(k)+1}, Tx) &= d(Tx_{n(k)}, Tx) \leq \varphi\left(M(x_{n(k)}, x)\right) \\ &= \varphi\left(\max\left\{d(x_{n(k)}, x), d(x_{n(k)}, Tx_{n(k)}), d(x, Tx)\right\}\right) \\ &= \varphi\left(\max\left\{d(x_{n(k)}, x), d(x_{n(k)}, x_{n(k)+1}), d(x, Tx)\right\}\right). \end{aligned} \tag{3.18}$$

Next we show that $Tx = x$ reasoning by contradiction. Taking into account that $\{d(x_{n(k)}, x)\}_{k \in \mathbb{N}} \to 0$ and $\{d(x_{n(k)}, x_{n(k)+1})\}_{k \in \mathbb{N}} \to 0$, if $d(x, Tx) > 0$, there exists $k_0 \in \mathbb{N}$ such that $\max\left\{d(x_{n(k)}, x), d(x_{n(k)}, x_{n(k)+1}), d(x, Tx)\right\} = d(x, Tx)$ for all $k \geq k_0$. Therefore $d(x_{n(k)+1}, Tx) \leq \varphi\left(d(x, Tx)\right)$ for all $k \geq k_0$. Letting $k \to \infty$, we deduce that $d(x, Tx) \leq \varphi\left(d(x, Tx)\right)$, but this is only possible when $d(x, Tx) = 0$, which contradicts $Tx \neq x$. As a consequence, x is a fixed point of T, that is, a best proximity point.

Case 2: $z \neq x$. In this case, we are going to prove that $z = Tx$. Notice that $d(A,B) = d(x,z) > 0$, so $r < 1$. Indeed, as

$$\lim_{n\to\infty} rd(x_n, Tx_n) = rd(x,z) \quad \text{and}$$
$$\lim_{n\to\infty} (d(x_n,x) + d(A,B)) = d(A,B) = d(x,z),$$

there exists $n_0 \in \mathbb{N}$ such that

$$rd(x_n, Tx_n) \le d(x_n, x) + d(A,B) \quad \text{for all } n \ge n_0.$$

Applying the contractivity condition (3.1), for all $n \ge n_0$,

$$\begin{aligned} d(Tx_n, Tx) &\le \varphi\left(M(x_n,x) - d(A,B)\right) \\ &= \varphi\left(\max\left\{d(x_n,x), d(x_n,Tx_n), d(x,Tx)\right\} - d(A,B)\right). \end{aligned} \tag{3.19}$$

Consider the subsets

$$N_1'' = \{\ n \in \mathbb{N} :\ \text{the maximum in (3.19) is } d(x_n, x)\ \},$$

$$N_2'' = \{\ n \in \mathbb{N} :\ \text{the maximum in (3.19) is } d(x_n, Tx_n)\ \},$$

$$N_3'' = \{\ n \in \mathbb{N} :\ \text{the maximum in (3.19) is } d(x, Tx)\ \}.$$

Clearly $N_1'' \cup N_2'' \cup N_3'' = \mathbb{N}$, so both three subsets can not be finite at the same time.

- Suppose that N_1'' is not finite. Then there exists a partial subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that
$$\max\left\{d(x_{n(k)},x), d(x_{n(k)},Tx_{n(k)}), d(x,Tx)\right\} = d(x_{n(k)},x) \quad \text{for all } k.$$
In such a case,
$$d(Tx_{n(k)},Tx) \le \varphi\left(d(x_{n(k)},x) - d(A,B)\right) \le \varphi\left(d(x_{n(k)},x)\right) \le d(x_{n(k)},x),$$
and letting $k \to \infty$ we deduce that $d(z,Tx) = 0$, that is, $z = Tx$.
- If N_2'' is not finite, there exists a partial subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that, for all k,
$$\begin{aligned} M(x_{n(k)},x) - d(A,B) &= \max\left\{d(x_{n(k)},x), d(x_{n(k)},Tx_{n(k)}), d(x,Tx)\right\} - d(A,B) \\ &= d(x_{n(k)},Tx_{n(k)}) - d(A,B) \\ &\le d(x_{n(k)},x_{n(k)+1}) + d(x_{n(k)-1},Tx_{n(k)}) - d(A,B) = d(x_{n(k)},x_{n(k)+1}), \end{aligned}$$
and also letting $k \to \infty$ in
$$\begin{aligned} d(Tx_{n(k)},Tx) &\le \varphi\left(M(x_{n(k)},x) - d(A,B)\right) \\ &\le M(x_{n(k)},x) - d(A,B) \le d(x_{n(k)},x_{n(k)+1}), \end{aligned}$$
we deduce that $d(z,Tx) = 0$.
- Finally, if N_3'' is not finite, there exists a partial subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that, for all k,
$$\begin{aligned} &M(x_{n(k)},x) - d(A,B) \\ &\quad = \max\left\{d(x_{n(k)},x), d(x_{n(k)},Tx_{n(k)}), d(x,Tx)\right\} - d(A,B) \\ &\quad = d(x,Tx) - d(A,B) \le d(x,z) + d(z,Tx) - d(A,B) \\ &\quad = d(A,B) + d(z,Tx) - d(A,B) = d(z,Tx). \end{aligned}$$

Therefore, $d(Tx_{n(k)}, Tx) \leq \varphi\left(M(x_{n(k)}, x) - d(A, B)\right) \leq \varphi(d(z, Tx))$ for all k. Letting $k \to \infty$ we conclude that $d(z, Tx) \leq \varphi(d(z, Tx)) \leq d(z, Tx)$, and the equality $\varphi(d(z, Tx)) = d(z, Tx)$ holds that $d(z, Tx) = 0$.

This means that, in any case, $d(x, Tx) = d(x, z) = d(A, B)$, that is, x is a best proximity point of T. The uniqueness of x can be proved as in Theorem 3.1. □

Taking into account that the pair (X, X) satisfies the P-property, if we put $A = B = X$ in Theorems 3.1 and 3.3, we obtain the following counterparts in the fixed point theory.

Corollary 3.4. *Let $T : X \to X$ be a mapping form a complete metric space (X, d) into itself and suppose that there exists $\varphi \in \mathcal{F}$ and $r \in \mathbb{R}$ verifying that, for all $x, y \in X$,*

$$rd(x, Tx) \leq d(x, y) \implies d(Tx, Ty) \leq \varphi\left(M(x, y)\right),$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$. Also assume that either

(b) T is a continuous mapping and $r = 1$, or

(b') r verifies $r < 1$ if $d(A, B) > 0$ and $r \leq 1/2$ if $d(A, B) = 0$.

Then T has a unique fixed point.

Next, we particularize Theorems 3.1 and 3.3 to the case in which $\varphi(t) = kt$ for all $t \geq 0$, where $k \in [0, 1)$.

Corollary 3.5. *Let A and B two closed subsets of a complete metric space (X, d) such that $A_0 \neq 0$ and let $T : A \to B$ be a mapping such that $TA_0 \subseteq B_0$. Suppose that there exists $k \in [0, 1)$ and $r \in \mathbb{R}$ verifying that, for all $x \in A_0$ and all $y \in A$,*

$$rd(x, Tx) \leq d(x, y) + d(A, B) \implies d(Tx, Ty) \leq k\left(M(x, y) - d(A, B)\right),$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$. Also assume that either

(a) T is continuous, $r = 1$ and (A, B) has the weak P*-property, or*

(a') (A, B) has the P*-property and r verifies $r < 1$ if $d(A, B) > 0$ and $r \leq 1/2$ if $d(A, B) = 0$.*

Then T has a unique best proximity point.

Acknowledgements

The authors would like to thank the referees for their valuable commentaries and suggestions, which have improved the final version of this work.

References

[1] A. Abkar and M. Gabeleh, *Proximal quasi-normal structure and a best proximity point theorem*, J. Nonlinear Convex Anal. **14** (2013), 653–659.

[2] M. A. Al-Thagafi and N. Shahzad, *Convergence and existence results for best proximity points*, Nonlinear Anal. **70** (2009), 3665–3671.

[3] J. Anuradha and P. Veeramani, *Proximal pointwise contraction*, Topology and its Applications **156** (2009), 2942–2948.

[4] J. Bogin, *A generalization of a fixed point theorem of Goebel, Kirk and Shimi*, Canad. Math. Bull. **19** (1976), 7–12.

[5] A. A. Eldred and P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl. **323** (2006), 1001–1006.

[6] Z. Kadelburg and S. Radenović, *A note on some recent best proximity point results for nonself mappings*, Gulf Journal of Mathematics **1** (2013), 36–41.
[7] W. A. Kirk, S. Reich and P. Veeramani, *Proximal retracts and best proximity pair theorems*, Numer. Funct. Anal. Optim. **24** (2003), 851–862.
[8] P. Kumam, H. Aydi, E. Karapınar and W. Sintunavarat, *Best proximity points and extension of Mizoguchi-Takahashi's fixed point theorems*, Fixed Point Theory Appl. 2013, 2013:242.
[9] C. Mongkolkeha and P. Kumam, *Best proximity point theorems for generalized cyclic contractions in ordered metric Spaces*, J. Optim. Theory Appl. **155** (2012), 215–226.
[10] C. Mongkolkeha and P. Kumam, *Some common best proximity points for proximity commuting mappings*, Optimization Letters **7** (2013), 1825–1836.
[11] C. Mongkolkeha and P. Kumam, *Best proximity points for asymptotic proximal pointwise weaker Meir-Keeler-type ϕ-contraction mappings*, J. Egypt. Math. Soc. **21** (2013), 87–90.
[12] C. Mongkolkeha, Y. J. Cho and P. Kumam, *Best proximity points for generalized proxinal C-contraction mappings in metric spaces with partial orders*, J. Ineq. Appl. 2013, 2013:94.
[13] C. Mongkolkeha, Y. J. Cho and Poom Kumam, *Best proximity points for Geraghty's proximal contraction mappings*, Fixed Point Theory Appl. 2013, 2013:180.
[14] H. K. Nashine, C. Vetro and P. Kumam, *Best proximity point theorems for rational proximal contractions*, Fixed Point Theory Appl. 2013, 2013:95.
[15] O. Popescu, *Two generalizations of some fixed point theorems*, Comput. Math. Appl. **62** (2011), 3912–3919.
[16] O. Popescu, *Fixed points for (ψ, φ)-weak contractions*, Appl. Math. Lett. **24** (2011), 1–4.
[17] W. Sanhan, C. Mongkolkeha and P. Kumam, *Generalized proximal ϕ-contraction mappings and Best proximity points*, Abstr. Appl. Anal. 2012, Article ID 896912, 19 pages.
[18] V. Sankar Raj, *A best proximity theorem for weakly contractive non-self mappings*, Nonlinear Anal. **74** (2011), 4804–4808.
[19] G. Sankara Raju Kosuru and P. Veeramani, *On existence of best proximity pair theorems for relatively nonexpansive mappings*, J. Nonlinear Convex Anal. **11** (2010), 71–77.
[20] S. Sadiq Basha, *Best proximity points: global optimal aproximate solutions*, J. Glob. Optim. **49** (2011), 15–21.
[21] W. Sintunavarat and P. Kumam, *Coupled best proximity point theorem in metric spaces*, Fixed Point Theory Appl. 2012, 2012:93.

Manuscript received November 26, 2013
revised March 25, 2014

A. Almeida
Department of Mathematics, University of Las Palmas de Gran Canaria, Campus Universitario de Tafira, Las Palmas de Gran Canaria, 35017, Spain
E-mail address: aalmeida@dma.ulpgc.es

A. Roldán
Department of Mathematics, University of Jaén, Campus las Lagunillas s/n, 23071, Jaén, Spain
E-mail address: afroldan@ujaen.es,aroldan@ugr.es

Kishin Sadaranganii
Department of Mathematics, University of Las Palmas de Gran Canaria, Campus Universitario de Tafira, Las Palmas de Gran Canaria, 35017, Spain
E-mail address: ksadaran@dma.ulpgc.es